

Stability in L^r for the Navier-Stokes Flow
in a n -dimensional Bounded Domain

By

Hideo KOZONO (*)

名古屋大学工学部 小園 英雄
Paderborn 大学

Fachbereich Mathematik-Informatik
der Universität-Gesamthochschule
Paderborn, D-4790 Paderborn
Federal Republic of Germany

Tohru OZAWA

名古屋大学理学部 小澤 徹

Department of Mathematics
Nagoya University
Nagoya 464
Japan

(*) On leave of absence from:

Department of Applied Physics
Nagoya University
Nagoya 464, Japan

Introduction

The purpose of this paper is to investigate the stability for an incompressible fluid motion in a bounded domain in $\mathbb{R}^n$.

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial\Omega$. The motion of the fluid occupying Ω is governed by the Navier-Stokes equations:

$$\left. \begin{aligned}
 -\Delta w + w \cdot \nabla w + \nabla q &= f, \quad \nabla \cdot w = 0 \quad \text{in } \Omega, \\
 w|_{\partial\Omega} &= 0,
 \end{aligned} \right\} (S)$$

where $w = w(x) = (w^1(x), \dots, w^n(x))$ and $q = q(x)$ denote the velocity and the pressure of the fluid, respectively, and $f = f(x) = (f^1(x), \dots, f^n(x))$ denotes the external force. If $w(x)$ and $f(x)$ are perturbed by $a(x)$ and $g(x, t)$, respectively, then the perturbed flow $v(x, t)$ is governed by the following time-dependent Navier-Stokes equations:

$$\left. \begin{aligned}
 \partial_t v - \Delta v + v \cdot \nabla v + \nabla \pi &= f + g \quad \text{in } Q := \Omega \times (0, \infty), \\
 \nabla \cdot v &= 0 \quad \text{in } Q, \\
 v|_{\partial\Omega} &= 0, \\
 v|_{t=0} &= w + a.
 \end{aligned} \right\} (N. S)$$

There are many papers concerning the stability problem for the solutions of the Navier-Stokes equations. See, e.g., Ladyzenskaya (10), Heywood (6) (7), Masuda (11) and Sattinger (12). These results, however, are obtained in L^2 -setting or require some regularity assumptions on the perturbed flow at the initial time. Making use of the method developed by Giga & Miyakawa (5), we consider the perturbed flow in L^r and take such assumptions away.

To state our results, we need some preliminaries. For $m \in \mathbb{R}$ and $r > 1$, $W^{m,r}(\Omega)$ denotes the Sobolev space of order m , so that $W^{0,r}(\Omega) = L^r(\Omega)$. We set $W^{m,r}(\Omega) = W^{m,r}(\Omega) \otimes \mathbb{C}^n$, $L^r(\Omega) = L^r(\Omega) \otimes \mathbb{C}^n$. For $k \in \mathbb{N} \cup \{0\}$, a Banach space X and an interval $I \subset \mathbb{R}$, $C^k(I; X)$ denotes the space of continuously differentiable functions from I into X . For $0 < \mu < 1$, $C^\mu(I; X)$ denotes the space of functions in $C^0(I; X)$ satisfying the Hölder condition with exponent μ on compact subintervals of I . We set $BC(I; X) = C^0(I; X) \cap L^\infty(I; X)$. $C_{0,\sigma}^\infty(\Omega)$ denotes the set of all C^∞ -vector fields φ with compact support in Ω such that $\nabla \cdot \varphi = 0$. For $r > 1$, X_r denotes the completion of $C_{0,\sigma}^\infty(\Omega)$ with respect to the $L^r(\Omega)$ -norm $\|\cdot\|_r$. Then by Fujiwara & Morimoto (1), we have the following decomposition:

$$L^r(\Omega) = X_r \oplus G_r \text{ (direct sum),}$$

where $G_r = \{\nabla \pi; \pi \in W^{1,r}(\Omega)\}$. Let P_r be the projection operator from $L^r(\Omega)$ onto X_r associated with this decomposition. We define the Stokes operator A_r by $A_r = -P_r \Delta$ with domain $D(A_r) = X_r \cap \{u \in W^{2,r}(\Omega); u|_{\partial\Omega} = 0\}$. Applying P_r to both sides of (S) and (N.S), we have the equations in X_r :

$$A_r w + P_r w \cdot \nabla w = P_r f. \quad (S)'$$

$$\frac{dv}{dt} + A_r v + P_r v \cdot \nabla v = P_r (f + g), \quad t > 0,$$

$$v(0) = a + w.$$

$$\left. \begin{array}{l} \\ \\ \end{array} \right\} (N.S)'$$

Our main results now read:

Theorem 1. Let $r > \max(n/3, 1)$ and $f \in \mathbb{L}^r(\Omega)$. Then there is a positive number $\lambda = \lambda(r)$ such that (S)' has a unique solution w in $D(A_r)$ if $\|P_r f\|_r \leq \lambda$.

