

THE FAST AND SLOW GROWING HIERARCHIES
AND THE INDUCTIVE DEFINITIONS

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§0. INTRODUCTION

The aim of subrecursive hierarchy theory is to assign ordinal notations to computable functions in such a way as to reflect their computational complexity. We shall consider here, as the complexity measure, the termination proofs of some algorithms for computing them, in particular, the proofs given in $ID_{<\omega}$ ($= \bigcup_{\nu < \omega} ID_\nu$; the theory of finitely iterated inductive definitions). Then the function whose termination proof is given in $ID_{<\omega}$ is called *provably computable* in $ID_{<\omega}$.

On the relation between termination proofs and subrecursive hierarchies, Wainer[15],[16] introduced a subrecursive inaccessible ordinal τ , so that for $x > 0$,

$$G_\tau(x) < F_\tau(x) \leq G_\tau(x+1)$$

where G_τ and F_τ are the slow and fast-growing functions at τ , respectively. This means that the slow-growing hierarchy catches up with the fast-growing one at stage τ . Then he stated that the ordinal height of τ is $\sup\{|ID_\nu| : \nu < \omega\}$ where $|ID_\nu|$ is the proof-theoretic ordinal of ID_ν , based on the results of Girard[8].

In this article, we shall demonstrate the following (I)-(III)

on the relation between termination proofs in $ID_{<\omega}$ and the slow and fast-growing hierarchies:

(I) We introduce an ordinal τ' such that for $x > 3$,

$$G_{\tau'}(x) < F_{\tau'}(x) \leq G_{\tau'}(G_{\tau'}(x)).$$

This means also that the slow-growing hierarchy catches up with the fast-growing hierarchy at τ' . The reason why we change the definition of τ is to show the collapsing lemma in Section 5.

(II) For each $\alpha < \tau'$, the function F_{α} is provably computable in $ID_{<\omega}$.

(III) If a computable function $f: \omega \rightarrow \omega$ is provably computable in $ID_{<\omega}$, then f is *dominated* by F_{α} for some $\alpha < \tau'$ (i.e., there is an $m < \omega$ such that $f(x) < F_{\alpha}(x)$ for $x > m$).

Our demonstration here is based on the results of Buchholz[4] on the functions provably computable in $ID_{\nu}(\nu \leq \omega)$. On the other hand, Arai[2] had already studied these functions by means of the slow-growing hierarchy. Here we shall first prove the relation of (I) which is a direct estimation of the fast-growing function at τ' by the slow-growing hierarchy at τ' , following the idea of [16]; secondly, we shall prove (II) and (III) which imply that τ' corresponds to the proof-theoretic ordinal of $ID_{<\omega}$. Moreover, we shall consider only the case of ID_{ν} where $\nu < \omega$. The author does not know how to construct τ' which implies (I) and corresponds to the proof theoretic ordinal of ID_{ω} .

§1. A SUBRECURSIVE INACCESSIBLE ORDINAL τ'

1A. In this section, we shall introduce a (tree-) ordinal τ' and prove that

$$(I) \quad G_{\tau'}(x) < F_{\tau'}(x) \leq G_{\tau'}(G_{\tau'}(x)) \quad \text{for } x > 3.$$

The definition of τ' is slightly changed from that of τ in [15], [16]. The reason why we change the definition of τ is to apply directly Buchholz' method in [4] to our case; we need this change to prove the collapsing lemma of Section 5.

In the following, the letters k, m, n, p, x denote non-negative integers.

1B. **TREE ORDINALS AND (p)-BUILT-UPNESS.** The hierarchies of number-theoretic functions considered here are defined by recursions over the set of countable ordinals which has an assignment of *fundamental sequences* at limit stages. For a countable limit ordinal λ , we call $\langle \lambda[x] \rangle_{x < \omega}$ a fundamental sequence for λ when it satisfies:

$$(i) \quad \lambda[0] < \lambda[1] < \lambda[2] < \cdots < \lambda,$$

$$(ii) \quad \sup\{\lambda[x] : x < \omega\} = \lambda.$$

Following [6], here we shall define the set Ω of countable tree-ordinals which is constructed by assigning the arbitrary chosen fundamental sequences at limit stages as follows:

DEFINITION 1.1 (Countable tree-ordinals). The set Ω of the *countable tree-ordinals* consists of the infinitary terms generated inductively by:

- (i) $0 \in \Omega$.
- (ii) If $\alpha \in \Omega$, then $\alpha+1 \in \Omega$.
- (iii) If $\alpha_x \in \Omega$ for all $x < \omega$, then $(\alpha_x)_{x<\omega} \in \Omega$.

For a given $\alpha \in \Omega$ such that $\alpha = (\alpha_x)_{x<\omega}$, we call α 'limit' and write $\alpha[x]$ for α_x . According to the inductive definition of Ω , proofs and definitions will usually be by induction over the well-founded 'sub-tree' partial ordering on Ω which is denoted $<$ and defined as the transitive closure of

- (i) $0 \leq \beta$,
- (ii) $\beta < \beta+1$,
- (iii) $\beta[x] < \beta$ for all $x < \omega$ if β is limit.

In order to ensure that $<$ -predecessors of α are linearly and hence well-ordered, and to develop basic domination properties; we need to restrict attention to tree-ordinals α possessing additional structure.

DEFINITION 1.2 ((p)-built-up tree-ordinals). For a given $p < \omega$, the subset $\Omega^{(p)\text{-bu}} \subset \Omega$ of (p)-built-up tree-ordinals consists of those $\alpha \in \Omega$ satisfying that:

$$\lambda[x] <_p \lambda[x+1] \text{ for all limit } \lambda \leq \alpha \text{ and } x < \omega,$$

where the relation $<_p$ on Ω is defined as the transitive closure of (i) $0 \leq_p \beta$, (ii) $\beta <_p \beta+1$, (iii) $\beta[p] < \beta$ if β is limit.

Built-upness and the other related notions on fundamental sequences are studied in [1],[11],[12],[13]. In [16], Wainer used the notion of structuredness (or niceness in [6]) as bases

to develop his theory of subrecursive inaccessible ordinals. From the author and Aoyama's study of [11], we can prove the same results to Wainer[16] when we use the notion of (p)-built-upness instead of the structuredness.

LEMMA 1.3. Let $p < \omega$ and $\alpha \in \Omega^{(p)\text{-bu}}$. Then the following hold.

- (1) If $\beta <_m \alpha$ and $p \leq m < n$, then $\beta <_n \alpha$.
- (2) If $\beta < \alpha$, then $\beta <_m \alpha$ for some $m < \omega$.
- (3) If $p \leq m$ and $\beta <_m \alpha$, then $\beta+1 \leq_{m+1} \alpha$.

Proof. By induction on α . See Lemma 2.3 and Cor.2.8 in [11]. $\square$

PROPOSITION 1.4 ([16]). For each $p < \omega$ and $\alpha \in \Omega^{(p)\text{-bu}}$, the set $\{\gamma: \gamma < \alpha\}$ is linearly and hence well-ordered by $<$. Furthermore, if $\gamma < \alpha$ then $\gamma+1 \leq \alpha$.

Proof: If $\gamma < \alpha$ and $\delta < \alpha$, choose any m such that $\gamma <_m \alpha$ and $\delta <_m \alpha$. Then we have $\gamma = \delta$ or $\gamma <_m \delta$ or $\delta <_m \gamma$. Hence we have $\gamma = \delta$ or $\gamma < \delta$ or $\delta < \gamma$. Furthermore, if $\gamma < \alpha$, then $\gamma <_m \alpha$ for some $m < \omega$. Hence $\gamma+1 \leq_{m+1} \alpha$ by 1.3. Therefore $\gamma+1 \leq \alpha$. $\square$

1C. HIERARCHIES $\{F_\alpha\}_{\alpha \in \Omega}$, $\{G_\alpha\}_{\alpha \in \Omega}$, $\{F'_\alpha\}_{\alpha \in \Omega}$. We define the fast-growing $\{F_\alpha\}_{\alpha \in \Omega}$ and slow-growing $\{G_\alpha\}_{\alpha \in \Omega}$ hierarchies as follows:

$$\begin{aligned} F_0(x) &= x+1, & G_0(x) &= 0, \\ F_{\alpha+1}(x) &= F_\alpha^x(F_\alpha(x)), & G_{\alpha+1}(x) &= G_\alpha(x)+1, \\ F_\lambda(x) &= F_{\lambda[x]}(x), & G_\lambda(x) &= G_{\lambda[x]}(x), \end{aligned}$$

where λ is limit and the superscript x denotes iteration x -times

of F_α (i.e., if $F: \omega \rightarrow \omega$ then $F^0(x) = x$, $F^{m+1}(x) = F(F^m(x))$).

Moreover, we introduce an auxiliary fast-growing hierarchy $\{F'_\alpha\}_{\alpha \in \Omega}$ as follows:

$$F'_0(x) = x+1,$$

$$F'_{\alpha+1}(x) = F'^X_\alpha(F'_\alpha(x)),$$

$$F'_\lambda(x) = F'_{\lambda[z]}(x), \text{ where } z = F'_{\lambda[1]}(x).$$

Then we have the following proposition which state that these hierarchies indexed by (p)-built-up ordinals have elementary properties on increase and domination.

PROPOSITION 1.5. *For some $p < \omega$, we assume $\alpha \in \Omega^{(p)\text{-bu}}$. Then the following holds:*

(1) $F_\alpha(x) < F_\alpha(x+1)$, $G_\alpha(x) \leq G_\alpha(x+1)$ and $F'_\alpha(x) < F'_\alpha(x+1)$ for $p \leq x+1$.

(2) If $\beta <_m \alpha$ for $p \leq m$, then $F_\beta(x) < F_\alpha(x)$, $G_\beta(x) < G_\alpha(x)$ and $F'_\beta(x) < F'_\alpha(x)$ for $x > m$.

Proof. By induction on α . See Theorem 3.1 of [11]. □

1D. SUBRECURSIVE INACCESSIBILITY. Now let us define the subrecursive inaccessibility on these hierarchies:

DEFINITION 1.6. Let $p < \omega$. We call $\alpha \in \Omega^{(p)\text{-bu}}$ *subrecursive inaccessible* (or *s-inaccessible* for short) if for all $x > p$,

$$G_\alpha(x) < F_\alpha(x) \leq F'_\alpha(x) \leq G_\alpha(G_\alpha(x)).$$

This definition slightly differs from the original subrecur-

sive inaccessibility in [15],[16], but they have the same meaning which the slow-growing function at α catches up with the fast-growing one.

LEMMA 1.7. Let $p < \omega$ and $\alpha \in \Omega^{(p)\text{-bu}}$.

- (1) For all $x > p$, $G_\alpha(x) < F_\alpha(x) \leq F'_\alpha(x)$.
 (2) If α is s -inaccessible, then α is limit and G_α^2 dominates every F'_β with $\beta < \alpha$ (i.e., for all but finitely many x , $F'_\beta(x) < G_\alpha(G_\alpha(x))$).

Proof. (1) Induction on α . (2) Clearly α cannot be 0. For any $\beta+1 \in \Omega^{(p)\text{-bu}}$ and $x > p$,

$$\begin{aligned} G_{\beta+1}(x) &= G_\beta(x)+1 < F'_\beta(x)+1 \leq F'_\beta(x+1) \leq F'_\beta(F'_\beta(x)) \leq F_{\beta}^{x+1}(x) \\ &= F'_{\beta+1}(x). \end{aligned}$$

Hence α must be limit. On the other hand, we can prove that if $\beta < \alpha$, then there is an $m < \omega$ such that $\beta <_m \alpha$ since α is (p) -built-up. Hence, F_α dominates every F_β with $\beta < \alpha$. Therefore G_α^2 dominates every F'_β with $\beta < \alpha$. $\square$

PROPOSITION 1.8 ([16,p.215]). Let $p < \omega$ and $\alpha \in \Omega^{(p)\text{-bu}}$ satisfy that

$$G_{\alpha[n+1]} = F'_{\alpha[n]} \text{ for all } n < \omega.$$

Then α is s -inaccessible and, if $\alpha[0]$ is finite (i.e., $\alpha[0] = 0+1+\dots+1$), then no $\beta < \alpha$ is s -inaccessible.

