

Quantisation on the Two-Handled Sphere

P. Crehan

Dept. of Mathematics, Faculty of Science, University of Kyoto

PAT@KUSM.KYOTO-U.AC.JP

Abstract

There is a class of polygonal billiards which can be mapped into classical flows on the surface of a g -handled sphere. In particular the $\pi/3$ -rhombus billiard flow is equivalent to a flow on the surface of a two-handled sphere. The quantisation of this system is as yet an unsolved problem.

1 Introduction

Flows in phase-space of compact two-dimensional fully integrable Hamiltonian systems have a second conservation law and are confined to the surface of two-dimensional tori. Generic systems are not confined in this way. Their attractors may have fractal dimension greater than two, the orbits may fill up parts of the three dimensional energy surface and in extreme cases they may even be ergodic. Many working in chaos-theory have found it interesting to quantise generic Hamiltonian systems and to consider the asymptotic properties of their spectra. These systems are not solvable as a rule and such work has to be done numerically. Generally the asymptotic energy gaps of fully integrable systems have a Poisson distribution, whereas chaotic systems with time reversal invariance are modelled by the Gaussian-Orthogonal-Ensemble. In between fully chaotic-ergodic systems and the fully integrable regular systems lie the mixed systems whose level-spacings are sometimes described by the Berry-Robnic distribution. These are not mathematically exact statements. They are reasonable approximations born out by numerical experiment, and the reader is referred to [1] for more information, as well as to [2] and [3] for general background on these topics. The question as to whether or not quantum chaos exists is a circular one, so anyone who insists on asking it is referred to [4] to find out why it does, as well as to [5] to find out why it does not!

The overall picture is that there are nice regular fully integrable systems which we can solve explicitly, whose classical motions lie on tori, and whose quantum level statistics are Poisson. Examples of these are provided by the integrable billiards. At the other end of the scale are chaotic systems which are easy to understand classically because they are ergodic, whose quantum mechanics can be solved if we are lucky, and whose level spacings have statistics which are GOE. An important example of which is the Sinai billiard [6]. In between lie systems which are very complicated, which we can not solve explicitly and which are qualitatively hard to understand. An interesting family of such in-between systems is provided by a class of billiard flows equivalent to flows on

g-handled spheres. These have an extra conservation law, so naively we might expect to analytically compute the energy levels and wavefunctions of the quantum problem. Although classically they are very like fully integrable systems, their quantum properties are very like those of classically chaotic systems. Even in the simplest case that of the $\pi/3$ -rhombus billiard, a full analytic solution to the quantum problem has not yet been found. The difficulty stems from the non-trivial topology of the invariant manifolds in phase-space.

The purpose of my talk is simply to try to provide some classical intuition which might bear on the problems of quantisation on a manifold with non-trivial topology, as well as an appreciation of the non-trivial nature of the task.

2 Classical flows on the g-handled sphere

First we take a look at some general features of classical flows on g-handled spheres. Most of what I will say can be found in [7]. Many elegant theorems relate the topology of these surfaces to properties of functions, vector fields or metrics with which they may be endowed. In particular if $\mathcal{M}$ is a compact orientable two dimensional manifold we have

$$\chi(\mathcal{M}) = V - E + F \quad (1)$$

$$= 2 - 2g \quad (2)$$

$$= \sum_{\alpha} i_X(p_{\alpha}) \quad (3)$$

$$= A - B + C \quad (4)$$

$$= \frac{1}{2\pi} \int_{\mathcal{M}} K dA. \quad (5)$$

If V is the number of vertices, E the number of edges and F the number of faces in a simplicial decomposition of $\mathcal{M}$, equation (1) defines the Euler characteristic $\chi(\mathcal{M})$. The Euler formula (2) relates this to the genus of $\mathcal{M}$. These are related to the properties of vector fields on $\mathcal{M}$ through the Hopf-index theorem (3). In this formula X is a vector field with a singular point at each p_{α} whose index is denoted $i_X(p_{\alpha})$. The index is a topological quantity which essentially counts the number of separatrices going in and going out of a small circle around the singular point p_{α} . For example it can be calculated according to

$$i_X(p_{\alpha}) = 1 - \frac{1}{2\pi} \int_{S^1} da$$

where S^1 is a small circle around p_{α} which does not contain any other singular point and da is the angle between the tangent to the circle and the vector X . In the case of non-degenerate singular points, the index of a node and a

saddle point is $+1$ or -1 respectively. If the vector field is the gradient of a smooth function $f(x, y)$ on $\mathcal{M}$ with non-degenerate critical points, then the Morse-relation is given by (4) where A is the number of maxima of $f(x, y)$, B the number of saddle points and C the number of minima. The last relation (5) is the Gauss-Bonnet formula and this holds whenever $\mathcal{M}$ is provided with a Riemannian structure whose curvature is K and the volume element dA .

