

Optimal parameters for damped Sine-Gordon equation

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1 Introduction

In this paper, we study an identification problem for physical parameters α, β and δ appearing in the one-dimensional damped sine-Gordon equation

$$\left. \begin{aligned} \frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial y}{\partial t} - \beta \Delta y + \delta \sin y &= g, \quad x \in (0, L), t \in (0, T), \\ y(t, 0) = y(t, L) &= 0, \quad t \in (0, T), \\ y(0, x) = y_0(x) \text{ and } \frac{\partial y}{\partial t}(0, x) &= y_1(x), \quad x \in (0, L). \end{aligned} \right\} \quad (1.1)$$

The identification problem for (1.1) consists in finding the parameters α, β and δ such that the solution of (1.1) exhibits the desired behavior. More precisely, the parameter estimation problem for (1.1) is described as follows. Let $P = \{q = (\alpha, \beta, \delta) \in \mathbb{R}^3 \mid \beta > 0\}$ be equipped with the Euclidean norm. Let $P_{ad} \subset P$ be an admissible set of parameters and define the cost functional $J(q)$ by

$$J(q) = \int_0^T \int_0^L (y(q; t, x) - z_d(t, x))^2 dx dt, \quad q \in P, \quad (1.2)$$

where z_d is a given function on $(0, T) \times (0, L)$. The data z_d can be thought of as the targeted behavior of (1.1). The parameter identification problem for (1.1) with the objective function (1.2) is to find $q^* = (\alpha^*, \beta^*, \delta^*) \in P_{ad}$ satisfying

$$J(q^*) = \inf_{q \in P_{ad}} J(q), \quad P_{ad} \subset P. \quad (1.3)$$

Since q^* is a set of constants, the bang bang control law can be derived from the state system (1.1) and the related adjoint state system. That is, if one chooses P_{ad} to be a closed subset in $\mathbb{R}^3$, then, under certain conditions, q^* is uniquely determined by the extremal values of the parameters in P_{ad} . These results were obtained in [5] and they will be reviewed in Theorem

3.1. It is meaningful to check the conditions on a, b and c which yield the bang bang control law (see Theorem 3.1). Unfortunately, it may be difficult to find q^* numerically by the bang bang control law, since one observes that all the parameter values approach zero.

In this paper we focus on examining the optimal values of a, b and d . The Powell's minimization method is used for the minimization of the cost functional J . The numerical solution of (1.1) is obtained by a Spectral Method [6].

The paper is organized as follows. In Section 2 we review error bounds for the solution of (1.1) and its approximation in a finite dimensional spectral space. In Section 3 we treat the parameter identification problem subject to (1.3) with (1.1). Finally, in Section 4 we present numerical results for the bang bang control law and the parameter estimation problem using the Powell's minimization method.

2 Weak solutions for the damped Sine-Gordon system

Let $I = (0, L)$, $Q = I \times (0, T)$, $H = L^2(I)$, and $H_0^1(I)$ be the Sobolev space on I with the norm $\|v\|_r$. Let the Hilbert space H have the norm $|v|$ and the inner product (u, v) . When $r = 1$, we denote the inner product in $H_0^1(I)$ by $((u, u)) = (\nabla u, \nabla u)$, and its norm by $\|u\|$. Let $\langle u, v \rangle$ denote the duality pairing between $V = H_0^1(I)$ and $V' = H^{-1}(I)$. Then we can define a self-adjoint operator A with the domain $D(A) = H_0^1(I) \cap H^2(I)$ by the relation $\langle Au, v \rangle = ((u, v))$, and $Au = -\Delta u$ for $u \in D(A)$.