Proof. If $G_{\alpha[n+1]} = F'_{\alpha[n]}$ for each n and $z = F_{\alpha[1]}(x)$, then

$$z = F_{\alpha[1]}(x) = G_{\alpha[2]}(x) < G_{\alpha[x]}(x) = G_\alpha(x)$$

for $x > \max(2,p)$. Hence we have that

$$\begin{aligned}
F'_\alpha(x) &= F'_{\alpha[z]}(x) = G_{\alpha[z+1]}(x) \leq G_{\alpha[G_\alpha(x)]}(x) \\
&\leq G_{\alpha[G_\alpha(x)]}(G_\alpha(x)) = G_\alpha(G_\alpha(x)) = G_\alpha^2(x).
\end{aligned}$$

(Since α is limit and (p) -built-up, we have $G_\alpha(x) \geq x$ ($x > p$).) So α is s -inaccessible. If also $\alpha[0]$ is finite and $\beta < \alpha$ were s -inaccessible then $\alpha[0] < \beta$ since β is limit. So $\alpha[n] < \beta \leq \alpha[n+1]$ for some n . By 1.7, for sufficient large x ,

$$G_{\alpha[n+1]}^2(x) = F_{\alpha[n]}^2(x) \leq F_{\alpha[n]}^{x+1}(x) = F_{\alpha[n]+1}(x) < G_\beta^2(x)$$

since $\alpha[n]+1 < \beta$. This is a contradiction, since $\beta \leq \alpha[n+1]$ and therefore $G_\beta^2(x) \leq G_{\alpha[n+1]}^2(x)$ for sufficiently large x . $\square$

This proposition suggests a method for constructing a minimal s -inaccessible which we shall denote

$$\tau' = (\tau'[x])_{x < \omega}.$$

First choose $\tau' = 3$ for which F_3 dominates all functions elementary in $\{F_\beta : \beta < 3\}$. Then if $\tau'[0], \dots, \tau'[n]$ have already defined, choose $\tau'[n+1]$ so that $G_{\tau'[n+1]} = F_{\tau'[n]}$.

1E. A MINIMAL S -INACCESSIBLE τ' . We introduce a minimal s -inaccessible τ' as in [15],[16]. Just as the fast-growing hierarchy uses countable tree-ordinals α to name big number-theoretic functions F'_α , we can use uncountable tree ordinals α to name big ordinal-functions $\varphi(\alpha): \Omega \longrightarrow \Omega$. These can be used to name bigger number-theoretic functions $F'_{\varphi(\alpha)(\beta)}$ etc. This idea leads to a collection of higher level fast-growing hierarchies $\varphi_n(\alpha): \Omega_n \longrightarrow \Omega_n$ where α ranges over the next higher tree class Ω_{n+1} .

DEFINITION 1.9 ([15]). The sets Ω_n of *higher level tree-*

ordinals are defined by induction similarly to the case of Ω :

- (i) $0 \in \Omega_n$.
- (ii) If $\alpha \in \Omega_n$, then $\alpha+1 \in \Omega_n$.
- (iii) If $\alpha_\gamma \in \Omega_n$ for all $\gamma \in \Omega_k$ ($k < n$), then $(\alpha_\gamma)_{\gamma \in \Omega_k} \in \Omega_n$.

As in the case of Ω , we call $(\alpha_\gamma)_{\gamma \in \Omega_k}$ limit and write $\alpha[\gamma]$ instead of α_γ . We shall identify Ω_0 with ω , and Ω_1 with Ω , in the following.

DEFINITION 1.10 ([15, Definition 5]). The level n fast-growing hierarchies of functions $\varphi_n: \Omega_{n+1} \times \Omega_n \rightarrow \Omega_n$ is defined by:

- (i) $\varphi_n(0, \beta) = \beta+1$,
- (ii) $\varphi_n(\alpha+1, \beta) = \varphi_n^\beta(\alpha, \varphi_n(\alpha, \beta))$,
- (iii) $\varphi_n(\lambda, \beta) = (\varphi_n(\lambda[\gamma], \beta))_{\gamma \in \Omega_k}$ for $\lambda = (\lambda[\gamma])_{\gamma \in \Omega_k}$ ($k < n$),
- (iv) $\varphi_n(\lambda, \beta) = \varphi_n(\lambda[z], \beta)$, $z = \varphi_n(\lambda[1], \beta)$ for $\lambda = (\lambda[\gamma])_{\gamma \in \Omega_n}$

where φ_n^β denotes the iteration β -times of φ_n (i.e., if $\psi: \Omega_{n+1} \times \Omega_n \rightarrow \Omega_n$, then $\psi^0(\alpha, \beta) = \beta$, $\psi^{\delta+1}(\alpha, \beta) = \psi(\alpha, \psi^\delta(\alpha, \beta))$, $\psi^\lambda(\alpha, \beta) = (\psi^{\lambda[\gamma]}(\alpha, \beta))_{\gamma \in \Omega_m}$ for $\lambda = (\lambda[\gamma])_{\gamma \in \Omega_m}$).

Note that, in the case $n = 0$, $\varphi_0(\alpha, \beta) = F_\alpha(\beta)$ for $\alpha \in \Omega_1$ and $\beta \in \Omega_0 (= \omega)$. We define $\omega_k \in \Omega_n$ by $\omega_k = (\gamma)_{\gamma \in \Omega_k}$ (i.e., $\omega_k[\gamma] = \gamma$).

DEFINITION 1.11 ([15, Definition 7]). The sets T_n ($\subset \Omega_n$) of *named tree-ordinals* are defined inductively by:

- (i) $0, 1, \omega_0, \dots, \omega_{n-1} \in T_n$.

- (ii) $T_k \subset T_n$ for $k < n$.
 (iii) If $\alpha \in T_{n+1}$ and $\beta, \gamma \in T_n$, then $\varphi_n^\gamma(\alpha, \beta) \in T_n$.

THEOREM 1.12 (Collapsing theorem [15]). *Let $x < \omega$, $\alpha \in T_2$ and $\beta \in T_1$. Then*

$$G_{\varphi_1}(\alpha, \beta)(x) = F'_{c\alpha}(G_\beta(x)),$$

where the function c ($= c_x$) which collapses each T_{n+1} to T_n is defined by: $c0 = 0$, $c1 = 1$, $c\omega_0 = x$, $c\omega_{k+1} = \omega_k$,
 $c(\varphi_{k+1}^\gamma(\delta, \xi)) = \varphi_k^{c\gamma}(c\delta, c\xi)$, $c(\varphi_0^\gamma(\delta, \xi)) = \varphi_0^\gamma(\delta, \xi)$. Hence, in particular, if α is generated in T_2 without reference to ω_0 then, as $G_{\omega_0}(x) = x$, we have $G_{\varphi_1}(\alpha, \omega_0) = F'_{c\alpha}$.

We shall prove this theorem in Section 3 by using the strong normalization theorem in Section 2. Together with Proposition 1.8, we can construct a minimal s -inaccessible ordinal as follows:

DEFINITION 1.13 ([15, Example 4]). We define $\tau' = (\tau'[x])_{x < \omega}$ by setting $\tau'[0] = 3$,

$$\tau'[n+1] = \varphi_1(\dots \varphi_n(\varphi_{n+1}(3, \omega_n), \omega_{n-1}), \dots, \omega_0).$$

THEOREM 1.14. τ' is a minimal s -inaccessible tree-ordinal.

Proof. From Section 4, τ' is (3)-built-up. Hence 1.8 and the collapsing theorem(1.12) complete the proof. $\square$

§2. PROVABLE COMPUTABILITY OF F_α ($\alpha < \tau'$)

2A. In this section, we shall prove Theorem 2.10;

(II) for $\alpha < \tau'$, F_α and F'_α are provably computable in $ID_{<\omega}$.

(For the definition of $ID_{<\omega}$, see Section 5.) As the corollaries of this proof, we shall prove also that the collapsing theorem in Section 3, and (3)-built-upness of τ' in Section 4 which we had used to prove (I) in Section 1.

To prove (II) above, we shall introduce the term structures for the sets T_n ($n < \omega$). Then we shall prove the strong normalization theorem for the structures. Our method here is the same as that of [4, Section 2] and our results of this section (and of Sections 3, 4 below) comes from those of [10].

2B. THE TERM STRUCTURES. We introduce term structures $\langle \bar{T}_n, NT_n, \cdot [\cdot], \longrightarrow \rangle$ ($n < \omega$) by considering each element in T_n as a finitary term and each defining equation of φ_n (Definition 1.10) as a rewrite (or reduction) rule of the terms. Let $\bar{0}$, $\bar{1}$, $\bar{\omega}_0$, $\bar{\omega}_1$, ..., $\bar{\varphi}_0$, $\bar{\varphi}_1$, ... be formal symbols.

DEFINITION 2.1. The sets $\bar{T}_n$ of *terms* are defined inductively by:

- (i) $\bar{0}$, $\bar{1}$, $\bar{\omega}_0$, $\bar{\omega}_1$, ..., $\bar{\omega}_{n-1} \in \bar{T}_n$.
- (ii) $\bar{T}_k \subset \bar{T}_n$ for $k < n$.
- (iii) If $a \in \bar{T}_{n+1}$ and $b, c \in \bar{T}_n$, then $\bar{\varphi}_n^c(a, b) \in \bar{T}_n$.

Naturally, terms in $\bar{T}_n$ are interpreted as tree-ordinals by

the function $\text{ord}: \bar{T}_n \longrightarrow T_n$ such that (i) $\text{ord}(\bar{0}) = 0$, $\text{ord}(\bar{1}) = 1$, $\text{ord}(\bar{\omega}_k) = \omega_k$, (ii) $\text{ord}(\bar{\varphi}_n^c(a,b)) = \varphi_n^{\text{ord}(c)}(\text{ord}(a), \text{ord}(b))$.

Abbreviations. $\bar{\varphi}_n(a,b) = \bar{\varphi}_n^{\bar{1}}(a,b)$, $b+1 = \bar{\varphi}_n(\bar{0}, b)$.

DEFINITION 2.2 (Normal terms). The sets NT_n of *normal terms* in $\bar{T}_n$; $\text{dom}(a) \in \{\emptyset, \{\bar{0}\}, \bar{T}_0, \dots, \bar{T}_{n-1}\}$ and $a[s]$ for $a \in NT_n$, $s \in \text{dom}(a)$ are defined inductively by:

(N1) $\bar{0} \in NT_n$; $\text{dom}(\bar{0}) = \emptyset$.

(N2) $\bar{1} \in NT_n$; $\text{dom}(\bar{1}) = \{\bar{0}\}$, $\bar{1}[\bar{0}] = \bar{0}$.

(N3) $\bar{\omega}_1 \in NT_n$ ($1 < n$); $\text{dom}(\bar{\omega}_1) = \bar{T}_1$, $\bar{\omega}_1[s] = s$.

(N4) $NT_k \subset NT_n$ for $k < n$.

(N5) Let $a \in NT_{n+1}$, $b, c \in NT_n$ and $A = \bar{\varphi}_n^c(a,b)$. Then $A \in NT_n$ if one of the following holds:

(i) $c = \bar{1}$ and $a = \bar{0}$ (i.e., $A = b+1$); $\text{dom}(A) = \{\bar{0}\}$, $A[s] = b$.

(ii) $\text{dom}(c) = \bar{T}_k$ ($k < n$); $\text{dom}(A) := \text{dom}(c)$, $A[s] = \bar{\varphi}_n^{c[s]}(a,b)$.

(iii) $c = \bar{1}$ and $\text{dom}(a) = \bar{T}_k$ ($k < n$); $\text{dom}(A) = \text{dom}(a)$,

$A[s] = \bar{\varphi}_n(a[s], b)$.

Next we introduce a *term-rewriting system* (S) (see e.g., Dershowitz[7] as for the definition) so that, for every term in $\bar{T}_n$ which is not normal, some rewrite rule in (S) is applicable to it.