Theorem 2. Let $r > \max(n/3, 1)$ and $0 < \mu < 1$. Let σ satisfy $\sigma = n/2r - 1/2$ for $n/3 < r < n/2$, $\sigma = 1/2 + \varepsilon$ for $r \geq n/2$, where $0 < \varepsilon < \min(1/2, n/2r)$. Let γ and δ satisfy $n/2r - 1/2 \leq \gamma < 1$, $\delta \geq 0$ and $-\gamma < \delta < \min((1 - |\gamma|)/2, 1 - \sigma)$. Let $\lambda(r)$ be the number given by Theorem 1. Then, there are positive numbers $\lambda' \leq \lambda(r)$ and $\eta = \eta(r, n, \gamma, \delta)$ such that for any

$(a, f, g) \in D(A_r^\gamma) \times \mathbb{L}^r(\Omega) \times C^\mu((0, \infty); \mathbb{L}^r(\Omega))$ with $\|P_r f\|_r \leq \lambda'$, $\|A_r^\gamma a\|_r + \sup_{t>0} t^{1-\gamma-\delta} \|A_r^{-\delta} P_r g\|_r \leq \eta$, (N.S)' has a unique solution v satisfying:

- (1) $v \in C^0((0, \infty); D(A_r^\gamma)) \cap C^1((0, \infty); X_r)$;
- (2) $v(t) - w \in D(A_r)$ for $t > 0$, $A_r(v - w) \in C^0((0, \infty); X_r)$, where w is the unique solution given by Theorem 1;
- (3) $\|A_r^\alpha(v(t) - w)\|_r = O(t^{\gamma-\alpha})$ as $t \rightarrow \infty$ for $\gamma \leq \alpha < 1 - \delta$.

In section 1, we shall prove Theorem 1. In the special case $n \leq 4$, every weak solution w of (S) in $W_0^{1,2}(\Omega)$ belongs to $W^{2,r}(\Omega)$ if $f \in \mathbb{L}^r(\Omega)$. See Temam (14, p. 172, Remark 1.4) and Gerhardt (2). Little has been known, however, about the existence of strong solution of (S) in the case $n \geq 5$. Using the properties of the

fractional powers of the Stokes operator developed by Giga (4), we shall construct a *strong solution* of (S) in any dimension for f small enough. In section 2, we shall prove Theorem 2. Let $w \in D(A_r)$ be the solution in Theorem 1. Setting $u(t) = v(t) - w$, we have the following equation:

$$\left. \begin{aligned} \frac{du}{dt} + A_r u + B_r u + P_r u \cdot \nabla u &= P_r g, \quad t > 0, \\ u(0) &= a, \end{aligned} \right\} \text{(N.S)}''$$

where $B_r u = P_r (w \cdot \nabla u + u \cdot \nabla w)$. Then, the stability problem for (S) can be reduced to obtaining the time-decay estimates for the solution of (N.S)''. In order to solve (N.S)'' *globally in time*, we make some modifications of the argument in Giga & Miyakawa (5). This requires the analysis of the perturbed operator $A_r + B_r$. From our view-point, the result of (5) may be regarded as the stability theorem in $L^r(\Omega)$ around the rest fluid motion, i.e., $w \equiv 0$ in Ω . To characterize the domains of the fractional powers of the perturbed operators plays an important role in our case.