In the case of the torus $g = 1$ and it is not hard to avoid dealing with curvature or singularities in the classical flow. As handles are added however $g \geq 2$, every flow must have some singular point and every metric must have some curvature. In this sense the g -handled spheres for $g > 1$ can not support free-particles in the normal sense of the word, not even classically.

3 Polygonal Billiard Flows

Consider a polygon in 2-dimensional Euclidean space which contains a billiard moving freely and colliding elastically with each of the sides. This provides an example of geodesic flow on a manifold with boundary. Simple as they seem such systems are very challenging mathematically and there remain many natural yet unsolved problems. It is not yet known if every polygonal billiard possesses a periodic orbit. On the other hand it is possible to say that if there exists a periodic orbit, it can be classified as odd or even depending on the number of segments of which it is made, and that the odd ones occur in isolation whereas the even ones occur in bands. There are many interesting known facts, conjectures as well as unsolved problems relating to the existence of periodic orbits, their distribution, and the transitivity, minimality or ergodicity of the flows. Due to considerations of space all I can do is refer the reader to an article by Gutkin [8], and references therein.

Now let us consider the so-called rational billiards - billiards in p -sided polygons whose contained-angles are all of the form $m_i\pi/n_i$ for $i = 1, \dots, p$, where each pair of integers n_i , and m_i is relatively prime. Later we use N to denote the LCM of the n_i . The configuration space of this system is the polygon itself and since the total momentum is conserved, the reduced momentum-space is a circle. The momentum remains constant except when the particle collides with a wall. For this reason the evolution of momentum is described by a finite set of maps on the circle, one for each side of the polygon. With respect to any set of axes if ϕ describes the direction of momentum of a billiard which collides with a side whose perpendicular has direction θ_i , then the direction of momentum is transformed by the circle map $\phi \rightarrow 2\theta_i - \phi$. We choose to ignore what happens when a billiard collides with a vertex. The momentum state at anytime is therefore of the form $\phi = \pm\phi + 2\sum_{i=1}^p k_i\theta_i \bmod 2\pi$ where k_i is a positive integer. General billiard flows fill up the circle, but in the case of rational billiards the set of possible directions of momentum is finite.

The phase-space accessible to a rational billiard is covered by a finite number

of copies of the original polygon, one copy for each of these momentum directions. In general the number of copies required is $2N$, and inequivalent flows on the same polygon can be labelled by the angle $\alpha \in [0, \pi/N]$. All angles in the interior of this interval correspond to what are known as generic flows, and for irrational α they are spatially dense. The flows corresponding to the end points are termed exceptional flows. In their case only N copies of the original polygon is required.

In the case of generic flows, the $2N$ copies of the original polygon are connected along their edges, and they join up to form a compact, 2-dimensional orientable invariant-manifold $\mathcal{M}_\alpha$ without boundary whose genus is given by

$$g(\mathcal{M}_\alpha) = 1 + \frac{N}{2} \sum_{i=1}^{i=p} \frac{m_i - 1}{n_i}.$$

$\mathcal{M}_\alpha$ is therefore a $g(\mathcal{M}_\alpha)$ -handled sphere. When a given flow is transferred from the polygon to the g -handled sphere, it is found that where there used to be vertices there are now saddle points. Each vertex v_i gives rise to N/n_i multi-saddle points of index $1 - m_i$, saddle points with m_i incoming and m_i outgoing separatrices.

The invariant manifolds of the exceptional flows cover the polygon N times. They have a boundary, they may or may not be orientable and they are often referred to in the literature as invariant *knots*.