Definition of the rewrite rules of (S): For normal a, b, c ;

(R1) $\bar{\varphi}_n^{\bar{0}}(a,b) \longrightarrow b$, (R2) $\bar{\varphi}_n^{\bar{1}}(b) \longrightarrow \bar{\varphi}_n^b(\bar{0}, \bar{\varphi}_n(\bar{0}, b))$,

(R3) $\bar{\varphi}_n^{a+1}(b) \longrightarrow \bar{\varphi}_n^b(a, \bar{\varphi}_n(a, b))$,

$$(R4) \quad \bar{\varphi}_n^{c+1}(a,b) \longrightarrow \bar{\varphi}_n(a, \bar{\varphi}_n^c(a,b)),$$

$$(R5) \quad \bar{\varphi}_n(a,b) \longrightarrow \bar{\varphi}_n(a[z],b) \text{ with } z = \varphi_n(a[1],b) \\ \text{if } \text{dom}(a) = \bar{T}_n.$$

Every rule in (R1)-(R5) may be applied to a term $A \in \bar{T}_n$ if A contains a subterm of the left-hand side of the rule. Then the rule is used by replacing the subterm to the right-hand side of the rule. We write $A \xrightarrow{1} B$ to indicate that the term B is obtained from the term A by a single application of some rule. We have the following fundamental proposition.

PROPOSITION 2.3. (1) *For every $a \in \bar{T}_n$, $a \in NT_n$ if and only if there is no $b \in \bar{T}_n$ such that $a \xrightarrow{1} b$.*

(2) (i) *If $a \in NT_n$ and $a = b+1$ for some b , then $\text{ord}(a) = \text{ord}(b)+1$.*

(ii) *If $a \in NT_n$ and $\text{dom}(a) = \bar{T}_k$ ($k < n$), then $\text{ord}(a) = (\text{ord}(a)[\gamma])_{\gamma \in \Omega_k}$ and $\text{ord}(a[b]) = \text{ord}(a)[\text{ord}(b)]$ for $a \in \bar{T}_k$.*

(iii) *If $a \in \bar{T}_n$ and $a \xrightarrow{1} b$, then $\text{ord}(a) = \text{ord}(b)$.*

Proof. Induction on the length of a . □

2C. THE STRONG NORMALIZATION THEOREM. Now we say that a term $A \in \bar{T}_n$ is *strongly normalizable* if every derivation sequence starting at A (i.e., $A \xrightarrow{1} A' \xrightarrow{1} A'' \xrightarrow{1} \dots$) is finite (cf.[9]). Then we prove the following theorem:

THEOREM 2.4 (Strong normalization theorem[10, Theorem 1]).
Every term a in $\bar{T}_n$ is strongly normalizable.

We can also show that the term rewriting system (S) has the Church-Rosser property (i.e., if $A \Rightarrow B$ and $A \Rightarrow C$, then there is a D such that $B \Rightarrow D$ and $C \Rightarrow D$, where $a \Rightarrow b$ indicates that b is obtained from a by a finite (perhaps empty) series of reduction " $\xrightarrow{1}$ "). This can be shown by induction on the length of terms A . However, we do not need this property in the article.

We devote the rest of this section to prove the strong normalization theorem and, as a corollary, prove Theorem 2.10. First, we introduce the subsets W_n of $\bar{T}_n$ which express all strongly normalizable terms of $\bar{T}_n$. Then we prove that $\bar{T}_n = W_n$ in $ID_{<\omega}$.

DEFINITION 2.5. For $n < \omega$, the sets W_n ($\subset \bar{T}_n$) are defined inductively by:

- (W1) $\bar{0} \in W_n$.
- (W2) If $a \in \bar{T}_n$ is normal and $a[s] \in W_n$ for all $s \in \text{dom}(a)$, then $a \in W_n$.
- (W3) If $a \in \bar{T}_n$ is not normal and $b \in W_n$ for all b such that $a \xrightarrow{1} b$, then $a \in W_n$.

We can easily show that every term in W_n is strongly normalizable as follows. From the inductive definition of W_n , the following partial ordering $<<$ on $\bigcup_{n<\omega} W_n$ is well-founded: $<<$ is defined as the transitive closure of

- (i) $\bar{0} \leq a$,
- (ii) $a[s] << a$ where a is normal and $s \in \text{dom}(a)$,
- (iii) $b << a$ where a is not normal and $a \xrightarrow{1} b$.

Hence, if $A \in W_n$, there is no infinite sequence $\langle A_i : i < \omega \rangle$ such

that $A = A_0$, $A_{i+1} \ll A_i$. Thus, in particular, every term in W_n is strongly normalizable.

We remark here that, as usual, we can extend $\ll$ to the lexicographic orderings $\ll$ on $W_{n+1} \times W_n$ and $W_{n+1} \times W_n \times W_n$ which are also well-founded. To prove the strong normalization theorem, we show the following theorem.

THEOREM 2.6. *For each $a \in \bar{T}_n$, " $a \in W_n$ " is provable in $ID_{<\omega}$.*

LEMMA 2.7. $(ID_{<\omega})$ *Let $a \in W_{n+1}$ and $b, c \in W_n$. If $\bar{\varphi}_n(a, d) \in W_n$ for all $d \in W_n$, then $\bar{\varphi}_n^c(a, b) \in W_n$.*

Proof. By induction on $(a, b, c) \in W_{n+1} \times W_n \times W_n$ over $\ll$. Let $A = \bar{\varphi}_n^c(a, b)$. We have the following cases:

Case 1. $A \in NT_n$ and $\text{dom}(A) = \{\bar{0}\}$: Then $A = \bar{\varphi}_n(\bar{0}, b)$. By the assumption, $A \in W_n$.

Case 2. $A \in NT_n$ and $\text{dom}(A) = \bar{T}_k (k < n)$: Let $s \in \bar{T}_k$.

(i) $\text{dom}(c) = \bar{T}_k$: Then $c[s] \ll c$. By I.H. (= induction hypotheses), $A[s] = \bar{\varphi}_n^{c[s]}(a, b) \in W_n$.

(ii) $c = \bar{1}$ and $\text{dom}(a) = \bar{T}_k$: By the assumption, $A \in W_n$. Hence $A[s] \in W_n$ by (W2). Hence, $A \in W_n$ by (W2).

Case 3. $A \in \bar{T}_n \setminus NT_n$: Let $A \xrightarrow{1} B$. We will show $B \in W_n$.

(i) $A = \bar{\varphi}_n^{\bar{0}}(a, b)$ and $B = b$: Then $B \in W_n$.

(ii) $A = \bar{\varphi}_n(a, b)$: Since $A \in W_n$ by the assumption, $B \in W_n$ by (W3).

(iii) $A = \bar{\varphi}_n^{e+1}(a, b)$ and $B = \bar{\varphi}_n(a, \bar{\varphi}_n^e(a, b))$: From $e \ll e+1$ and I.H., $\bar{\varphi}_n^e(a, b) \in W_n$. Hence $B \in W_n$ by the assumption.

(iv) In all other cases (e.g., $A = \bar{\varphi}_n^c(a, b)$, $B = \bar{\varphi}_n^c(a', b)$ and

$a \xrightarrow{1} a')$, $B \in W_n$ follows immediately from I.H. Hence $A \in W_n$ by (W3). $\square$

LEMMA 2.8. ($ID_{<\omega}$) For $a \in W_{n+1}$ and $b \in W_n$, $\bar{\varphi}_n(a,b) \in W_n$.

Proof. By induction on $(a,b) \in W_{n+1} \times W_n$ over $<<$. Let $A = \bar{\varphi}_n(a,b)$.

Case 1. $A \in NT_n$ and $\text{dom}(A) = \{\bar{0}\}$: Then $A = \bar{\varphi}_n(\bar{0},b)$ and $A[\bar{0}] = b \in W_n$. Hence $A \in W_n$ by (W2).

Case 2. $A \in NT_n$ and $\text{dom}(A) = \bar{T}_k (k < n)$: Then $\text{dom}(a) = \bar{T}_n$ and $A[s] = \bar{\varphi}_n(a[s],b)$ for $s \in \text{dom}(A)$. From $a[s] << a$ and I.H., $A[s] \in W_n$. Hence $A \in W_n$ by (W2).

Case 3. $A \in \bar{T}_n \setminus NT_n$: Let $A \xrightarrow{1} B$. We will show $B \in W_n$.

(i) $A = \bar{\varphi}_n(a'+1,b)$ and $B = \bar{\varphi}_n^b(a', \bar{\varphi}_n(a',b))$: By I.H., $\bar{\varphi}_n(a',d) \in W_n$ for all $d \in W_n$. Hence, $\bar{\varphi}_n(a',b) \in W_n$ and $B \in W_n$ by 2.7.

(ii) $A = \bar{\varphi}_n(a,b)$ and $B = \bar{\varphi}_n(a[z],b)$ where $a \in NT_{n+1}$, $\text{dom}(a) = \bar{T}_n$, $z = \bar{\varphi}_n(a[1],b)$: Since $a \in W_{n+1}$, $b \in W_n$ and $1 \in W_n$, we have $a[1] \in W_{n+1}$ by (W2). So $z \in W_n$ from I.H. and $a[1] << a$. Hence $a[z] \in W_{n+1}$ by (W2). Therefore $B \in W_n$ from $a[z] << a$ and I.H.

(iii) In all other cases (e.g., $A = \bar{\varphi}_n(a,b)$, $B = \bar{\varphi}_n(a',b)$ and $a \xrightarrow{1} a'$), $B \in W_n$ follows immediately from I.H. Hence $A \in W_n$ by (W3). $\square$

LEMMA 2.9. ($ID_{<\omega}$) For $a \in W_{n+1}$ and $b,c \in W_n$, $\bar{\varphi}_n^c(a,b) \in W_n$.

Proof. Immediate from 2.7 and 2.8.

Proof of Theorem 2.6. By induction on the length of $a \in T_n$.

Clearly, $\bar{0}, \bar{1}, \bar{\omega}_0, \dots, \bar{\omega}_{n-1} \in W_n$ and $W_k \subset W_n$ for $k < n$. By 2.9, $\bar{\varphi}_n^c(d, b) \in W_n$ for $d \in W_{n+1}$ and $b, c \in W_n$. This completes the proof. $\square$

Proof of Theorem 2.4 (Strong normalization theorem). From 2.6, we have $T_n = W_n$. Hence, if we consider the well-founded ordering $<<$ on W_n defined above, it is also the well-founded ordering on T_n . If there were an infinite sequence $\{a_i\}_{i < \omega}$ such that $a_0 \xrightarrow{1} a_1 \xrightarrow{1} a_2 \xrightarrow{1} \dots$, then it is an infinite descending sequence on $<<$ such that $\dots << a_2 << a_1 << a_0$. This contradicts the well-foundedness of $<<$ on T_n . Hence the proof of the strong normalization theorem is completed. $\square$

THEOREM 2.10. *For each $\alpha < \tau'$, F_α and F'_α are provably computable in $ID_{<\omega}$.*

Proof. Let $\alpha < \tau'$. Then $\alpha < \tau'[m] \in T_1$ for some $m < \omega$. Hence $\alpha \in T_1$ and there is an $a \in \bar{T}_1$ such that $\alpha = \text{ord}(a)$. From 2.9, $\forall x (\bar{\varphi}_0(a, \bar{x}) \in W_0)$ is provable in $ID_{<\omega}$ where $\bar{x}$ is the numeral of x (i.e., if $x = 0$, then $\bar{x}$ is the numeral of x (i.e., if $x = 0$ then $\bar{x} = \bar{0}$; if $x = 1$ then $\bar{x} = \bar{1}$; if $x > 1$ then $\bar{x} = \bar{\varphi}_0(0, \overline{x-1})$). Hence $\forall x \exists y (\bar{\varphi}_0(a, \bar{x}) \xrightarrow{1} \dots \xrightarrow{1} \bar{y})$ is provable in $ID_{<\omega}$. On the other hand, $\text{ord}(\bar{\varphi}_0(a, \bar{x})) = F'_\alpha(x)$ and $\text{ord}(\bar{y}) = y$. And we have that if $b \xrightarrow{1} d$ then $\text{ord}(b) = \text{ord}(d)$. Hence $\forall x \exists y (\bar{\varphi}_0(a, \bar{x}) \xrightarrow{1} \dots \xrightarrow{1} \bar{y})$ equals to $\forall x \exists y (F'_\alpha(x) = y)$. Therefore F'_α is provably computable in $ID_{<\omega}$. Moreover, we have $F_\alpha(x) \leq F'_\alpha(x)$. Hence $\forall x \exists y (F'_\alpha(x) = y)$ implies $\forall x \exists y (F_\alpha(x) = y)$ in $ID_{<\omega}$. Therefore F_α is also provably computable in $ID_{<\omega}$. $\square$

§3. COLLAPSING THEOREM

3A. As a corollary of the strong normalization theorem proved above, we shall prove the collapsing theorem (Theorem 1.12) used in Section 1:

THEOREM 1.12 (Collapsing Theorem [15]). *Let $x < \omega$, $\alpha \in T_2$ and $\beta \in T_1$. Then*

$$G_{\varphi_1}(\alpha, \beta)(x) = F'_{c\alpha}(G_{\beta}(x)),$$

where the function $c (= c_x)$ which collapses each T_{n+1} to T_n is defined by: $c0 = 0$, $c1 = 1$, $c\omega_0 = x$, $c\omega_{k+1} = \omega_k$,
 $c(\varphi_{k+1}^{\gamma}(\delta, \xi)) = \varphi_k^{c\gamma}(c\delta, c\xi)$, $c(\varphi_0^{\gamma}(\delta, \xi)) = \varphi_0^{\gamma}(\delta, \xi)$. Hence, in particular, if α is generated in T_2 without reference to ω_0 then, as $G_{\omega_0}(x) = x$, we have $G_{\varphi_1}(\alpha, \omega_0) = F'_{c\alpha}$.