1. Proof of Theorem 1

In what follows, different positive constants might be denoted by the letter C . Since A_r has the bounded inverse A_r^{-1} in X_r , (S)' is equivalent to the following equation in X_r :

$$w + A_r^{-1} P_r w \cdot \nabla w = A_r^{-1} P_r f. \quad (S)''$$

We consider $D(A_r)$ as a Banach space with the norm $\|\cdot\|_{D(A_r)}$, given by $\|u\|_{D(A_r)} := \|A_r u\|_r$ for $u \in D(A_r)$. Without loss of generality, we may assume $f \in X_r$, i.e., $P_r f = f$. For $f \in X_r$ and $w \in D(A_r)$, we define

$$F(f, w) := w + A_r^{-1} P_r w \cdot \nabla w - A_r^{-1} f.$$

Then we have:

Proposition 1.1. *Let $r > \max(n/3, 1)$. Then,*

- (1) $F: (f, w) \mapsto F(f, w)$ is continuous from $X_r \times D(A_r)$ into $D(A_r)$.
- (2) For each $f \in X_r$, the map $F(f, \cdot): D(A_r) \ni w \mapsto F(f, w) \in D(A_r)$ is of class C^1 .

Proof. We choose $\theta = \theta(n, r)$ and $\rho = \rho(n, r)$ satisfying $0 < \theta < 1$, $1/2 < \rho < 1$ and $\theta + \rho = n/2r + 1/2$. By Giga & Miyakawa (5, Lemma 2.2), we have

$$\|P_r u \cdot \nabla v\|_r \leq C \|A_r^\theta u\|_r \|A_r^\rho v\|_r \leq C \|A_r u\|_r \|A_r v\|_r, \quad u, v \in D(A_r). \quad (1.1)$$

Hence $F(f, w) \in D(A_r)$ for all $f \in X_r$ and $w \in D(A_r)$. Since $\|F(f_1, w) - F(f_2, w)\|_{D(A_r)} = \|P_r(f_1 - f_2)\|_r$ for $f_i \in X_r$, $i = 1, 2$, and $w \in D(A_r)$, part (1) will follow if we can show part (2). For

each $w \in D(A_r)$, we define a linear operator K_w by

$$K_w u = u + A_r^{-1} P_r (w \cdot \nabla u + u \cdot \nabla w) \quad \text{for } u \in D(A_r).$$

By (1.1), K_w is in the space $B(D(A_r))$ of all bounded operators in $D(A_r)$. Moreover, for each $f \in X_r$, we have

$$\begin{aligned} & \|F(f, w + u) - F(f, w) - K_w u\|_{D(A_r)} \\ &= \|A_r^{-1} u \cdot \nabla u\|_r \leq C \|A_r u\|_r^2 = C \|u\|_{D(A_r)}^2. \end{aligned}$$

This shows that the Fréchet derivative $D_w F(f, w)$ at $(f, w) \in X_r \times D(A_r)$ is equal to K_w . Since again by (1.1), the inequality

$$\|K_{w_1} v - K_{w_2} v\|_{D(A_r)} \leq C \|A_r v\|_r \|A_r (w_1 - w_2)\|_r$$

holds for all $w_i, v \in D(A_r)$, $i = 1, 2$, we see that the map $w \mapsto K_w$ is continuous from $D(A_r)$ into $B(D(A_r))$. This completes the proof of Proposition 1.1. $\square$

By the proof of this proposition, we have $F(0, 0) = 0$, $D_w F(0, 0) = K_0 = \text{identity on } D(A_r)$. Therefore it follows from the implicit function theorem that there is a unique *continuous* mapping w from a neighborhood $U_\lambda = \{f \in X_r; \|f\|_r < \lambda\}$ of 0 into $D(A_r)$ such that

$$w(0) = 0, \quad F(f, w(f)) = 0 \quad \text{for } f \in U_\lambda. \quad (1.2)$$

(1.2) shows that $w(f)$ is a unique solution of (S)".

2. Proof of Theorem 2

We define the operator B_r by $B_r u = P_r(w \cdot \nabla u + u \cdot \nabla w)$ for $u \in D(B_r) := D(A_r^\sigma)$, where w is the solution obtained in Theorem 1. Then it follows that $D(A_r) \subset D(B_r)$ and

$$\|B_r u\|_r \leq C \|A_r w\|_r \|A_r^\sigma u\|_r, \quad u \in D(B_r). \quad (2.1)$$

Indeed, by the choice of σ , we have $1/2 < \sigma < 1$ and $1 + \sigma \geq n/2r + 1/2$. Then (2.1) follows from Giga & Miyakawa (5, Lemma 2.2).

The following propositions play an important role in this section.

Proposition 2.1. Let $L_r := A_r + B_r$ with domain $D(L_r) = D(A_r)$.