As in [1] the variational formulation for the weak solutions of (1.1) is given by

$$\left. \begin{aligned} \langle \frac{\partial^2 y}{\partial t^2}, v \rangle + \alpha \langle \frac{\partial y}{\partial t}, v \rangle + \beta((y, v)) + \delta(f(y), v) &= (g(t), v), \quad v \in V, \quad t \in (0, T), \\ y(0) = y_0 \quad \text{and} \quad \frac{\partial y}{\partial t}(0) &= y_1. \end{aligned} \right\} \quad (2.1)$$

Here we considered a general nonlinear function $f : V \rightarrow H$ instead of $\sin(y)$, having in mind other results involving more general equations, including the ones considered in (1.1). Assume that f is a Lipschitz continuous function with $f(0) = 0$. Problem (2.1) is an initial value problem for a formal abstract second-order differential equation in H :

$$\left. \begin{aligned} y'' + \alpha y' + \beta Ay + \delta f(y) &= g, \quad t \in (0, T), \\ y(0) = y_0, \quad y'(0) &= y_1, \end{aligned} \right\} \quad (2.2)$$

where $' = d/dt$ and $'' = d^2/dt^2$. The weak solutions of (2.1) are the solutions of (2.2) sought in the Hilbert space

$$W(0, T) = \{u \mid u \in L^2(0, T; V), u' \in L^2(0, T; H), u'' \in L^2(0, T; V')\}.$$

The existence, uniqueness and regularity results for the weak solutions of (2.2) are summarized in Theorem 2.1, see [4] for the proofs.

Theorem 2.1 Let $\alpha, \delta \in R$, $\beta > 0$ and let us assume that

$$y_0 \in V, \quad y_1 \in H, \quad \text{and} \quad g \in L^2(0, T; H). \quad (2.3)$$

Then there exists a unique weak solution $y \in L^2(0, T; V)$ of (2.2). This solution satisfies $y \in C([0, T]; V) \cap W(0, T)$, $y' \in C([0, T]; H)$, and

$$\|y(t)\|^2 + |y'(t)|^2 \leq C_1 \left[\|y_0\|^2 + |y_1|^2 + \|g\|_{L^2(0, T; H)}^2 \right], \quad \forall t \in [0, T], \quad (2.4)$$

where C_1 is a constant.

Furthermore, if

$$y_0 \in D(A), \quad y_1 \in V \quad \text{and} \quad g' \in L^2(0, T; H), \quad (2.5)$$

then $y \in C([0, T]; D(A))$ and $y' \in C([0, T]; V)$.

Let N be a positive integer. Now we establish error bounds for finite spectral approximations $y_N(t)$. Let S_N be the subspace of H spanned by the sine functions $\{u_n(x) := \sin(n\pi x/L)\}$, $n = 1, \dots, N$. Let $y_N(t) = y_N(\cdot, t) \in S_N$ be the solution of

$$\left. \begin{aligned} \left(\frac{\partial^2 y_N}{\partial t^2}, v \right) + \alpha \left(\frac{\partial y_N}{\partial t}, v \right) + \beta((y_N, v)) + \delta(f(y_N), v) &= (g(t), v), \\ v \in S_N, \quad t \in (0, T), \\ ((y_N(0) - y(0), v)) &= 0, \quad \left(\frac{\partial y_N}{\partial t}(0) - y_1, v \right) = 0, \quad v \in S_N. \end{aligned} \right\} \quad (2.6)$$

We need the following well-known error estimate [6]: for any $s, r \in R$ with $0 \leq s \leq r$,

$$\|P_N u - u\|_s \leq C_0 (1 + N^2)^{(s-r)/2} \|u\|_r \quad \text{for } u \in H_0^r(I), \quad (2.7)$$

where $P_N : H \rightarrow S_N$ is the projection operator, and C_0 is a constant dependent on L . Using P_N the initial value problem (2.6) can be written in an equivalent form

$$\left. \begin{aligned} y_N'' + \alpha y_N' + \beta A y_N + \delta P_N f(y_N) &= P_N g, \quad t \in (0, T), \\ y_N(0) &= P_N y_0, \quad y_N'(0) = P_N y_1. \end{aligned} \right\} \quad (2.8)$$

The following Theorem for the error estimate is established in [3].