We introduce a function $\bar{c}$ which represents the function c on the terms as follows: (for each fixed $x < \omega$) (i) $\bar{c}0 = \bar{0}$, $\bar{c}1 = \bar{1}$, $\bar{c}\bar{\omega}_0 = \bar{x}$, $\bar{c}\bar{\omega}_{k+1} = \bar{\omega}_k$,

$$(ii) \quad \bar{c}(\bar{\varphi}_{n+1}^{\gamma}(\delta, \xi)) = \bar{\varphi}_n^{\bar{c}\gamma}(\bar{c}\delta, \bar{c}\xi) \quad \text{and} \quad \bar{c}(\bar{\varphi}_0^{\gamma}(\delta, \xi)) = \bar{\varphi}_0^{\gamma}(\delta, \xi),$$

where $\bar{x}$ is the numeral of x (i.e., if $x = 0$ then $\bar{x} = \bar{0}$; if $x = 1$ then $\bar{x} = \bar{1}$; if $x > 1$ then $\bar{x} = \bar{\varphi}_0(\bar{0}, \overline{x-1}) (= \overline{x-1+1})$).

LEMMA 3.1. *Let $a \in \bar{T}_n$ and $x < \omega$. Then the following hold.*

- (1) *If $a = b+1$ for some b , then $\bar{c}(b) = \bar{c}b+1$.*
- (2) *If $a \in NT_n$ and $\text{dom}(a) = \bar{T}_0$, then $\bar{c}(a[\bar{x}]) = \bar{c}a$ and $\text{ord}(a[\bar{x}]) = \text{ord}(a)$.*
- (3) *If $a \in NT_n$ and $\text{dom}(a) = \bar{T}_k$ for some $k > 0$, then*

$\text{ord}(a[b]) = \text{ord}(a)[\text{ord}(b)]$ and

$\text{ord}(\bar{c}(a[b])) = \text{ord}(\bar{c}a)[\text{ord}(\bar{c}b)]$ for $b \in \text{dom}(a)$.

(4) If $a \xrightarrow{1} b$, then $\text{ord}(a) = \text{ord}(b)$ and $\text{ord}(\bar{c}a) = \text{ord}(\bar{c}b)$.

Proof. (1)-(4) Induction on the length of a . □

LEMMA 3.2. If $x < \omega$ and $a \in \bar{T}_1$, then $G_{\text{ord}(a)}(x) = \text{ord}(\bar{c}a)$.

Proof. From the strong normalization theorem(Theorem 2.4), the proof is proceeded by transfinite induction on a over the well-founded ordering $<<$ (where $<<$ on $\bar{T}_n$ is defined as the transitive closure of (i) $\bar{0} \leq b$, (ii) $b[z] << b$ for normal b with $z \in \text{dom}(b)$, (iii) $d << b$ for non-normal b with $b \xrightarrow{1} d$).

Case 1. $a = \bar{0}$. This case is trivial.

Case 2. $a \in \text{NT}_1$ and $\text{dom}(a) = \{\bar{0}\}$. Then $a = \bar{1}$ or $b+1$ for some $b \in \bar{T}_1$. If $a = \bar{1}$, the assertion is trivial. If $a = b+1$, then

$$G_{\text{ord}(a)}(x) = G_{\text{ord}(b)}(x)+1 = \text{ord}(\bar{c}b)+1 = \text{ord}(\bar{c}a)$$

by I.H. and 3.1(1).

Case 3. $a \in \text{NT}_1$ and $\text{dom}(a) = \bar{T}_0$. By 3.1(2) and I.H.,

$$G_{\text{ord}(a)}(x) = G_{\text{ord}(a[\bar{x}])}(x) = \text{ord}(\bar{c}(a[\bar{x}])) = \text{ord}(\bar{c}a).$$

Case 4. $a \xrightarrow{1} b$ for some b . By 3.1(4) and I.H.,

$$G_{\text{ord}(a)}(x) = G_{\text{ord}(b)}(x) = \text{ord}(\bar{c}b) = \text{ord}(\bar{c}a). \quad \square$$

Proof of the collapsing theorem(Theorem 1.12). For $a \in \bar{T}_2$ and $b \in \bar{T}_1$, we have $\bar{c}(\bar{\varphi}_1(a,b)) = \bar{\varphi}_0(\bar{c}a, \bar{c}b)$ and hence $\text{ord}(\bar{c}(\bar{\varphi}_1(a,b))) = \varphi_0(\text{ord}(\bar{c}a), \text{ord}(\bar{c}b))$. Thus we have

$$\begin{aligned} G_{\varphi_1(\text{ord}(a), \text{ord}(b))}(x) &= G_{\text{ord}(\bar{\varphi}_1(a,b))}(x) \\ &= \text{ord}(\bar{c}(\bar{\varphi}_1(a,b))) \quad \text{by 3.2} \end{aligned}$$

$$\begin{aligned}
&= \varphi_0(\text{ord}(\bar{c}a), \text{ord}(\bar{c}b)) \\
&= F_{\text{ord}(\bar{c}a)}(\text{ord}(\bar{c}b)) \\
&= F_{\text{ord}(\bar{c}a)}(G_{\text{ord}(b)}(x)) \text{ by 3.2.}
\end{aligned}$$

For given $\alpha \in T_2$ and $\beta \in T_1$, we choose a and b above such that (i) $\text{ord}(a) = \alpha$, $\text{ord}(\bar{c}a) = c\alpha$ and (ii) $\text{ord}(b) = \beta$ (we can choose such a and b since the elements of T_n are constructed by the same way as to the element in $\bar{T}_n$). This completes the proof. $\square$

We recall that (Definition 1.13);

$$\tau'[0] = 3, \quad \tau'[n+1] = \varphi_1(\dots \varphi_n(\varphi_{n+1}(3, \omega_n), \omega_{n-1}), \dots, \omega_0).$$

We have the following figure from the collapsing theorem and Proposition 1.8:

$$\tau'[0] = 3 \quad \dots\dots\dots G_{\tau', [0]}(x) = 3$$

$$\tau'[1] = \varphi_1(3, \omega_0) \quad \dots\dots\dots G_{\tau', [1]}(x) = F'_{\tau', [0]}(x)$$

$$\tau'[2] = \varphi_1(\varphi_2(3, \omega_1), \omega_0) \quad \dots\dots\dots G_{\tau', [2]}(x) = F'_{\tau', [1]}(x)$$

$$\tau'[3] \quad \dots\dots\dots G_{\tau', [3]}(x) = F'_{\tau', [2]}(x)$$

$$\begin{array}{ccc}
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\cdot & & \cdot
\end{array}$$

$$\tau' = (\tau'[x])_{x < \omega} \quad \dots\dots\dots G_{\tau'}(x) < F_{\tau'}(x) \leq F'_{\tau'}(x) \leq G_{\tau'}^2(x)$$

$$(\tau' \text{ is minimal } s\text{-inaccessible}) \quad \quad \quad (x > p).$$

Figure 3.1.

§4. (3)-BUILT-UPNESS OF τ'

4A. In this section we shall prove the following theorems:

THEOREM 4.9. *Every element in T_1^+ is (k) -built-up for all $k < \omega$.*

THEOREM 4.10. *τ' is (3) -built-up.*

This corollary completes the proof of Theorem 1.14 that τ' is minimal s -inaccessible.

The notion of built-upness of fundamental sequences is first introduced by Schmidt[12] and the author and Aoyama[11] studied some other related notions of built-upness including (p) -built-upness.

In this section, we shall also introduce the sets of T_n^* ($\subseteq T_n^+$ $\subseteq T_n$) for the use of the next section. To begin with, we prove the following proposition which is needed to prove our theorems below.

PROPOSITION 4.1 ([10, Lemma 3.4]). *Let $\alpha \in T_n$ and $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m}$. Then $\alpha[\gamma] \in T_n$ for every $\gamma \in T_m$. Moreover, if $\gamma \in T_m \setminus \{0\}$, then $\alpha[\gamma] \in T_n \setminus \{0\}$.*

Proof. For a given $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m} \in T_n$, there is a normal $a \in \bar{T}_n$ such that $\text{ord}(a) = \alpha$ by 2.3(2)(iii) and the strong normalization theorem. We fix such an $a \in T_n$ with the minimal length. The proof of this proposition can be proceeded by induction on the length of this term a for α . □

It follows from this proposition that we can use transfinite induction on the terms in T_n ($n < \omega$) over the ordering $<$ of T_n which is defined in the same way as $<$ in Ω ; i.e., $<$ is the transitive closure of (i) $0 \leq \alpha$, (ii) $\alpha < \alpha+1$, (iii) $\alpha[\gamma] < \alpha$ for all $\gamma \in T_n$ if $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_n}$. Next we extend the relation $<_k$ ($k < \omega$) on Ω to higher level tree-ordinals from this proposition.

DEFINITION 4.2. The *step-down relations* $<_k$ ($k < \omega$) on $\cup_{n < \omega} T_n$ are defined inductively as follows: For $\alpha, \beta \in T_n$,

$\alpha <_k \beta$ if $\beta \neq 0$ and one of the following holds;

- (i) $\alpha \leq_k \gamma$ if $\beta = \gamma + 1$,
- (ii) $\alpha \leq_k \beta[k]$ if $\beta = (\beta[x])_{x \in \Omega_0}$,
- (iii) $\alpha <_k \beta[\gamma]$ for all $\gamma \in T_m \setminus \{0\}$ if $\beta = (\beta[\gamma])_{\gamma \in \Omega_m}$ ($m > 0$).

where $\alpha \leq_k \delta$ means that $\alpha <_k \delta$ or $\alpha = \delta$.

Note that if $\alpha, \beta \in T_1$ then the relations $<_k$ defined above are the same as ones defined in Definition 1.2.

LEMMA 4.3. For $\alpha \in T_{n+1}$, $\beta \in T_n$ and $\gamma \in T_n \setminus \{0\}$, $\beta <_k \varphi_n^\gamma(\alpha, \beta)$.

Proof. The lemma immediately follows from the two claims. $\square$

CLAIM 1. Let $\alpha \in T_{n+1}$ and $\beta \in T_n$. If $\delta <_k \varphi_n(\alpha, \delta)$ for all $\delta \in T_n$, then $\beta <_k \varphi_n^\gamma(\alpha, \beta)$ for $\gamma \in T_n \setminus \{0\}$.

Proof of Claim 1. Transfinite induction on $\gamma \in T_n$.

Case 1. $\gamma = \eta + 1$. Then $\beta \leq_k \varphi_n^\eta(\alpha, \beta) <_k \varphi_n(\alpha, \varphi_n^\eta(\alpha, \beta)) = \varphi_n^\gamma(\alpha, \beta)$ by I.H.

Case 2. $\gamma = (\gamma[x])_{x \in \Omega_0}$. Then $\beta \leq_k \varphi_n^{\gamma[k]}(\alpha, \beta) = \varphi_n^\gamma(\alpha, \beta)[k]$ by I.H. Hence $\beta <_k \varphi_n^\gamma(\alpha, \beta)$.