The $\mathcal{M}_\alpha$ are all of dimension two in the four dimensional phase-space. This means that there exists a second conservation law and for this reason these systems are called pseudo-integrable or quasi-integrable. Special cases of these are the almost-integrable billiards. They are more integrable than the pseudo-integrable billiards [8]. Finally there are the integrable billiards. These arise when the vertices of the polygon are such that $m_i = 1$ for all $i = 1, \dots, p$. In such cases $g = 1$ and the invariant manifolds are tori. There are only a small number of integrable billiards, these are the rectangles and the triangles with vertex-angles $\{\pi/3, \pi/3, \pi/3\}$, $\{\pi/2, \pi/4, \pi/4\}$, and $\{\pi/2, \pi/3, \pi/6\}$.

The simplest case of a rational billiard with $g = 2$ is the $\pi/3$ -rhombus billiard. In this case the polygon has four sides with opposite angles $\pi/3$ and $2\pi/3$. The circle maps corresponding to collisions with the sides are

$$\phi \rightarrow 2\pi/3 - \phi \quad \phi \rightarrow 4\pi/3 - \phi.$$

Under the action of these transformations, any initial direction of momentum ϕ will become either one of

$$\pm\phi, \quad \pm\phi + 2\pi/3, \quad \pm\phi + 4\pi/3.$$

These values degenerate for the exceptional flows $\phi = 0$ or $2\pi/3$, into 0 , $2\pi/3$ and $4\pi/3$. The generic orbit therefore has length six and the invariant manifold corresponding to a generic flow covers the rhombus six times.

To construct these manifolds from six copies of the rhombus, lay down the first copy labelling all of the vertices and all of the sides. Reflecting the rhombus in one of the sides uses up one of the remaining five and induces a labelling of its sides and vertices. The original rhombus has become a polygon made of two glued rhombi all of whose vertices and edges are labelled. Now choose one of these rhombi, and one of its exterior edges, and reflect the chosen rhombus in this edge. This uses up another one of the copies and induces a labelling of its vertices and edges. Continue in this way until all six copies are used up. The labelling on the exterior edges and vertices tells you how to glue these together into a closed compact orientable two-dimensional surface.

There is no unique way of doing this, but the resulting surface is unique - in this case a two-handled sphere. Furthermore the total angle around the each obtuse vertex will add up to 4π testifying to the presence of negative curvature concentrated at that point. If you go to even more trouble and draw the flow lines on each of the copies of the rhombus, you will find saddle points in place of these vertices. Each with two ingoing and two outgoing separatrices.

Although a little tedious, it is well worth the trouble to verify all of this at least in the case of the $\pi/3$ rhombus. Further hints and contructions can be found in [8] or [11]. The picture which should emerge is that the flow in phase-space of a free particle moving inside the $\pi/3$ -rhombus and colliding elastically with its walls, is equivalent to a flow on the surface of a two-handled sphere. This flow has a pair of ordinary saddle points. It resembles an integrable system in the sense that the motion is confined to a two dimensional surface in the four dimensional phase-space. It can also be shown that in such systems neighbouring trajectories do not diverge exponentially as they do in chaotic systems, and that the Kolmogorov entropy is zero. The only hint of irregularity is the higher genus of the invariant surface and the consequent presence of saddle points which cause bifurcations in the flow.

4 Quantisation

This simply means solving the Schrodinger equation on the polygon

$$\partial_{xx}^2 \Phi(x, y) + \partial_{yy}^2 \Phi(x, y) + E \Phi(x, y) = 0$$

subject to the vanishing of $\Phi(x, y)$ on the boundary. Richens and Berry have considered this problem for the $\pi/3$ -rhombus [11]. They note that it is easy to find half of the solutions, those which vanish along the short diagonal. These solutions correspond to energy levels deduced from a WKB argument. They point out that the WKB method is highly inadequate for higher genus surfaces. Analytic solutions have still not been found for the other eigenfunctions of the $\pi/3$ -rhombus, although they have been studied numerically by Biswas and Jain [9]. In the absense of analytic solutions, and in view of the failure of the WKB method even to provide useful asymptotic information, other strategies have

been tried. These are based on periodic-orbit methods. Mathematically the connection between periodic orbits of a classical dynamical system and eigenvalues of the Laplacian on a Riemann surface is established by Selberg's trace formula, a readable account of which has been provided by Mc Keane [12]. The importance of periodic orbits for understanding both classical and quantum dynamics can easily be appreciated from an article by Cvitanović [13]. A more general introduction to this subject can be found in [2] and [3]. With a view to applying these techniques to the case of the $\pi/3$ -rhombus, its periodic orbits have been studied by Jain and Parab [14] and Parab and Jain [15].