Theorem 2.2 Let $r > 0$. If the solution y of (2.2) satisfy $y \in H_0^r(I)$, then there is a C_1 such that

$$|y(t) - y_N(t)| \leq C_1 (1 + N^2)^{-r/2}, \quad \forall t \in [0, T].$$

If the solution y of (2.2) satisfy $y \in H_0^{r+1}(I)$, then there is a constant $C_2 > 0$ such that

$$\|y(t) - y_N(t)\| \leq C_2 (1 + N^2)^{-r/2}, \quad \forall t \in [0, T].$$

3 Parameter identification problem

In this section we study a parameter identification problem for the one dimensional damped sine-Gordon equation of the form

$$\left. \begin{aligned} y'' + \alpha y' + \beta Ay + \delta \sin y &= g, \quad t \in (0, T), \\ y(0) &= y_0, \quad y'(0) = y_1. \end{aligned} \right\} \quad (3.1)$$

We will always assume that the conditions (2.3) in Theorem 2.1 are satisfied for the initial data y_0, y_1 and the forcing term g . Recall that $P = \{q = (\alpha, \beta, \delta) \in \mathbb{R}^3 \mid \beta > 0\}$ with the Euclidean norm. By Theorem 2.1 we have a well-defined solution map from P into $W(0, T) \subset C([0, T]; H)$, denoted by $y(q)$, which is the solution of (3.1).

With the solution $y(q)$ of (3.1) let us define the cost functional by

$$J(q) = \int_0^T |y(q; t) - z_d(t)|^2 dt, \quad z_d \in L^2(Q), \quad q \in P. \quad (3.2)$$

The parameter identification problem for (3.1) with the objective function (3.2) is to find $q^* = (\alpha^*, \beta^*, \delta^*) \in P_{ad}$, which is an admissible subset of P , satisfying

$$J(q^*) = \inf_{q \in P_{ad}} J(q). \quad (3.3)$$

The parameter q^* is called an optimal parameter. It is well known that the map $q \rightarrow y(q)$ from P into $C([0, T]; H)$ is continuous, see [5]. Hence it is clear that the minimization problem (3.3) has at least one solution, provided P_{ad} is bounded and closed.

The following Theorem and Corollary are proved in [5].

Theorem 3.1 The optimal parameter q^* for (3.3) with (3.1) is characterized by two equations and one constraint

$$\left\{ \begin{aligned} y'' + \alpha^* y' + \beta^* Ay + \gamma^* \sin y &= g \quad \text{in } (0, T), \\ y(0) &= y_0, \quad y'(0) = y_1, \end{aligned} \right. \quad (3.4)$$

$$\left\{ \begin{aligned} w'' - \alpha^* w' + \beta^* Aw + \gamma^* \cos(y)w &= y - z_d \quad \text{in } (0, T), \\ w(T) &= 0, \quad w'(T) = 0, \end{aligned} \right. \quad (3.5)$$

$$\int_0^T ((\alpha^* - \alpha)y' + (\beta^* - \beta)Ay + (\gamma^* - \gamma) \sin y + g, w) dt \geq 0, \quad \forall q \in P_{ad}. \quad (3.6)$$

The constraint (3.6) is known to express the necessary condition for q^* . One can obtain the formula for q^* under the assumptions in Corollary 3.1. This is called the bang bang control law.

Corollary 3.1 Assume that the admissible set is given

$$P_{ad} = [\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \times [\gamma_1, \gamma_2], \quad \beta_1 > 0.$$

Then the optimal parameter $q^* = (\alpha^*, \beta^*, \delta^*)$ subject to (1.2) and (1.1) is determined by the formulas

$$\begin{aligned}\alpha^* &= \frac{1}{2}\{\text{sign}(a) + 1\}\alpha_2 - \frac{1}{2}\{\text{sign}(a) - 1\}\alpha_1, \\ \beta^* &= \frac{1}{2}\{\text{sign}(b) + 1\}\beta_1 - \frac{1}{2}\{\text{sign}(b) - 1\}\beta_1, \\ \gamma^* &= \frac{1}{2}\{\text{sign}(c) + 1\}\gamma_2 - \frac{1}{2}\{\text{sign}(c) - 1\}\gamma_1\end{aligned}$$

provided that

$$\begin{aligned}a &= \int_Q \frac{\partial y}{\partial t}(x, t) w(x, t) \, dx dt \neq 0, \\ b &= \int_Q \nabla y(x, t) \cdot \nabla w(x, t) \, dx dt \neq 0, \\ c &= \int_Q \sin y(t, x)(x, t) w(x, t) \, dx dt \neq 0.\end{aligned}$$