Case 3. $\gamma = (\gamma[\delta])_{\delta \in \Omega_m}$ ($0 < m < n$). From 4.1, $\gamma[\delta] \in T_n \setminus \{0\}$ for $\delta \in T_m \setminus \{0\}$. Hence $\beta <_k \varphi_n^{\gamma[\delta]}(\alpha, \beta) = \varphi_n^\gamma(\alpha, \beta)[\delta]$ for $\delta \in T_m \setminus \{0\}$ by I.H. Therefore $\beta <_k \varphi_n^\gamma(\alpha, \beta)$. $\square$

CLAIM 2. Let $\alpha \in T_{n+1}$. Then $\beta <_k \varphi_n(\alpha, \beta)$ for all $\beta \in T_n$.

Proof of Claim 2. Transfinite induction on $\alpha \in T_{n+1}$.

Case 1. $\alpha = 0$. Then $\beta <_k \beta + 1 = \varphi_n(\alpha, \beta)$.

Case 2. $\alpha = \gamma + 1$. Then $\delta <_k \varphi_n(\gamma, \delta)$ for all $\delta \in T_n$ by I.H. Hence, by Claim 1, $\beta <_k \varphi_n(\gamma, \beta) \leq_k \varphi_n^\beta(\gamma, \varphi_n(\gamma, \beta)) = \varphi_n(\alpha, \beta)$.

Case 3. $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m}$ ($m < n$). By I.H., $\beta <_k \varphi_n(\alpha[\gamma], \beta) = \varphi_n(\alpha, \beta)[\gamma]$ for $\gamma \in T_m$. Hence $\beta <_k \varphi_n(\alpha, \beta)$.

Case 4. $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_n}$. By I.H., $\beta <_k \varphi_n(\alpha[z], \beta) = \varphi_n(\alpha, \beta)$ where $z = \varphi_n(\alpha[1], \beta)$. $\square$

LEMMA 4.4. Let $\alpha \in T_{n+1}$ and $\beta, \delta, \gamma \in T_n$. If $\gamma <_k \delta$, then $\varphi_n^\gamma(\alpha, \beta) <_k \varphi_n^\delta(\alpha, \beta)$.

Proof. Transfinite induction on $\delta \in T_n$.

Case 1. $\delta = 0$. This case is trivial.

Case 2. $\delta = \eta + 1$. By I.H. and 4.3, $\varphi_n^\gamma(\alpha, \beta) \leq_k \varphi_n^\eta(\alpha, \beta) <_k \varphi_n(\alpha, \varphi_n^\eta(\alpha, \beta)) = \varphi_n^\delta(\alpha, \beta)$.

Case 3. $\delta = (\delta[x])_{x \in \Omega_0}$. By I.H., $\varphi_n^\gamma(\alpha, \beta) \leq_k \varphi_n^{\delta[k]}(\alpha, \beta) = \varphi_n^\delta(\alpha, \beta)[k]$. Hence $\varphi_n^\gamma(\alpha, \beta) <_k \varphi_n^\delta(\alpha, \beta)$.

Case 4. $\delta = (\delta[\xi])_{\xi \in \Omega_m}$ ($0 < m < n$). Then $\varphi_n^\gamma(\alpha, \beta) <_k \varphi_n^{\delta[\xi]}(\alpha, \beta) =$

$\varphi_n^\delta(\alpha, \beta)[\xi]$ for $\xi \in T_m \setminus \{0\}$ by I.H. Hence $\varphi_n^\gamma(\alpha, \beta) <_k \varphi_n^\delta(\alpha, \beta)$. $\square$

LEMMA 4.5. Let $\alpha, \gamma \in T_{n+1}$, $\beta \in T_n \setminus \{0\}$ and $n > 0$. If $\gamma <_k \alpha$, then $\varphi_n(\gamma, \beta) <_k \varphi_n(\alpha, \beta)$.

Proof. Transfinite induction on $\alpha \in T_n$.

Case 1. $\alpha = 0$. This case is trivial.

Case 2. $\alpha = \eta + 1$. By I.H. and 4.3, $\varphi_n(\gamma, \beta) \leq_k \varphi_n(\eta, \beta) <_k \varphi_n^\beta(\eta, \varphi_n(\eta, \beta)) = \varphi_n(\alpha, \beta)$ since $\beta \neq 0$.

Case 3. $\alpha = (\alpha[x])_{x \in \Omega_0}$. By I.H., $\varphi_n(\gamma, \beta) \leq_k \varphi_n(\alpha[k], \beta) = \varphi_n(\alpha, \beta)[k]$. Hence $\varphi_n(\gamma, \beta) <_k \varphi_n(\alpha, \beta)$.

Case 4. $\alpha = (\alpha[\xi])_{\xi \in \Omega_m} \ (0 < m < n)$. By I.H., $\varphi_n(\gamma, \beta) <_k \varphi_n(\alpha[\xi], \beta) = \varphi_\alpha(\beta)[\xi]$ for $\xi \in T_m \setminus \{0\}$. Hence $\varphi_n(\gamma, \beta) <_k \varphi_n(\alpha, \beta)$.

Case 5. $\alpha = (\alpha[\xi])_{\xi \in \Omega_n}$. By I.H., $\varphi_n(\gamma, \beta) <_k \varphi_n(\alpha[z], \beta) = \varphi_n(\alpha, \beta)$ for $\beta \in T_n \setminus \{0\}$ where $z = \varphi_n(\alpha[1], \beta)$. $\square$

4B. THE SUBSETS T_n^+ OF $T_n \ (n < \omega)$. We shall define the subset T_n^+ for each $n < \omega$, and prove that every element of T_1^+ is built-up.

DEFINITION 4.6. The subset $T_n^+ \subseteq T_n$ are defined inductively as follows:

- (i) $0, 1, \omega_0, \omega_1, \dots, \omega_{n-1} \in T_n^+$.
- (ii) $T_k^+ \subseteq T_n^+$ for $k < n$.
- (iii) If $\alpha \in T_{n+1}^+$, $\gamma \in T_n^+$ and $\beta \in T_n^+ \setminus \{0\}$, then $\varphi_n^\gamma(\alpha, \beta) \in T_n^+$.

Note that the definition of T_n^+ above differs from that of T_n only in the restriction on β in (iii).

PROPOSITION 4.7 (cf.4.1). Let $\alpha \in T_n^+$ and $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m}$. Then $\alpha[\gamma] \in T_n^+$ for every $\gamma \in T_m^+$. Moreover, if $\gamma \in T_m^+ \setminus \{0\}$, then $\alpha[\gamma] \in T_n^+ \setminus \{0\}$.

Proof. It is proceeded in the same way as 4.1. First, we introduce the subsets $\bar{T}_n^+ \subseteq \bar{T}_n$ of terms of T_n^+ as 2.1:

$$(i) \quad \bar{0}, \bar{1}, \bar{\omega}_0, \bar{\omega}_1, \dots, \bar{\omega}_{n-1} \in \bar{T}_n^+.$$

$$(ii) \quad \bar{T}_k^+ \subseteq \bar{T}_n^+ \text{ for } k < n.$$

$$(iii) \quad \text{If } \alpha \in \bar{T}_{n+1}^+, \gamma \in \bar{T}_n^+ \text{ and } \beta \in \bar{T}_n^+ \setminus \{\bar{0}\}, \text{ then } \varphi_n^\gamma(\alpha, \beta) \in \bar{T}_n^+.$$

Then we can prove that for each $\alpha \in T_n^+$, there is a term $a \in \bar{T}_n^+$ such that $\text{ord}(a) = \alpha$. And the strong normalization theorem on $\bar{T}_n^+$ holds since if $a \xrightarrow{1} a'$ and $a \in \bar{T}_n^+$, then $a' \in \bar{T}_n^+$. Hence we can prove this proposition in the same way as 4.1. $\square$

THEOREM 4.8 ([10, Theorem 3]). Let $\alpha \in T_n^+$ and $\alpha = (\alpha[\xi])_{\xi \in \Omega_m}$. If $\gamma, \delta \in T_m$ and $\gamma <_k \delta$, then $\alpha[\gamma] <_k \alpha[\delta]$.

Proof. From the proof of 4.7., for a given $\alpha \in T_n^+$, we can take a normal term $a \in \bar{T}_n^+$ with the minimal length such that $\text{ord}(a) = \alpha$. The proof of this theorem is proceeded by induction on the length of this term a . We have the following cases:

Case 1. $a = \bar{\omega}_m$. Then $\alpha = \omega_m$. We have $\alpha[\gamma] = \gamma <_k \delta = \alpha[\delta]$.

Case 2. $a = \bar{\varphi}_n(d, b)$ and $\text{dom}(d) = \bar{T}_m$. Then $\alpha = \varphi_n(\lambda, \beta)$ so that $\lambda = (\lambda[\xi])_{\xi \in \Omega_m} = \text{ord}(d)$ and $\beta = \text{ord}(b) \in T_n^+ \setminus \{0\}$ from the definition of T_n^+ above and $a \in \bar{T}_n^+$. Hence, by I.H. $\lambda[\gamma] <_k \lambda[\delta]$ and 4.5, $\varphi_n(\lambda, \beta)[\gamma] = \varphi_n(\lambda[\gamma], \beta) <_k \varphi_n(\lambda[\delta], \beta) = \varphi_n(\lambda, \beta)[\delta]$.

Case 3. $a = \bar{\varphi}_n^e(d, b)$ and $\text{dom}(e) = \bar{T}_m$. This case is treated similarly to Case 2, using 4.4. $\square$

THEOREM 4.9. Each $\alpha \in T_1^+$ is (k) -built-up for all $k < \omega$.

Proof. For each $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m} \in T_n^+$ and $\gamma \in T_m^+$, $\alpha[\gamma] \in T_n^+$ from 4.7. Hence for each $\alpha \in T_1^+$ and limit $\lambda \leq \alpha$, we have $\lambda \in T_1^+$. Thus by 4.8, $\lambda[x] <_k \lambda[x+1]$ for all k , $x < \omega$ and limit $\lambda \leq \alpha \in T_1^+$. $\square$

The reason why we introduce the set T_n^+ is that (k) -built-upness does not hold for some element in T_1 since, if we put $\alpha = \varphi_1(\omega_0, 0)$, then $\alpha[x] = \varphi_1(x, 0) = 1$ for all $x < \omega$.

THEOREM 4.10 ([10, Corollary 3.1]). τ' is (3) -built-up.

Proof. Let $x < \omega$. From the definition of τ' (1.13), $\tau'[x] \in T_1^+$. By 4.9, $\tau'[x]$ is (3) -built-up. Hence it is sufficient to prove that $\tau'[x] <_3 \tau'[x+1]$. For this, we have

$$\begin{aligned}
 \tau'[x] &= \varphi_1(\dots \varphi_x(3, \omega_{x-1}) \dots, \omega_0) \\
 &<_3 \varphi_1(\dots \varphi_x(\omega_0, \omega_{x-1}) \dots, \omega_0) \\
 &<_0 \varphi_1(\dots \varphi_x(\varphi_1(z, \omega_0), \omega_{x-1}) \dots, \omega_0) \\
 &\quad \text{where } z = \varphi_2(\dots \varphi_x(1, \omega_{x-1}) \dots, \omega_1) \\
 &= \varphi_1(\dots \varphi_x(\omega_1, \omega_{x-1}) \dots, \omega_0) \\
 &<_0 \varphi_1(\dots \varphi_x(\varphi_2(z', \omega_1), \omega_{x-1}) \dots, \omega_0) \\
 &\quad \text{where } z' = \varphi_2(\dots \varphi_x(1, \omega_{x-1}) \dots, \omega_1) \\
 &= \varphi_1(\dots \varphi_x(\omega_2, \omega_{x-1}) \dots, \omega_0) \\
 &\quad \vdots \\
 &= \varphi_1(\dots \varphi_x(\omega_x, \omega_{x-1}) \dots, \omega_0) \\
 &<_3 \varphi_1(\dots \varphi_x(\varphi_{x+1}(3, \omega_x), \omega_{x-1}) \dots, \omega_0)
 \end{aligned}$$

$$= \tau'[x+1]$$

from $3 <_3 \omega_0$ and from 3.5, 3.3. □

4C. THE SUBSETS T_n^* OF $T_n^+(n < \omega)$. Here we shall introduce the sets T_n^* of terms which are used in the next section.