There is a positive constant $C_* = C_*(\Omega, n, r)$ such that if

$\|A_r w\|_r \leq C_*$, then $\Sigma_+ := \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq 0\} \subset \rho(-L_r)$ (the resolvent set of $-L_r$) and

$$\|(\lambda + L_r)^{-1}\|_{B(X_r)} \leq M_r (1 + |\lambda|)^{-1} \quad \text{for all } \lambda \in \Sigma_+ \quad (2.2)$$

with a positive constant M_r independent of λ .

Proof. It follows from Giga (3) (see also Wahl (15, Chapter III)) that $\Sigma_+ \subset \rho(-A_r)$ and $\|(A_r + \lambda)^{-1}\|_{B(X_r)} \leq N_r(1 + |\lambda|)^{-1}$ for all $\lambda \in \Sigma_+$ with $N_r > 0$ independent of λ . Since $L_r + \lambda = (1 + B_r(A_r + \lambda)^{-1})(A_r + \lambda)$ for $\lambda \in \Sigma_+$, it is sufficient to prove that there is a constant $k_r \in (0, 1)$ such that $\|B_r(A_r + \lambda)^{-1}\|_{B(X_r)} \leq k_r$ for all $\lambda \in \Sigma_+$. Indeed, by (2.1) and the moment inequality (see Tanabe (13, Proposition 2.3.3)), we have

$$\begin{aligned}
 \|B_r(A_r + \lambda)^{-1}f\|_r &\leq C\|A_rw\|_r\|A_r^\sigma(A_r + \lambda)^{-1}f\|_r \\
 &\leq C\|A_rw\|_r\|A_r(A_r + \lambda)^{-1}f\|_r^\sigma\|(A_r + \lambda)^{-1}f\|_r^{1-\sigma} \\
 &\leq C\|A_rw\|_r(N_r + 1)\|f\|_r^\sigma(N_r(1 + |\lambda|)^{-1}\|f\|_r)^{1-\sigma} \\
 &\leq C(N_r + 1)\|A_rw\|_r\|f\|_r
 \end{aligned} \tag{2.3}$$

for all $\lambda \in \Sigma_+$ and all $f \in X_r$. Therefore taking C_* so that $0 < C_* < 1/C(N_r + 1)$ and $k_r := C(N_r + 1)C_*$, we see, under the condition $\|A_rw\|_r \leq C_*$, that $\|B_r(A_r + \lambda)^{-1}\|_{B(X_r)} \leq k_r < 1$. $\square$

An immediate consequence of this proposition is as follows.

Corollary 2.2. *Let $\|A_rw\|_r \leq C_*$. Then, $-L_r$ generates a uniformly bounded holomorphic semi-group $(e^{-tL_r})_{t \geq 0}$ of class C_0 in X_r .*

Moreover, we can define the fractional power L_r^α of L_r for

any $\alpha \in \mathbb{R}$. Concerning the domains of fractional powers L_r^α and A_r^α , we have the following:

Proposition 2.3. Suppose that $\|A_r w\|_r \leq C_*$ (see Proposition 2.1).

(1) For $0 < \alpha < 1$, the identity $D(A_r^\alpha) = D(L_r^\alpha)$ holds and there is a constant $K = K(\alpha, r)$ such that

$$K^{-1} \|L_r^\alpha u\|_r \leq \|A_r^\alpha u\|_r \leq K \|L_r^\alpha u\|_r \quad \text{for all } u \in D(L_r^\alpha). \quad (2.4)$$

(2) For $\kappa > 0$ with $\kappa + \sigma \leq 1$, there is a constant $K' = K'(\kappa, \sigma, r)$ such that

$$\|L_r^{-\kappa} u\|_r \leq K' \|A_r^{-\kappa} u\|_r \quad \text{for all } u \in X_r. \quad (2.5)$$

Proof. (1) We first prove that $D(A_r^\alpha) \subset D(L_r^\alpha)$. For simplicity, we write $A = A_r$, $B = B_r$ and $L = L_r$. Note that

$$\begin{aligned} A^{-\alpha} &= \pi^{-1} \sin \pi \alpha \int_0^\infty \lambda^{-\alpha} (A + \lambda)^{-1} d\lambda \\ &= \pi^{-1} \sin \pi \alpha \int_0^\infty \lambda^{-\alpha} (A + B + \lambda)^{-1} (A + B + \lambda) (A + \lambda)^{-1} d\lambda \\ &= \pi^{-1} \sin \pi \alpha \int_0^\infty \lambda^{-\alpha} (L + \lambda)^{-1} (1 + B(A + \lambda)^{-1}) d\lambda \\ &= L^{-\alpha} + S_\alpha, \end{aligned} \quad (2.6)$$