Now for a numerical analysis let us introduce the cost functional corresponding to (3.2). It can be give by the form

$$J_N(q) = \int_0^T |y_N(q; t) - z_d(t)|^2 dt, \quad q \in P, \quad (3.7)$$

where $y_N(q)$ is the weak solution of (2.6) when $f(y) = \sin y$. Similarly to (3.3), the parameter identification problem for (3.7) is to find $q_N^* \in P_{ad}$ such that

$$J_N(q_N^*) = \min_{q \in P_{ad}} J_N(q). \quad (3.8)$$

As in [5], one can easily prove that the cost functional (3.8) is continuous on P_{ad} . Therefore the minimization problem admits a minimum in P_{ad} .

The following Lemma and Theorem are proved in [3].

Lemma 3.1 There exists $C_3 > 0$ independent on N such that

$$|J_N(q) - J(q)| \leq C_3(1 + N^2)^{-r}.$$

Theorem 3.2 Let $\{q_N^*\}$ be a sequence satisfying (3.8) and q^* be its limit point. Then $J(q^*) = \min_{q \in P_{ad}} J(q)$.

4 Numerical results

For our numerical experiments we chose to use a spectral method for the solution of the initial and boundary value problems (3.1) and (3.5), and Powell's minimization method for the minimization of the cost functional. See [6] for a detailed discussion of spectral methods and see [7,2] for the Powell's minimization method.

To accommodate the zero boundary conditions in (3.1) functions $u_n(x) = \sin(\pi nx/L)$, $n = 1, 2, \dots$ are chosen as a (non-normalized) basis in $H = L_2(I)$. Let P_N be the projection operator onto $S_N = \text{span}\{w_n, n = 1, 2, \dots, N\}$ in H , see (2.6)-(2.8) with $f(y) = \sin y$.

Expanding the functions in (2.6) with $f(y) = \sin y$ into the Fourier sine series, and using $v = w_k$, $k = 1, 2, \dots, N$ there we get

$$\left. \begin{aligned} Y_k'' + \alpha Y_k' + \beta_k Y_k + \delta S_k(t) &= F_k(t), \quad t \in (0, T), \\ Y_k(0) &= Y_{k_0}, \quad Y_k'(0) = Y_{k_1}. \end{aligned} \right\} \quad (4.1)$$

where $\beta_k = \beta k^2 \pi^2 / L^2$, $S_k(t)$ is the k -th Fourier sine coefficient of $P_N \sin y_N(t)$, and $Y_k(t)$, $F_k(t)$, Y_{k_0} , and Y_{k_1} are the Fourier coefficients of the solution $y_N(t)$ and the corresponding functions in (2.6). Finally the approximate solutions $y_N(t) \in S_N$ of (3.4) are given. Similarly one can define the approximate solutions $w_N(t) \in S_N$ of (3.5) by the equations

$$\left. \begin{aligned} W_k'' - \alpha W_k' + \beta_k W_k + \delta C_k(t) W_k &= Y_k(t) - Z_k(t), \quad t \in (0, T), \\ W_k(T) &= 0, \quad W_k'(T) = 0, \end{aligned} \right\} \quad (4.2)$$

where $C_k(t)$ is the k -th Fourier sine coefficient of $P_N \cos y_N(t)$.

To test the assumptions on a, b, c in Corollary 3.1 and obtain q^* let $z_d(t) = P_N z_d(t) = \sum Z_k(t) w_k$ and introduce the time-discretized cost functional $J_N(q)$ defined by

$$J_N(q) = \frac{L}{2} \sum_{i=1}^M \sum_{k=1}^N [Y_k(q; t_i) - Z_k(t_i)]^2, \quad q \in P_{ad}, \quad (4.3)$$

where $Y_k(q; t)$ is the solution $Y_k(t)$ of (4.1) for the given values of the parameters $q = (\alpha, \beta, \delta) \in P_{ad}$. Lemma 3.1 and Theorem 3.2 hold for the cost functional (4.3), see [3].

The minimization problem for $J_N(q)$ is solved using a modification of Powell's minimization method. The modified method for solving our problem is described in [3].

