

On manifolds whose geodesic flows are integrable

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The geodesic flow of an n -dimensional riemannian manifold is said to be (completely) integrable if it admits n first integrals (including the hamiltonian) that mutually commute with respect to the Poisson bracket and that are functionally independent almost everywhere. In this talk I will give an exposition of three kinds of riemannian manifolds whose geodesic flows are integrable, which I studied recently. Two of those are called Liouville manifolds and Kähler-Liouville manifolds respectively. The third one is a certain kind of two-dimensional riemannian manifolds, which are diffeomorphic to the sphere, and whose geodesic flows admit first integrals that are fiberwise homogeneous polynomials of degree greater than two.

1. **Liouville manifolds** Liouville manifolds are, roughly speaking, riemannian manifolds whose geodesic equations have similar forms as those of ellipsoids. Since such systems were studied by Liouville, and since in two-dimensional case they have been called “Liouville surfaces”, we call them Liouville manifolds. The precise definition is as follows: Let M be an n -dimensional complete riemannian manifold, and let E be the associated energy function (the hamiltonian of the geodesic flow). Let $\mathcal{F}$ be an n -dimensional vector space of functions on the cotangent bundle T^*M that are fiberwise homogeneous polynomials of degree two. We say that the pair $(M, \mathcal{F})$ is a Liouville manifold if the following four conditions are satisfied:

- (L.1) $E \in \mathcal{F}$;
- (L.2) $\{F, H\} = 0$ for any $F, H \in \mathcal{F}$;
- (L.3) F_p ($F \in \mathcal{F}$) are simultaneously normalizable for any $p \in M$;
- (L.4) $\dim \{F_p \mid F \in \mathcal{F}\} = n$ at some point $p \in M$;

where $F_p = F|_{T_p^*M}$. We assume some non-degeneracy condition, called properness, and obtain the notation of “rank”. It may be said that proper Liouville manifolds of rank one are fairly well understood.

2. Kähler-Liouville manifolds The notion of Kähler-Liouville manifold is a hermitian version (or a complexification) of that of Liouville manifold. The definition is as follows: Let M be a complete Kähler manifold of complex dimension n , I its complex structure, and E its energy function (the hamiltonian of the geodesic flow). Let $\mathcal{F}$ be an n -dimensional vector space of functions on the cotangent bundle T^*M that are fiberwise homogeneous polynomials of degree two. Then we say that $(M, \mathcal{F})$ is a Kähler-Liouville manifold if it satisfies the following conditions:

- (KL.1) $E \in \mathcal{F}$;
- (KL.2) $\{F, H\} = 0$ for any $F, H \in \mathcal{F}$;
- (KL.3) $F_p = F|_{T_p^*M}$ is a hermitian form for every $p \in M$ and $F \in \mathcal{F}$;
- (KL.4) F_p ($F \in \mathcal{F}$) are simultaneously normalizable for every $p \in M$;
- (KL.5) $\{F_p \mid F \in \mathcal{F}\}$ is n -dimensional at some $p \in M$.

Note that only n first integrals are given in the definition. However, it will turn out that other n first integrals appear automatically if some non-degeneracy condition is assumed. They appear as infinitesimal automorphisms of $(M, \mathcal{F})$. If M is compact, then those yield a holomorphic $(C^\infty)^n$ -action on M , and M becomes a toric variety with this action. A typical example is a complex projective space with the standard Kähler metric.

3. Two-dimensional manifolds Let M be a two-dimensional riemannian manifold diffeomorphic to the sphere, and suppose that its geodesic flow has a nontrivial first integral F , which is a homogeneous polynomial of degree k on each cotangent space. Such (M, F) is well-understood when $k = 1$ or 2 : If $k = 1$, M is a surface of revolution and F is a Killing vector field generating the revolution (identified with a function on T^*M); and if $k = 2$, then (M, F) is a Liouville surface. Also, two one-parameter families are known for $k = 3$ and 4 by Bolsinov and Fomenko. Here we introduce a family of such (M, F) for every $k \geq 3$, which we found recently. They form a family parametrized by functions in one variable for each k , and each of them is a $C_{2\pi}$ manifold, namely, all geodesics are closed and have the same length 2π .

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