where $S_\alpha = \pi^{-1} \sin \pi \alpha \int_0^\infty \lambda^{-\alpha} (L + \lambda)^{-1} B(A + \lambda)^{-1} d\lambda$. Suppose that $u \in D(A^\alpha)$. Setting $v = A^\alpha u$, we have by (2.6) $u = L^{-\alpha} v + S_\alpha v$. Therefore it is enough to show $S_\alpha v \in D(L^\alpha)$. By (2.2), (2.3), Krein (9, p.115 (5.15)) and $CC_*(N_r + 1) < 1$, we have

$$\|L^\alpha (L + \lambda)^{-1}\|_{B(X_r)} \leq M(1 + \lambda)^{\alpha-1}, \quad \|B(A + \lambda)^{-1}\|_{B(X_r)} \leq (1 + \lambda)^{\sigma-1}$$

for all $\lambda \geq 0$. This gives

$$\begin{aligned} & \int_0^\infty \|L^\alpha \lambda^{-\alpha} (L + \lambda)^{-1} B(A + \lambda)^{-1} v\|_r d\lambda \\ & \leq \int_0^\infty \lambda^{-\alpha} \|L^\alpha (L + \lambda)^{-1}\|_{B(X_r)} \|B(A + \lambda)^{-1}\|_r d\lambda \\ & \leq M \int_0^\infty \lambda^{-\alpha} (1 + \lambda)^{\sigma+\alpha-2} d\lambda \|v\|_r. \end{aligned}$$

Since $\sigma < 1$, the last integrand above converges and we obtain $S_\alpha v \in D(L^\alpha)$. We next prove that $D(L^\alpha) \subset D(A^\alpha)$. Similarly we have $L^{-\alpha} =$

$$A^{-\alpha} + T_\alpha, \text{ where } T_\alpha = -\pi^{-1} \sin \pi \alpha \int_0^\infty \lambda^{-\alpha} (A + \lambda)^{-1} B(A + B + \lambda)^{-1} d\lambda.$$

Hence it suffices to show that $T_\alpha v \in D(A^\alpha)$ for $v \in X_r$. By the proof of Proposition 2.1, we see that $1 + B(A + \lambda)^{-1}$ is invertible and $\|(1 + B(A + \lambda)^{-1})^{-1}\|_{B(X_r)} \leq (1 - k_r)^{-1}$ for all $\lambda \geq 0$.

Therefore

$$\begin{aligned}
\|B(A + B + \lambda)^{-1}\|_{\mathbb{B}(X_r)} &= \|B(A + \lambda)^{-1}(1 + B(A + \lambda)^{-1})^{-1}\|_{\mathbb{B}(X_r)} \\
&\leq \|B(A + \lambda)^{-1}\|_{\mathbb{B}(X_r)} \|(1 + B(A + \lambda)^{-1})^{-1}\|_{\mathbb{B}(X_r)} \\
&\leq (1 - k_r)^{-1} (1 + \lambda)^{\sigma-1}
\end{aligned}$$

for all $\lambda \geq 0$ and we get as before

$$\begin{aligned}
&\int_0^\infty \|A^\alpha \lambda^{-\alpha} (A + \lambda)^{-1} B(A + B + \lambda)^{-1} v\| d\lambda \\
&\leq \int_0^\infty \lambda^{-\alpha} \|A^\alpha (A + \lambda)^{-1}\|_{\mathbb{B}(X_r)} \|B(A + B + \lambda)^{-1} v\|_r d\lambda \\
&\leq N_r (1 - k_r)^{-1} \int_0^\infty \lambda^{-\alpha} (1 + \lambda)^{\sigma+\alpha-2} d\lambda \|v\|_r < \infty.
\end{aligned}$$

This shows that $T_\alpha v \in D(A^\alpha)$ for all $v \in X_r$. After all we obtain $D(A^\alpha) = D(L^\alpha)$. Since $0 \in \rho(A) \cap \rho(L)$, (2.4) is an immediate consequence of this identity.