DEFINITION 4.11. The subset $T_n^* \subseteq T_n^+$ are defined inductively as follows:

- (i) $0, 1, \omega_0, \omega_1, \dots, \omega_{n-1} \in T_n^*$.
- (ii) $T_k^* \subseteq T_n^*$ for $k < n$.
- (iii) If $\alpha \in T_{n+1}^*$, $\gamma \in T_n^*$ and $\beta \in T_n^* \setminus T_{n-1}^*$, then $\varphi_n^\gamma(\alpha, \beta) \in T_n^*$ where $T_{-1}^* = \{0\}$.

Note that similarly to the case of the sets T_n^+ , the definition of T_n^* above differs from that of T_n^+ only in the restriction on β in (iii). We can prove the same propositions as the case of T_n^+ in the same way as the corresponding proofs.

PROPOSITION 4.12 (cf.4.1). Let $\alpha \in T_n$ and $\alpha = (\alpha[\gamma])_{\gamma \in \Omega_m}$. Then $\alpha[\gamma] \in T_n^*$ for every $\gamma \in T_m^*$. Moreover, if $\gamma \in T_m^* \setminus T_{m-1}^*$, then $\alpha[\gamma] \in T_n^* \setminus T_{m-1}^*$.

DEFINITION 4.13 (cf.4.2). The step-down relations $<_k^*$ ($k < \omega$) on $\cup_{n < \omega} T_n^*$ are defined inductively as follows: For $\alpha, \beta \in T_n^*$,

$\alpha <_k^* \beta$ if $\beta \neq 0$ and one of the following holds;

- (i) $\alpha \leq_k^* \gamma$ if $\beta = \gamma + 1$,
- (ii) $\alpha \leq_k^* \beta[k]$ if $\beta = (\beta[x])_{x \in \Omega_0}$,
- (iii) $\alpha <_k^* \beta[\gamma]$ for all $\gamma \in T_m^* \setminus (T_{m-1}^* \cup \{\omega_{m-1}\})$

if $\beta = (\beta[\gamma])_{\gamma \in \Omega_m}$ ($m > 0$).

LEMMA 4.14 (cf.4.3). For $\alpha \in T_{n+1}^*$, $\beta \in T_n^* \setminus T_{n-1}^*$ and $\gamma \in T_n^* \setminus \{0\}$, we have $\beta <_k^* \varphi_n^\gamma(\alpha, \beta)$.

LEMMA 4.15 (cf.4.4). Let $\alpha \in T_{n+1}^*$, $\delta, \gamma \in T_n^*$ and $\beta \in T_n^* \setminus T_{n-1}^*$. If $\gamma <_k^* \delta$, then $\varphi_n^\gamma(\alpha, \beta) <_k^* \varphi_n^\delta(\alpha, \beta)$.

LEMMA 4.16 (cf.4.5). Let $\alpha, \gamma \in T_{n+1}^*$, $\beta \in T_n^* \setminus T_{n-1}^*$ and $n > 0$. If $\gamma <_k^* \alpha$, then $\varphi_n(\gamma, \beta) <_k^* \varphi_n(\alpha, \beta)$.

THEOREM 4.17 (cf.4.8). Let $\alpha \in T_n^*$ and $\alpha = (\alpha[\xi])_{\xi \in \Omega_m}$. If $\gamma, \delta \in T_m^*$ and $\gamma <_k^* \delta$, then $\alpha[\gamma] <_k^* \alpha[\delta]$.

The next lemma is used in the next section.

LEMMA 4.18. (1) If $\alpha \in T_{m+1}^* \setminus (T_m^* \cup \{\omega_m\})$, then $\omega_m <_k^* \alpha$ for all $k < \omega$.

(2) $k <_k^* \omega_0$ for $k < \omega$.

(3) $\omega_i <_0^* \omega_n$ for $i < n$.

Proof. (1) If $\alpha \in T_{m+1}^* \setminus (T_m^* \cup \{\omega_m\})$, then α is of the form: $\varphi_{m+1}^\gamma(\dots, \omega_m)\dots$. Hence 4.14 completes the proof. (2) Trivial for the definition of $<_k^*$. (3) It is sufficient to prove that $\omega_i <_0^* \omega_{i+1}$. By (1) we have $\omega_i <_0^* \alpha$ for all $\alpha \in T_{i+1}^* \setminus (T_i^* \cup \{\omega_i\})$. From the definition of $<_0^*$ and $\omega_i[\alpha] = \alpha$, this completes the proof. $\square$

§5. PROVABLY COMPUTABLE FUNCTIONS IN $ID_{<\omega}$

5A. In this section we shall prove the following theorem:

THEOREM 5.1. *If a π_2^0 -sentence $\forall x \exists y A(x,y)$ ($A \in \Sigma_1^0$) is provable in $ID_v (v < \omega)$, then there is an $\alpha < \tau'[v+1]$ such that for all $n > 1$, there is an $k < F_\alpha(n)$ $A(n,k)$.*

Clearly, this theorem implies (III) in Introduction. Here we shall prove this theorem in the same way as Buchholz[4].

5B. THE SYSTEM $ID_v (v < \omega)$. We introduce the system ID_v for $v < \omega$ following [4, Section 4].

Preliminaries. Let L denote the first-order language consisting of the following symbols:

- (i) the logical constants $\neg, \wedge, \vee, \forall, \exists$,
- (ii) number variables (indicated by x, y),
- (iii) a constant 0 (zero) and a unary function symbol ' $'$ (successor),
- (iv) constants for primitive recursive predicates (among them the symbol $<$ for the arithmetic 'less' relation).

By $s, t, t_0, \dots$ we denote arbitrary L -terms. The constant terms $0, 0', 0'', \dots$ are called numerals; we identify numerals and natural numbers and denote them by i, j, k, m, n, u, v, w . A formula of the shape $Rt_1 \dots t_n$ or $\neg Rt_1 \dots t_n$, where R is a n -ary predicate symbol of L , is called an *arithmetic prime formula* (abbreviated by a.p.f.).

Let X be a unary and Y a binary predicate variable. A *positive operator form* is a formula $\mathcal{U}_y(X, Y, y, x)$ of $L(X, Y)$ in

which only X, Y, y, x occur free and all occurrences of X are positive. The language L_{ID} is obtained from L by adding a binary predicate constant $P^{\mathcal{U}}$ and a 3-ary predicate constant $P^{\mathcal{U}}_{<}$ for each positive operator form $\mathcal{U}$.

Abbreviations.

$$\begin{aligned} t \in P^{\mathcal{U}}_S &:= P^{\mathcal{U}}_S t := P^{\mathcal{U}}_S t, & t \notin P^{\mathcal{U}}_S &:= \neg(t \in P^{\mathcal{U}}_S), \\ P^{\mathcal{U}}_{<S} t_0 t_1 &:= P^{\mathcal{U}}_{<} t_0 t_1, & \mathcal{U}_S(X, x) &:= \mathcal{U}(X, P^{\mathcal{U}}_{<S}, s, x). \end{aligned}$$

The formal theory ID_{ν} with $\nu < \omega$ is an extension of Peano Arithmetic, formulated in the language L_{ID} , by the following axioms:

$$\begin{aligned} (P^{\mathcal{U}}.1) \quad & \forall y \forall x (\mathcal{U}_y(P^{\mathcal{U}}_y, x) \longrightarrow x \in P^{\mathcal{U}}_y). \\ (P^{\mathcal{U}}.2)_{<\nu} \quad & \forall x (\mathcal{U}_u(F, x) \longrightarrow F(x)) \longrightarrow \forall x (P^{\mathcal{U}}_u x \longrightarrow F(x)), \text{ for each} \\ & L_{ID}\text{-formula } F(x) \text{ and each } u < \nu. \\ (P^{\mathcal{U}}.3) \quad & \forall y \forall x_0 \forall x_1 (P^{\mathcal{U}}_{<y} x_0 x_1 \longleftrightarrow x_0 < y \wedge x_1 \in P^{\mathcal{U}}_{x_0}). \end{aligned}$$

4C. THE INFINITARY SYSTEM $\varphi ID^{\omega}_{<\omega}$. As in [4, Section 4], the infinitary system $\varphi ID^{\omega}_{<\omega}$ shall be formulated in the language $L_{ID}(N)$ which arises from L_{ID} by adding a new unary predicate symbol N . This is a technical tool which shall help us to keep control over the numerals n occurring in $\exists$ -inferences $A(n) \vdash \exists x A(x)$ of $\varphi ID^{\omega}_{<\omega}$ -derivations. Following Tait[14] we assume all formulas to be in negation normal form, i.e., the formulas are built up from atomic and negated atomic formulas by means of $\wedge, \vee, \forall, \exists$. If A is a complex formula we consider $\neg A$ as a notation for the corresponding negation normal form.

Definition of the length $|A|$ of a $L_{ID}(N)$ -formula A

1. $|Nt| := |\neg Nt| := 0$.
2. $|A| := 1$, if A is an a.p.f. or a formula $(\neg)P_s^u t$.
3. $|P_{<s}^u t_0 t_1| := |\neg P_{<s}^u t_0 t_1| := 2$.
4. $|A \wedge B| := |A \vee B| := \max\{|A|, |B|\} + 1$.
5. $|\forall x A| := |\exists x A| := |A| + 1$.

PROPOSITION 5.2. $|\neg A| = |A|$, for each $L_{ID}(N)$ -formula A .

As before we use the letters u, v to denote numbers $< \omega$.

Inductive definition of formula sets $\text{Pos}_v (v < \omega)$

1. All $L(N)$ -formulas belong to Pos_v .
2. All formulas $P_u^u t$, $(\neg)P_{<u}^u t_0 t_1$ with $u \leq v$ belong to Pos_v .
3. All formulas $\neg P_u^u t$ with $u < v$ belong to Pos_v .
4. If A and B belong to Pos_v , then the formulas $A \wedge B$, $A \vee B$, $\forall x A$, $\exists x A$ also belong to Pos_v .

REMARK 5.3. If $P_u^u t \in \text{Pos}_v$, then also $\neg P_u^u t \in \text{Pos}_v$.

Notations

- In the following A, B, C always denote closed $L_{ID}(N)$ -formulas.
- Γ, Γ', Δ denote finite sets of closed $L_{ID}(N)$ -formulas; we write, e.g., Γ, Δ, A for $\Gamma \cup \Delta \cup \{A\}$.
- A^N denotes the result of restricting all quantifiers in A to N .
- $t \in N := Nt$, $t \notin N := \neg Nt$.
- As before we use the letters $\alpha, \beta, \gamma, \delta$ to denote elements of T_n^* .

DEFINITION 5.4. $\gamma \prec_\Gamma \alpha \iff \gamma \prec_k \alpha$,

where $k := \max(\{3\} \cup \{3n : \neg Nn \in \Gamma\})$.

PROPOSITION 5.5. (1) $\gamma <_{\Gamma} \alpha$ and $\Gamma \subset \Delta \Rightarrow \gamma <_{\Delta} \alpha$. ((0)-built-upness of all elements of T_n^* : Theorem 4.17.)

(2) $\gamma <_{\Gamma \cup \{0 \notin N\}} \alpha \Rightarrow \gamma <_{\Gamma} \alpha$.

Basic inference rules

($\wedge$) $A_0, A_1 \vdash A_0 \wedge A_1$.

($\vee$) $A \vdash A \vee B; B \vdash A \vee B$.

($\forall^\omega$) $(A(n))_{n \in \omega} \vdash \forall x A(x)$.

($\exists$) $A(n) \vdash \exists x A(x)$.

(N) $n \in N \vdash n' \in N$.

($P_{<u}^{\mathcal{U}}$) $P_{jn}^{\mathcal{U}} \vdash P_{<u}^{\mathcal{U}} jn$, if $j < u < \omega$.

($\neg P_{<u}^{\mathcal{U}}$) $\neg P_{jn}^{\mathcal{U}} \vdash \neg P_{<u}^{\mathcal{U}} jn$, if $j < u < \omega$.

Every instance $(A_i)_{i \in I} \vdash A$ of these rules is called a basic inference. If $(A_i)_{i \in I} \vdash A$ is a basic inference with $A \in \text{Pos}_V$, then $A_i \in \text{Pos}_V$ for all $i \in I$. This property will be used in the proof of 5.10.

The system $\phi ID_{<\omega}^\omega$ consists of the language $L_{ID}(N)$ and a certain derivability relation $\vdash_m^\alpha \Gamma$ (" Γ is derivable with order $\alpha \in T_n^*$ and cut degree $m \in \omega$ ") which we introduce below by an iterated inductive definition.