Apart from the question of finding analytic solutions, numerical work on the full set of eigenvalues has shown that even for the $\pi/3$ -rhombus the asymptotic distribution of energy level spacings closely fits the GOE [9]. As one moves up through higher genus this fit quickly improves [10]. Richens and Berry have also considered a 'square torus billiard' [11]. In this case the flow in phase-space is equivalent to a flow on a 5-handled sphere which has 4 saddle-points, each with 3 ingoing and 3 outgoing separatrices. Numerically they found that the energy level spacings exhibit the phenomenon of level-crossing avoidance. Both this and GOE statistics of the level spacings are properties associated with the quantum mechanics of classically chaotic systems.

5 Conclusion

To conclude - there exists a class of flows on the g -handled spheres which are related to the rational billiards. Classically they are very like regular fully integrable systems, whereas quantum mechanically their properties are those associated with the classical chaos. In each case it is possible to consider solving the Schrodinger equation on the g -handled sphere, however an easier although not identical problem is to solve it on the polygon from which the g -handled sphere is constructed. In general the WKB method is of little use computing the energy levels. The simplest non-trivial case is the $\pi/3$ -rhombus. This corresponds to a class of flows on a two-handled sphere. In this case the WKB method provides only half of the eigenvalues. The corresponding wavefunctions are analytic. As yet the full quantum problem remains unsolved.

References

- [1] B. Eckhardt: Phys. Rep. 163. p.205-297 (1988): Quantum Mechanics of Classically Non-Integrable Systems.
- [2] Martin C. Gutzwiller: Chaos in Classical and Quantum Mechanics. Interdisciplinary Applied Mathematics Vol.1. Springer Verlag 1990.

- [3] Ozorio De Almeida: Hamiltonian Systems - Chaos and Quantisation. Cambridge Monographs in Mathematical Physics 1988.
- [4] F. Steiner: Desy 94-013 Jan. 1994 - chao-dyn/9402001: Quantum Chaos:
- [5] J. Ford & G. Mantica: Am. J. Phys. 60 (12) p.1086-1098. Dec.1992: Does Quantum Mechanics Obey the Correspondence Principle? Is it Complete?
- [6] M. V. Berry: Ann. Phys. (N.Y.) 131 p163-216 (1981): Quantizing a Classically Ergodic System - Sinai's Billiard and the KKR Method.
- [7] C. Godbillon: Dynamical Systems on Surfaces - Springer Verlag 1983
- [8] E. Gutkin: Physica 19D (1986) p.311-333: Billiards in Polygons.
- [9] D. Biswas & S. R. Jain: Phys. Rev. A Vol.42 N0.6 p.3170-3185 Sept. 1990: Quantum Description of a Pseudointegrable system - the $\pi/3$ -Rhombus Billiard.
- [10] Taksu Cheon & Thomas D. Cohen: Phys. Rev. Let. Vol.62 No.24 p.2769-2772 June 1989: Quantum Level Statistics of Pseudointegrable Billiards.
- [11] P. J. Richens & M. V. Berry: Physica 2D (1981) p.495-512: Pseudointegrable Systems in Classical and Quantum Mechanics.
- [12] H. P. McKeane: Comm. Pure and Applied Mathematics, Vol. XXV p.225-246 (1972): Selberg's Trace Formula as Applied to a Compact Riemann Surface.
- [13] P. Cvitanović: Physica D 51 (1991) p.138-151: Periodic Orbits as the Skeleton of Classical and Quantum Chaos.
- [14] S. R. Jain & H. D. Parab: J. Phys. A: Math & Gen. 25 (1992) p.6669-6683: Periodic Orbits of an Almost Integrable System.
- [15] H. D. Parab & S. R. Jain: Phys. Rev. E. Vol.47 I p.R776-9 1993: Law of Proliferation of Periodic Orbits in Pseudointegrable Billiards.