(2) By (2.6), it suffices to show

$$\|S_\kappa u\|_r \leq C \|A^{-\kappa} u\|_r \quad \text{for all } u \in X_r,$$

with $C > 0$ independent of u . For this purpose, we prove $\|S_\kappa A^\kappa v\|_r \leq C \|v\|_r$ for all $v \in D(A^\kappa)$. By (2.1) and Krein (9, p.115 (5.15)), we have

$$\|B(A + \lambda)^{-1} A^\kappa v\|_r \leq C \|Aw\|_r \|A^\sigma (A + \lambda)^{-1} A^\kappa v\|_r$$

$$\begin{aligned}
&= C \|Aw\|_r \|A^{\sigma+\kappa} (A + \lambda)^{-1} v\|_r \\
&\leq CC_*(N_r + 1) (1 + \lambda)^{\sigma+\kappa-1} \|v\|_r \leq (1 + \lambda)^{\sigma+\kappa-1} \|v\|_r
\end{aligned}$$

for all $v \in D(A^\kappa)$. Hence it follows from (2.2) that

$$\begin{aligned}
\|S_\kappa A^\kappa v\|_r &\leq \pi^{-1} \sin \pi \kappa \int_0^\infty \lambda^{-\kappa} \|(L + \lambda)^{-1}\|_{B(X_r)} \|B(A + \lambda)^{-1} A^\kappa v\|_r d\lambda \\
&\leq M \pi^{-1} \sin \pi \kappa \int_0^\infty \lambda^{-\kappa} (1 + \lambda)^{\kappa+\sigma-2} d\lambda \|v\|_r,
\end{aligned}$$

as required. □

Now, we solve (N.S)". We first construct a solution of the following integral equation:

$$u(t) = e^{-tL_r} a + \int_0^t e^{-(t-s)L_r} P_r (g(s) - u \cdot \nabla u(s)) ds. \quad (I.E)$$

In order to solve (I.E), we use the implicit function theorem similar to Kozono (8). Let r , γ and δ be as in Theorem 2. We define function spaces $\mathcal{X} = \mathcal{X}_{\gamma, \delta}^r$ and $\mathcal{Y} = \mathcal{Y}_\gamma^r$ by

$$\begin{aligned}
\mathcal{X}_{\gamma, \delta}^r &= \{f; \text{measurable functions on } (0, \infty) \text{ with values in } X_r, \\
&\quad t^{1-\gamma-\delta} L_r^{-\delta} f \in BC((0, \infty); X_r)\},
\end{aligned}$$

$$\mathcal{Y}_\gamma^r = \{u \in BC((0, \infty); D(L_r^\gamma)) \cap C^0((0, \infty); D(L_r^{(1+\gamma)/2}))\};$$

$$\sup_{t>0} t^{(1-\gamma)/2} \|L_r^\gamma u(t)\|_r < \infty,$$

respectively. Then $\mathfrak{X}_{\gamma, \delta}^r$ and $\mathfrak{Y}_\gamma^r$ are Banach spaces with norms

$$\|f\|_{\mathfrak{X}} = \|f\|_{\mathfrak{X}_{\gamma, \delta}^r} := \sup_{t>0} t^{1-\gamma-\delta} \|L_r^{-\delta} f(t)\|_r,$$

$$\|u\|_{\mathfrak{Y}} = \|u\|_{\mathfrak{Y}_\gamma^r} := \sup_{t>0} \|L_r^\gamma u(t)\|_r + \sup_{t>0} t^{(1-\gamma)/2} \|L_r^{(1+\gamma)/2} u(t)\|_r,$$

respectively. Without loss of generality, we may assume $P_r g = g$.

For $(a, g, u) \in D(L_r^\gamma) \times \mathfrak{X} \times \mathfrak{Y}$, we define

$$G(a, g, u)(t) := u(t) - e^{-tL_r} a - \int_0^t e^{-(t-s)L_r} (g(s) - P_r u \cdot \nabla u(s)) ds.$$

Then we have:

Proposition 2.4. Suppose that $\|A_r w\|_r \leq C_*$ (see Proposition 2.1).

(1) $G: (a, g, u) \mapsto G(a, g, u)$ is continuous from $D(L_r^\gamma) \times \mathfrak{X} \times \mathfrak{Y}$ into $\mathfrak{Y}$.

(2) For each $(a, g) \in D(L_r^\gamma) \times \mathfrak{X}$, the map

$G(a, g, \cdot): \mathfrak{Y} \ni u \mapsto G(a, g, u) \in \mathfrak{Y}$ is of class C^1 .