Inductive definition of $\vdash_m^\alpha \Gamma$ ($\alpha \in T_n^$, $m \in \omega$)*

(Ax1) $\vdash_m^\alpha \Gamma, A$, if A is a true a.p.f. or $A \equiv 0 \in N$ or $A \equiv \neg P_{<u}^{\mathcal{U}} jn$ with $u \leq j$.

(Ax2) $\vdash_m^\alpha \Gamma, \neg A, A$, if $A \equiv n \in N$ or $A \equiv P_u^{\mathcal{U}} n$.

(Bas) If $(A_i)_{i \in I} \vdash A$ is a basic inference with $A \in \Gamma$ and $\forall i \in I (\vdash_m^\alpha \Gamma, A_i)$, then $\vdash_m^{\alpha+1} \Gamma$.

- $(P_u^{\mathcal{U}}) \quad \vdash_m^\alpha \Gamma, n \in N \wedge \mathcal{U}_u^N(P_u^{\mathcal{U}}, n) \text{ and } P_u^{\mathcal{U}} n \in \Gamma \Rightarrow \vdash_m^{\alpha+3} \Gamma.$
 $(Cut) \quad \vdash_m^\alpha \Gamma, \neg C \text{ and } \vdash_m^\alpha \Gamma, C \text{ and } |C| < m \Rightarrow \vdash_m^{\alpha+1} \Gamma$
 $(\Omega_{u+1}) \quad \left. \begin{array}{l} \text{dom}(\alpha) = \Omega_{u+1} \text{ and } \vdash_m^{\alpha[1]} \Gamma, P_u^{\mathcal{U}} n \text{ and} \\ \forall z \in \Omega_{u+1} \forall \Delta \subset \text{Pos}_u (\vdash_1^z \Delta, P_u^{\mathcal{U}} n \Rightarrow \vdash_m^{\alpha[z]} \Delta, \Gamma) \end{array} \right\} \Rightarrow \vdash_m^{\alpha+1} \Gamma.$
 $(<) \quad \vdash_m^\beta \Gamma \text{ and } \beta <_\Gamma \alpha \Rightarrow \vdash_m^\alpha \Gamma.$

LEMMA 5.6. (1) $\vdash_m^\alpha \Gamma$ and $m \leq k, \Gamma \subset \Delta \Rightarrow \vdash_k^\alpha \Delta.$

(2) $\vdash_m^\alpha \Gamma \Rightarrow \vdash_m^{\gamma+\alpha} \Gamma. (\gamma+\alpha = \varphi_n^\alpha(0, \gamma).)$

(3) $\vdash_m^\alpha \Gamma, 0 \notin N \Rightarrow \vdash_m^\alpha \Gamma.$

Proof. (cf. [4, Lemma 4.2].) Induction on α using 5.5 and the relation that $(\gamma+\alpha)[\delta] = \gamma+\alpha[\delta]$ for all $\delta \in \Omega_k$ with $\alpha = (\alpha[\delta])_{\delta \in \Omega_k}.$ $\square$

LEMMA 5.7 (Inversion). Let $(A_i)_{i \in I} \vdash A$ be a basic inference $(\wedge), (\forall^\infty), (P_{<u}^{\mathcal{U}}), (\neg P_{<u}^{\mathcal{U}}).$ Then $\vdash_m^\alpha \Gamma, A$ implies $\forall i \in I (\vdash_m^\alpha \Gamma, A_i).$

Proof. Similar to [4, Lemma 4.3] by induction on $\alpha.$ $\square$

LEMMA 5.8 (Reduction). Suppose $\vdash_m^\alpha \Gamma_0, \neg C$ and $|C| \leq m$, where C is formula of the shape $A \vee B$ or $\exists x A(x)$ or $P_{<u}^{\mathcal{U}} n$ or $\neg P_{<u}^{\mathcal{U}} n$ or a false a.p.f. Then $\vdash_m^\beta \Gamma, C$ implies $\vdash_m^{\alpha+\beta} \Gamma_0, \Gamma.$

Proof. Similar to [4, Lemma 4.4] from induction on β and the relation that $\alpha+(\beta+1) = (\alpha+\beta)+1.$ $\square$

THEOREM 5.9 (Cutelimination). $\vdash_{m+1}^\alpha \Gamma$ and $\alpha \in T_{\nu+1}^*, \nu < \omega, m > 0 \Rightarrow \vdash_m^z \Gamma$ where $z = \varphi_{\nu+1}^\alpha(1, \varphi_{\nu+1}(1, \varphi_{\nu+1}^k(2, \omega_\nu)))$ for all $k < \omega.$

Proof. (cf. [4, Theorem 4.5].) Induction on α . Let $z = \varphi_{v+1}^\alpha(1, \varphi_{v+1}^k(1, \varphi_{v+1}^k(2, \omega_v)))$.

1. Suppose $\alpha = \gamma+1$, $A \in \Gamma$ and $\forall i \in I (\vdash_{m+1}^\gamma \Gamma, A_i)$, where $(A_i)_{i \in I} \vdash A$ is a basic inference ($\mathcal{J}$). Then by I.H. we have $\forall i \in I (\vdash_m^\beta \Gamma, A_i)$ where $\beta = \varphi_{v+1}^\gamma(1, \varphi_{v+1}^k(1, \varphi_{v+1}^k(2, \omega_v)))$. By ($\mathcal{J}$) we have $\vdash_m^{\beta+1} \Gamma$ and then $\vdash_m^z \Gamma$ since $\beta+1 = \varphi_{v+1}(0, \beta) <_0 \varphi_{v+1}(1, \beta) = z$ by 4.5.

2. Suppose $\alpha = \gamma+1$, $\vdash_{m+1}^\gamma \Gamma, \neg C$, $\vdash_{m+1}^\gamma \Gamma, C$ and $|C| = m$. Then by I.H. we have $\vdash_m^\beta \Gamma, \neg C$ and $\vdash_m^\beta \Gamma, C$ where β is as 1. We may assume that C fulfills the condition of 5.8. By ($\langle$) and 5.8, we have $\vdash_m^{(\beta+1)+\beta} \Gamma$. Hence $\vdash_m^z \Gamma$ since $(\beta+1)+\beta = \varphi_{v+1}^\beta(0, \varphi_{v+1}(0, \beta)) = \varphi_{v+1}(1, \beta) = z$.

3. Suppose $\alpha = \gamma+3$, $P_u^{\mathcal{U}} \in \Gamma$ and $\vdash_{m+1}^\gamma \Gamma, B$ with $B = n \in N \wedge \mathcal{U}_u(P_u^{\mathcal{U}}, n)$. Then by I.H. and ($\langle$) we have $\vdash_m^\beta \Gamma, B$ where $\beta = \varphi_{v+1}^{\gamma+2}(1, \varphi_{v+1}^k(1, \varphi_{v+1}^k(2, \omega_v)))$. By $(P_u^{\mathcal{U}})$ we get $\vdash_m^{\beta+3} \Gamma$ and hence $\vdash_m^z \Gamma$ since $\beta+3 = \varphi_{v+1}^3(0, \beta) <_2 \varphi_{v+1}^\beta(1, \varphi_{v+1}(0, \beta)) = \varphi_{v+1}(1, \beta) = z$.

4. In all other cases the assertion follows from I.H. and the fact that $\beta+1 = \varphi_{v+1}(0, \beta) <_0 \varphi_{v+1}(1, \beta) = z$ as in 1 above. $\square$

THEOREM 5.10 (Collapsing Lemma). $\vdash_1^\alpha \Gamma$ and $\Gamma \subset \text{Pos}_v$, $\alpha \in T_{v+2}^* \Rightarrow \vdash_1^z \Gamma$ where $z = \varphi_{v+1}(\alpha, \omega_v)$.

Proof. (cf. [4, Theorem 4.6].) Induction on α .

1. Suppose $\alpha = (\alpha[\delta])_{\delta \in \Omega_{u+1}}$, $\vdash_1^{\alpha[1]} \Gamma, P_u^{\mathcal{U}}$ and $\vdash_1^{\alpha[z]} \Delta, \Gamma$ for all $z \in T_{u+1}^*$, $\Delta \subset \text{Pos}_u$ with $\vdash_1^z \Delta, P_u^{\mathcal{U}}$. Then $u \leq v$.

Case 1. $u < v$. We have $\varphi_{v+1}(\alpha[z], \omega_v) = \varphi_{v+1}(\alpha, \omega_v)[z]$ for all $z \in T_{u+1}^*$. Hence the assertion follows by (Ω_{u+1}) .

Case 2. $u = v$. Then $\Gamma \cup \{P_u^{\mathcal{U}}\} \subset \text{Pos}_u$ and by I.H. $\vdash_1^\beta \Gamma, P_u^{\mathcal{U}}$ where $\beta = \varphi_{v+1}(\alpha[1], \omega_v)$. Since $\beta \in T_{u+1}^*$ we get $\vdash_1^z \Gamma$ where $z =$

$\varphi_{v+1}(\alpha[\beta], \omega_v)$. But $z = \varphi_{v+1}(\alpha[\beta], \omega_v) = \varphi_{v+1}(\alpha, \omega_v)$ from the definition of φ_{v+1} (see 1.10).

2. In all other cases the assertion follows from the I.H. $\square$

DEFINITION 5.11. $L(N)_+ := \{A: A \text{ is a sentence of } L(N) \text{ in which } N \text{ occurs only positively}\}$. For $\Gamma = \{A_1, \dots, A_n\} \subset L(N)_+$ we define:

$$\models \Gamma(k) : \Leftrightarrow \begin{cases} A_1 \vee \dots \vee A_n \text{ is true in the standard model} \\ \text{when } N \text{ is interpreted as } \{i < \omega : 3i < k\}. \end{cases}$$

$$\text{LEMMA 5.12. } \left. \begin{array}{l} \vdash_1^\alpha i_1 \notin N, \dots, i_m \notin N, \Gamma \text{ and } \alpha \in T_1^*, \\ \Gamma \subset L(N)_+, n \geq \max\{3, 3i_1, \dots, 3i_m\} \end{array} \right\} \Rightarrow \models \Gamma(F_\alpha(n)).$$

Proof. (cf. [4, Lemma 4.7].) Induction on α . Let $\Gamma_0 = \{i_1 \notin N, \dots, i_m \notin N\}$ and $k = \max\{3, 3i_1, \dots, 3i_m\} \leq n$.

1. $(Ax1) \vdash_1^\alpha \Gamma_0, \Gamma$. The assertion is trivial for $0 < F_\alpha(n)$.
2. $(Ax2) \vdash_1^\alpha \Gamma_0, \Gamma$. The assertion follows from $n < F_\alpha(n)$.
3. If $\vdash_1^\alpha \Gamma_0, \Gamma$ is the conclusion of a basic inference $\neq (N)$, then the assertion follows from the I.H. and the relation $F_\beta(n) < F_{\beta+1}(n)$.

4. Suppose $\alpha = \beta+1$, $N(j+1) \in \Gamma$, $\vdash_1^\beta \Gamma_0, \Gamma, Nj$. By I.H. we have $\models \Gamma \cup \{Nj\} (F_\alpha(n))$. Then we have $F_\beta(n) < F_\beta^2(n) < F_\beta^3(n) < F_\beta^4(n) \leq F_\beta^{n+1}(n) = F_\alpha(n)$. So, $F_\beta(n)+3 \leq F_\alpha(n)$. Hence $\models \Gamma(F_\beta(n))$.

5. Suppose $\vdash_1^\beta \Gamma_0, \Gamma$ with $\beta <_{\Gamma_0 \cup \Gamma} \alpha$. Then we have $F_\beta(n) < F_\alpha(n)$ since $n \geq k$. The assertion follows from the I.H.

6. Suppose $\alpha = \beta+1$, $\vdash_1^\beta \Gamma_0, \Gamma, i_0 \notin N$ and $\vdash_1^\beta i_0 \notin N, \Gamma_0, \Gamma$. Let $\hat{n} = F_\alpha(n)$. Then we have $n < \hat{n} < F_\beta(\hat{n}) = F_\beta^2(n) < F_\alpha(n)$.

6.1. $\hat{n} < 3i_0$. From $\vdash_1^\beta \Gamma_0, \Gamma, i_0 \notin N$ we obtain by the I.H.

$\models \Gamma \cup \{i_0 \in N\}(\hat{n})$ and then $\models \Gamma(\hat{n})$, since $\neg(3i_0 < \hat{n})$. Using $\hat{n} < F_\alpha(n)$ we get the assertion.