To simulate the data let $\hat{q} \in P_{ad}$. Since real data always contain some noise, we set

$$z_d(t, x) = y(\hat{q}; t, x) + \epsilon \eta(x), \quad (4.4)$$

where $\eta(x)$ is a random variable uniformly distributed on interval $[-1, 1]$, and ϵ is a small constant. If $\epsilon = 0$, then $z_d(t) = y(\hat{q}; t)$ for all $t \in [0, T]$. Therefore, in this case one can check the performance of the parameter identification algorithm (i.e. if the algorithm finds the original set of parameters $\hat{q}$) by choosing sufficiently large N and M in (3.7).

We conducted two sets of numerical simulations with $\epsilon = 0$. See [3] for $\epsilon \neq 0$. The problem is to identify three unknown parameters α, β and δ .

In all simulations the initial value problem (4.1) and (4.2) are solved using a Leap-Frog Method with the time step $h = 0.01$ as follows. For example, let $Y_k^j, k = 1, 2, \dots, N$ be defined by

$$\begin{aligned} Y_k^{-1} &= Y_{k_0} - h Y_{k_1}, \\ Y_k^{j+1} &= \frac{2Y_k^j - [\beta_k Y_k^j - F_k(t_j) + \delta S_k(t_j)h^2] + (1 - \alpha h/2)Y_k^{j-1}}{1 + \alpha h/2}, \end{aligned}$$

表 1: Parameter values for numerical simulations

Time and spatial intervals	$[0, T] \times [0, L] = [0, 4] \times [0, \pi]$
Admissible set	$P_{ad} = [0.001, 1] \times [0.1, 1] \times [0.1, 1] \times [0.1, 1]$
Initial conditions	$y_0(x) = 0$ $y_1(x) = \exp[-100(x - \pi/2)^2]$
Forcing function	$f(t, x) = 0.01$
N	16
Observation times	$t_i = (T/M)i, i = 1, 2, \dots, M$

for $j = 0, 1, 2, \dots$. Then Y_k^j is an approximation of $Y_k(t)$ at $t = t_j = hj$.

The number of observations M varied in different simulations, but it is fixed as $M = 400$. The results of various observations are in [3].

Finally, let $q_0 \in P_{ad}$ be an arbitrarily chosen set of parameters, and $q_1, q_2, \dots$ be the sequence of the sets of parameters iteratively obtained in the Powell's minimization method. The stopping criterion for this iterative process is

$$\frac{|J_N(q_m) - J_N(q_{m-1})|}{|J_N(q_0)|} < 10^{-6}. \quad (4.5)$$

Simulation 4.1 In this simulation let us consider $\hat{q} = (0.02, 0.7, 0.5)$ which is an interior point of P_{ad} , and z_d be computed according to (4.4). Let $q_N^* = q_m$ be the set of parameters attained when the Powell's minimization method was terminated according to the stopping criterion (4.5). The minimizers q_N^* together with the number of iterations m are shown in Tables 1 for the noise level $\epsilon = 0$, and the number of observations M .

Table 2		$\epsilon = 0$	
M	m	q_N^*	$J_N(q_N^*)$
400	5	(0.02000, 0.70000, 0.50001)	0.000000
a	b	c	
-0.101522×10^{-8}	0.101384×10^{-6}	-0.295462×10^{-9}	

Tables 2 shows the identification algorithm is successful. The excellent simulation results are given in [3] for a small number of observations. As we have mentioned in the Introduction one can observe that all the parameters a, b and c are almost equal to zero.

Simulation 4.2 In this simulation let us consider $\hat{q} = (0.01, 1, 0.1)$ which is a boundary point in P_{ad} . All the procedures are the same as in Simulation 4.1.

Table 3		$\epsilon = 0$	
M	m	q_N^*	$J_N(q_N^*)$
400	4	(0.010040, 0.999992, 0.100026)	0.000000
a	b	c	
-0.893024×10^{-7}	0.416599×10^{-7}	-0.517080×10^{-7}	

All the parameters a, b and c can be regarded as zeros for the error bound 10^{-6} . Based on the results shown in Tables 2 and 3, one can guess that the assumptions on the parameters a, b, c specified in Corollary 3.1 for finding q^* may be not suitable in these cases.

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