Proof. We first show that $G(a, g, u) \in \mathfrak{Y}$ for $(a, g, u) \in$

$D(L_r^\gamma) \times \mathfrak{X} \times \mathfrak{Y}$. By the moment inequality (Tanabe (13, Proposition 2.3.3)), we have

$$\|L_r^\alpha u\|_r \leq C_{\alpha, \gamma} \|L_r^\gamma u\|_r^{(1+\gamma-2\alpha)/(1-\gamma)} \|L_r^{(1+\gamma)/2} u\|_r^{2(\alpha-\gamma)/(1-\gamma)}$$

for $\gamma \leq \alpha \leq (1+\gamma)/2$ and $u \in D(L_r^{(1+\gamma)/2})$ with $C_{\alpha,\gamma}$ independent of u . Therefore it follows that

$$\|L_r^\alpha u(t)\|_r \leq C_{\alpha,\gamma} \|u\|_{\mathcal{V}} t^{\gamma-\alpha}, \quad t > 0, \quad \gamma \leq \alpha \leq (1+\gamma)/2 \quad (2.7)$$

for $u \in \mathcal{V}$. Now, we set $v_0(t) = e^{-tL_r} a$, $v_1(t) = \int_0^t e^{-(t-s)L_r} g(s) ds$ and $v_2(t) = \int_0^t e^{-(t-s)L_r} P_r u \cdot \nabla u(s) ds$. Note that by Corollary 2.2, the inequality

$$\|L_r^\alpha e^{-tL_r}\|_{B(X_r)} \leq C_\alpha t^{-\alpha} \quad \text{for all } \alpha \geq 0, \quad t > 0,$$

holds with C_α independent of t . Hence $v_0 \in \mathcal{V}$ since $a \in D(L_r^\gamma)$. Moreover, we have

$$\begin{aligned} \|L_r^\alpha v_1(t)\|_r &\leq \int_0^t \|L_r^\alpha e^{-(t-s)L_r} g(s)\|_r ds \\ &\leq \int_0^t \|L_r^{\alpha+\delta} e^{-(t-s)L_r}\|_{B(X_r)} \|L_r^{-\delta} g(s)\|_r ds \\ &\leq C_{\alpha+\delta} \int_0^t (t-s)^{-\alpha-\delta} \|g\|_{\mathcal{X}} s^{\gamma+\delta-1} ds \\ &\leq C_{\alpha+\delta} B(1-\alpha-\delta, \gamma+\delta) \|g\|_{\mathcal{X}} t^{\gamma-\alpha} \end{aligned} \quad (2.8)$$

for $\alpha < 1 - \delta$, where $B(\cdot, \cdot)$ is the beta function. Since $\gamma, (1 + \gamma)/2 < 1 - \delta$, there is a positive constant B such that

$$\sup_{t>0} \|L_r^\gamma v_1(t)\|_r + \sup_{t>0} t^{(1-\gamma)/2} \|L_r^{(1+\gamma)/2} v_1(t)\|_r \leq B \|g\|_{\mathfrak{X}}. \quad (2.9)$$

Hence $v_1 \in \mathfrak{V}$. Taking $\rho = (1 + \gamma)/2 - \delta/2$, we have $\rho > 0$, $\rho + \delta > 1/2$ and $\delta + 2\rho = 1 + \gamma \geq n/2r + 1/2$. Since $\delta + \sigma < 1$, we obtain, by Proposition 2.3, Giga & Miyakawa (5, Lemma 2.2) and (2.7),

$$\begin{aligned} \|L_r^\alpha v_2(t)\|_r &\leq \int_0^t \|L_r^{\alpha+\delta} e^{-(t-s)L_r} \|_{\mathcal{B}(X_r)} \|L_r^{-\delta} P_r u \cdot \nabla u(s)\|_r ds \\ &\leq C_{\alpha+\delta} K' \int_0^t (t-s)^{-\alpha-\delta} \|A_r^{-\delta} P_r u \cdot \nabla u(s)\|_r ds \\ &\leq C_{\alpha+\delta} K' \int_0^t (t-s)^{-\alpha-\delta} \|A_r^\rho u(s)\|_r^2 ds \\ &\leq C_{\alpha+\delta} K' K_\rho^2 \int_0^t (t-s)^{-\alpha-\delta} \|L_r^\rho u(s)\|_r^2 ds \\ &\leq C_{\alpha+\delta} K' K_\rho^2 \int_0^t (t-s)^{-\alpha-\delta} \|u\|_{\mathfrak{V}}^2 s^{2\gamma-2\rho} ds \\ &= C_{\alpha+\delta} K' K_\rho^2 \|u\|_{\mathfrak{V}}^2 \int_0^t (t-s)^{-\alpha-\delta} s^{\gamma+\delta-1} ds \\ &= C_{\alpha+\delta} K' K_\rho^2 B(1-\alpha-\delta, \gamma+\delta) \|u\|_{\mathfrak{V}}^2 t^{\gamma-\alpha} \end{aligned} \quad (2.10)$$