6.2. $3i_0 \leq \hat{n}$. From $\vdash_1^\beta i_0 \notin N, \Gamma_0, \Gamma$ and $\max\{k, 3i_0\} \leq \hat{n}$ we obtain by I.H. $\models \Gamma(F_\beta(\hat{n}))$ and then $\models \Gamma(F_\alpha(n))$. $\square$

THEOREM 5.13 (Bounding). *If $\vdash_1^\alpha \forall x \in N(\exists y \in N)A^N(x, y)$, where $0 < \alpha \in T_1^*$, $\nu \leq \omega$, $m > 0$ and $A(x, y)$ a Σ_1^0 -formula of the language L , then $\forall n > 1, \exists k < F_{\alpha+1}(n)(A(n, k))$.*

Proof. (cf. [4, Theorem 4.8].) From the premise we obtain $\vdash_1^\alpha n \notin N, \exists y \in N(A^N(n, y))$ for all $n < \omega$. Then by 5.12 we get $\models \{\exists y \in N(A^N(n, y))\}(F_\alpha(\hat{n}))$ for all $n < \omega$ and all $\hat{n} \geq \max\{3, 3n\}$. Hence $\forall n \exists k < F_\alpha(3n+3)A(n, k)$. From $3n+3 < 4n+2 = F_1^2(n)$ since $n > 1$. We have $F_\alpha(3n+3) < F_\alpha(F_1^2(n)) \leq F_\alpha^3(n) \leq F_\alpha^{n+1}(n) = F_{\alpha+1}(n)$ since $1 \leq_1 \alpha$ by $0 < \alpha$ and 1.3(3). $\square$

4C. EMBEDDING $ID_\nu(\nu < \omega)$ INTO $\varphi ID_{<\omega}^\infty$. In the remaining part of this section we show that $ID_\nu(\nu < \omega)$ can be embedded into $\varphi ID_{<\omega}^\infty$ and finally we prove the theorem that if a Π_2^0 -sentence $\forall x \exists y A(x, y) (A \in \Sigma_1^0)$ is provable in $ID_\nu(\nu < \omega)$ then there is an $\alpha < \tau'$ such that $\forall n > 1 \exists k < F_\alpha(n)(A(n, k))$.

Abbreviations. $k^\sim = \varphi_{\nu+1}^{k+1}(2, \omega_\nu)$.

$\alpha \dashrightarrow_n \beta : \iff \exists \alpha_0, \dots, \alpha_n (\alpha_0 = \alpha \wedge \alpha_n = \beta \wedge \forall i < n (\alpha_{i+1} \leq_2^* \alpha_{i+1}))$.

LEMMA 5.14. (1) $k^\sim + 1 <_0^* (k+1)^\sim$. (2) $k^\sim \dashrightarrow_9 (k+1)^\sim$.

Proof. (cf. [4, Lemma 4.9].) (1) From $\varphi_{\nu+1}(0, k^\sim) <_0^* \varphi_{\nu+1}(2, k^\sim) = (k+1)^\sim$ by 4.5. (2) From the relation that, since $2 <_2^* k^\sim$, $k^\sim + 3$

$$\begin{aligned} & \prec_2^* \varphi_{\nu+1}(1, k^{\sim}) \prec_2^* \varphi_{\nu+1}(1, k^{\sim})+3 \prec_2^* \varphi_{\nu+1}^2(1, k^{\sim}) \prec_2^* \varphi_{\nu+1}^2(1, k^{\sim})+3 \prec_2^* \\ & \varphi_{\nu+1}^3(1, k^{\sim}) \prec_2^* \varphi_{\nu+1}(2, k^{\sim}) = (k+1)^{\sim} \quad \square \end{aligned}$$

LEMMA 5.15. $\vdash_0^{k^{\sim}} \neg A, A$ where $k = |A|$.

Proof. Similar to [4, Lemma 1.10]. $\square$

LEMMA 5.16. $\vdash^{(k^{\sim}+1)+\omega_{\nu}} \neg F(0), \neg \forall x \in \mathbb{N} (F(x) \longrightarrow F(x')), n \notin \mathbb{N}, F(n)$
where $k = |F|$.

Proof. Similar to [4, Lemma 4.11]. $\square$

DEFINITION 5.17. For $A \in \text{Pos}_u$ let A^* denote the result of replacing all occurrences of $P_u^{\mathcal{U}}$ in A by $F(\cdot)$. $\{A_1, \dots, A_m\}^* = \{A_1^*, \dots, A_m^*\}$.

PROPOSITION 5.18. $\Gamma_0 \cup \Gamma \subset \text{Pos}_u$, $\alpha \in T_{u+1}^*(u+1 \leq \nu)$, $k = |F|$,
 $\vdash_1^{\alpha} \Gamma_0, \Gamma \Rightarrow \vdash_1^{(k^{\sim}+1)+\alpha} \Gamma_0, \neg(\forall x \in \mathbb{N} (\mathcal{U}_u^N(F, x) \longrightarrow F(x))), \Gamma^*$.

Proof. Similar to [4, P.151 Proposition]. $\square$

LEMMA 5.19. $\alpha \in T_{u+1}^*$, $\Delta \subset \text{Pos}_u$, $k = |F|$, $\vdash_1^{\alpha} \Delta$, $P_u^{\mathcal{U}} n \Rightarrow$
 $\vdash_1^{(k^{\sim}+1)+\alpha} \Delta, \neg(\forall x \in \mathbb{N} (\mathcal{U}_u^N(F, x) \longrightarrow F(x))), F(n)$.

Proof. From 5.18. $\square$

LEMMA 5.20. $\vdash_1^{(k^{\sim}+1)+\omega_{u+1}} \neg \forall x \in \mathbb{N} (\mathcal{U}_u^N(F, x) \longrightarrow F(x)), \neg P_u^{\mathcal{U}} n, F(n)$
with $k = |F|$.

Proof. Similar to [4, Lemma 4.13] from 5.19. $\square$

THEOREM 5.21. *If the sentence A is provable in $ID_v (v < \omega)$, then there is a $k < \omega$ such that $\vdash_k^z A^N$ where $z = \varphi_{v+1}^k(2, \omega_v)$.*

LEMMA 5.22. (1) $(k^{\sim}+1)+k^{\sim} \dashrightarrow_9 (k+2)^{\sim}$.

(2) $k^{\sim}+4 <_3^* (k+1)^{\sim}$.

Proof. (1) $(k^{\sim}+1)+k^{\sim} = \varphi_{v+1}(1, k^{\sim}) <_0 \varphi_{v+1}(2, k^{\sim}) = (k+1)^{\sim} \dashrightarrow_9 (k+2)^{\sim}$. (2) $k^{\sim}+4 = \varphi_{v+1}(0, k^{\sim})+3 <_3^* \varphi_{v+1}^{k^{\sim}}(0, \varphi_{v+1}(0, k^{\sim})) = \varphi_{v+1}(1, k^{\sim}) <_0^* (k+1)^{\sim}$. $\square$

PROPOSITION 5.23. *For every mathematical axiom $A(v_1, \dots, v_m)$ of ID_v , there is a $k < \omega$ such that $\vdash_1^{k^{\sim}} A(i_1, \dots, i_m)^N$ for all $i_1, \dots, i_m < \omega$. ($v_1, v_2, \dots$ denote variables of the language L .)*

Proof. Similar to [4, p.152 Proposition 1] from the relations $(k^{\sim}+1)+\omega_v <_0 (k+1)^{\sim} = \varphi_{v+1}(1, k^{\sim}) <_0 (k+1)^{\sim} \dashrightarrow_9 (k+2)^{\sim}$, $\omega_{u+1} \leq_0 \omega_v$ and $k^{\sim}+4 <_3 (k+1)^{\sim} \dashrightarrow_{18} (k+3)$. $\square$

PROPOSITION 5.24. *By PL1 we denote Tait's calculus for first-order predicate logic in the language L_{ID} (cf. [14]). If $\Gamma(v_1, \dots, v_m)$ is derivable in PL1, then there is a $k < \omega$ such that $\vdash_0^{k^{\sim}} i_1 \in N, \dots, i_m \in N, \Gamma(i_1, \dots, i_m)$ for all $i_1, \dots, i_m < \omega$.*

Proof. Similar to [4, p.152 Proposition 2]. $\square$

Proof of Theorem 5.21. Suppose $ID_v \vdash A$ (A closed). Then $PL1 \vdash$

$\neg(A_1 \wedge \dots \wedge A_n), A$ where every A_i is the universal closure of an axiom of ID_v . By 5.23 and 5.24, there is an $m < \omega$ such that $\vdash_1^m (A_1 \wedge \dots \wedge A_n)^N$ and $\vdash_0^m \neg(A_1 \wedge \dots \wedge A_n)^N, A^N$. By a cut with cut formula $(A_1 \wedge \dots \wedge A_n)^N$ we obtain now $\vdash_k^{k \sim N} A^N$ with $k = \max\{|(A_1 \wedge \dots \wedge A_n)^N|, m\} + 1$, since $\vdash_k^{k \sim +1} B \Rightarrow \vdash_k^{(k+1) \sim} B$. $\square$

Proof of Theorem 5.1. Suppose $ID_v \vdash A$ (A closed). Then by 5.21, $\vdash_k^\alpha A^N$ where $\alpha = \varphi_{v+1}^k(2, \omega_v)$ for some $0 < k < \omega$. If $k > 1$, then by 5.9(Cutelimination) $\vdash_{k-1}^{\alpha'} A^N$ where

$$\begin{aligned} \alpha' &= \varphi_{v+1}^\alpha(1, \varphi_{v+1}(1, \varphi_{v+1}^k(2, \omega_v))) = \varphi_{v+1}^\alpha(1, \varphi_{v+1}(1, \alpha)) \\ &= \varphi_{v+1}(2, \alpha) = \varphi_{v+1}^{k+1}(2, \omega_v). \end{aligned}$$

By iterating this argument, we obtain $\vdash_1^\beta A^N$ where

$$\beta = \varphi_{v+1}^{k+m}(2, \omega_v) \text{ for some } m < \omega.$$

Then by iterating 5.10(Collapsing) we have $\vdash_1^\gamma A^N$ where

$$\gamma = \varphi_1(\dots \varphi_v(\varphi_{v+1}^{k+m}(2, \omega_v), \omega_{v-1}), \dots, \omega_0).$$

And we have $\gamma < \tau'[v+1]$ since

$$\begin{aligned} \gamma &= \varphi_1(\dots \varphi_v(\varphi_{v+1}^{k+m-1}(2, \varphi_{v+1}(2, \omega_v)), \omega_{v-1}), \dots, \omega_0) \\ &< \varphi_1(\dots \varphi_v(\varphi_{v+1}^{\omega_0}(2, \varphi_{v+1}(2, \omega_v)), \omega_{v-1}), \dots, \omega_0) \\ &< \varphi_1(\dots \varphi_v(\varphi_{v+1}^{\varphi_1(z, \omega_0)}(2, \varphi_{v+1}(2, \omega_v)), \omega_{v-1}), \dots, \omega_0) \\ &\quad \text{where } z = \varphi_2(\dots \varphi_v(\varphi_{v+1}^1(2, \varphi_{v+1}(2, \omega_v)), \omega_{v-1}), \dots, \omega_1) \\ &= \varphi_1(\dots \varphi_v(\varphi_{v+1}^{\omega_1}(2, \varphi_{v+1}(2, \omega_v)), \omega_{v-1}), \dots, \omega_0) \\ &\quad \vdots \\ &\quad \vdots \\ &< \varphi_1(\dots \varphi_v(\varphi_{v+1}^{\omega_v}(2, \varphi_{v+1}(2, \omega_v)), \omega_{v-1}), \dots, \omega_0) \\ &= \varphi_1(\dots \varphi_v(\varphi_{v+1}(3, \omega_v), \omega_{v-1}), \dots, \omega_0) \\ &= \tau'[v+1]. \end{aligned}$$

Hence $\gamma < \tau'[v+1] < \tau'$. Also $\gamma+1 < \tau'[v+1]$ since $\gamma \in T_1^*$ and $\tau'[v+1]$ is (0)-built-up. By 5.13(Bounding) we have $\forall n > 1, \exists k < F_{\gamma+1}(n)(A(n, k))$. $\square$

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