for $\gamma \leq \alpha < 1 - \delta$. Hence there is a constant $B' > 0$ such that

$$\sup_{t>0} \|L_r^\gamma v_2(t)\|_r + \sup_{t>0} t^{(1-\gamma)/2} \|L_r^{(1+\gamma)/2} v_2(t)\|_r \leq B' \|u\|_{\mathfrak{V}}^2 \quad (2.11)$$

and we have $v_2 \in \mathfrak{V}$. After all we see that G maps $D(L_r^\gamma) \times \mathfrak{X} \times \mathfrak{V}$ into $\mathfrak{V}$. In view of Corollary 2.2 and (2.9), part (1) will follow if

we can show part (2). For $u, v \in \mathcal{V}$, we put

$$(T_u v)(t) := v(t) + \int_0^t e^{-(t-s)L_r} P_r (u \cdot \nabla v(s) + v \cdot \nabla u(s)) ds$$

In the same way as in (2.11), we see that $T_u \in \mathcal{B}(X_r)$ for $u \in \mathcal{V}$ and that $u \mapsto T_u$ is continuous from $\mathcal{V}$ into $\mathcal{B}(X_r)$. Moreover,

$$\|G(a, g, u+v) - G(a, g, u) - T_u v\|_{\mathcal{V}} \leq B' \|v\|_{\mathcal{V}}^2 \quad (2.12)$$

for $(a, g, u) \in D(L_r^\gamma) \times \mathcal{X} \times \mathcal{V}$ and $v \in \mathcal{V}$. Indeed, in the same way as in (2.10), we have

$$\begin{aligned} & \sum_{\alpha=\gamma, (\gamma+1)/2} t^{\alpha-\gamma} \|L_r^\alpha (G(a, g, u+v) - G(a, g, u) - T_u v)\|_r \\ &= \sum_{\alpha=\gamma, (\gamma+1)/2} t^{\alpha-\gamma} \left\| \int_0^t L_r^\alpha e^{-(t-s)L_r} P_r v \cdot \nabla v(s) ds \right\|_r \leq B' \|v\|_{\mathcal{V}}^2 \end{aligned}$$

for all $t > 0$.

(2.12) shows that the Fréchet derivative $D_u G(a, g, u)$ at $(a, g, u) \in D(L_r^\gamma) \times \mathcal{X} \times \mathcal{V}$ is equal to T_u . This completes the proof. $\square$

Since $G(0, 0, 0) = 0$, $D_u G(0, 0, 0) = \text{identity on } \mathcal{V}$, it follows from the implicit function theorem that there is a unique continuous map u from a neighborhood $V_\eta = \{(a, g) \in D(L_r^\gamma) \times \mathcal{X}; \|L_r^\gamma a\|_r + \|g\|_{\mathcal{X}} < \eta\}$ of $(0, 0)$ into $\mathcal{V}$ such that

$$u(0, 0) = 0, \quad G(a, g, u(a, g)) = 0 \quad \text{for } (a, g) \in V_\eta. \quad (2.13)$$

This shows that $u(a, g)$ is a unique solution of (I.E) for (a, g) .

Using the same method as in Giga & Miyakawa (5, Theorem 2.5), we see that $P_r u \cdot \nabla u$ for such a solution u is Hölder continuous on $(0, \infty)$ with values in X_r . Then it follows from Tanabe (13, Theorem 3.3.4) that u satisfies the *differential equation* (N.S)". $\square$

Remark. By Proposition 2.3, we can choose η in Theorem 2 so small that $(a, g) \in V_\eta$. Since the map $w : U_\lambda \ni f \mapsto w(f) \in D(A_r)$ is continuous (see Proposition 1.1), we can take $\lambda' (\leq \lambda)$ so that $\|A_r w\|_r \leq C_*$ if $\|f\|_r \leq \lambda'$.

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