AN ALGEBRAIC PROOF OF DELIGNE'S REGULARITY CRITERION

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ABSTRACT. Deligne's regularity criterion for an integrable connection ∇ on a smooth complex algebraic variety X says that ∇ is regular along the irreducible divisors at infinity in some fixed normal compactification of X if and only if the restriction of ∇ to every smooth curve on X is fuchsian (*i. e.* has only regular singularities at infinity). The "only if" part is the difficult implication. Deligne's proof is transcendental and uses Hironaka's resolution of singularities.

Following [1], we present a purely algebraic proof of this implication which does not use resolution beyond the case of plane curves. It relies upon a study of the formal structure of integrable connections on surfaces with (possibly irregular) singularities along a divisor with normal crossings.

1. INTRODUCTION: FUCHSIAN CONNECTIONS.

1.1. Let X be a connected algebraic complex manifold, and let $\mathcal{E}$ be an algebraic vector bundle on X endowed with an integrable connection ∇.

When X is a curve, the dichotomy between regular and irregular singularities (at infinity) goes back to the 19th century. The connection ∇ is said to be *fuchsian* if all singularities (at infinity) are regular, see Manin's classical paper [8]. This is obviously a birational notion.

In higher dimension, one may consider a normal compactification $\bar{X}$ of X and look at the irreducible components Z_j of $\partial\bar{X} = \bar{X} \setminus X$ of codimension one in $\bar{X}$. The local subring $\mathcal{O}_{X,Z_j}$ of $\mathbf{C}(X)$ is a discrete valuation ring with residue field $\mathcal{O}_{X,Z_j}/\mathfrak{m}_{X,Z_j} = \mathbf{C}(Z_j)$. One is then in the familiar one-dimensional situation, over $\mathbf{C}(Z_j)$ instead of $\mathbf{C}$, and the notion of regularity of ∇ along Z_j is defined in the usual way. Namely, consider a germ of vector field θ_j on X - viewed as a derivation of $\mathbf{C}(X)$ - which preserves $\mathfrak{m}_{X,Z_j}$, but does not send it into $\mathfrak{m}^2_{X,Z_j}$, and which acts trivially on the residue field. Then ∇ is said to be *regular along* Z_j if there is an $\mathcal{O}_{X,Z_j}$-lattice in the generic fiber $\mathcal{E}_{\mathbf{C}(X)}$ which is stable under $\nabla(\theta_j)$ (this condition does not depend on θ_j, cf. [2, I.3.3.4]).

In order to obtain a birational notion of fuchsianity, one is then led to say that ∇ is *fuchsian* if for any $(\bar{X}, Z_j)$ as before, ∇ is regular at Z_j.

At first look, this definition is rather forbidding, since it requires to consider all divisorial valuations of $\mathbf{C}(X)$ at the same time. Fortunately, it turns out that it suffices to consider only one normal compactification $\bar{X}$. Indeed, according to P. Deligne, one has the following characterizations of fuchsianity.

1.1.1**. Theorem.** [5, II.4.4, 4.6] *The following are equivalent:*

$i)$ ∇ *is fuchsian,*

$ii)$ *for some normal compactification* $\bar{X}$, ∇ *is regular along all irreducible components of* $\partial\bar{X}$ *of codimension one in* $\bar{X}$,

$iii)$ *for any smooth curve* C *and any locally closed embedding* $h : C \to X$, $h^*\nabla$ *is fuchsian,*

$iv)$ *for any smooth* Y *and any morphism* $f : Y \to X$, $f^*\nabla$ *is fuchsian,*

$v)$ *for some dominant morphism* $f : Y \to X$ *with* Y *smooth,* $f^*\nabla$ *is fuchsian.*

Note that we *do not* assume that $\partial\bar{X}$ has normal crossings. The difficult implication is $ii) \Rightarrow iii)$. The implication $iii) \Rightarrow i)$ is comparatively easy to establish (*cf. e.g.* [2, I. 3.4.7]), and the other implications follow very easily from these two. The difficulty with $ii) \Rightarrow iii)$ arises when the closure of C in $\bar{X}$ does not meet $\partial\bar{X}$ transversally. A closely related difficulty, with $ii) \Rightarrow i)$, is to show that ∇ remains regular at the exceptional divisor when one blows up a subvariety of $\partial\bar{X}$.

1.2. Deligne's proof (as contained in the erratum to [5]) is not elementary: it relies upon certain transcendental complex-analytic arguments on one hand, and upon Hironaka's resolution of singularities on the other hand.

Recently, inspired by part of Sabbah's work on asymptotic analysis in dimension 2 [9], the author has found a purely algebraic proof of Deligne's regularity criterion, as a consequence of a more general result about the semicontinuity of the Poincaré-Katz rank [1].

The aim of this text is to explain this argument in the simplified form needed for 1.1.1, *i.e.* in the context of algebraic connections which are regular at infinity. We do not use resolution of singularities beyond resolution of plane curves.

We refer to [4] for a nice introduction to the background and the eventful story of this problem. Its difficulty is related to the fact that the standard techniques of logarithmic lattices break down in this algebraic context; along the way, it becomes necessary to go beyond the framework of regular connections.

1.3. More precisely, we present an algebraic proof of the following local refined form of Deligne's regularity criterion[1].

1.3.1. **Theorem.** *Let X be a normal connected algebraic variety over* $\mathbf{C}$. *Let U be a smooth open subset of X, with complement $\partial X = X \setminus U$, and let Q be a closed point of ∂X. Let $Z_1, \ldots, Z_t$ be the irreducible components of ∂X of codimension one in X which pass through Q.*
Let C be a smooth connected curve, $P \in C$ be a closed point, and $h : C \to X$ be a morphism such that $h(C \setminus P) \subset U$ and $h(P) = Q$.
Let $\mathcal{E}$ be a vector bundle on U with an integrable connection ∇. If $(\mathcal{E}, \nabla)$ is regular along each Z_j, then the vector bundle $h^*(\mathcal{E}, \nabla)$ on $C \setminus P$ is regular at P.

1.4. **Warning.** What does regularity actually mean? *Quot capita tot sensus*: in the literature, one can find *analytic* definitions (moderate growth of solutions of meromorphic connections), *algebraic* definitions (such as the items in 1.1.1, reducedness of the characteristic variety, and many others [2]), as well as *mixed* definitions (of GAGA type: formal/algebraic versus analytic properties of connections).

The occurrence of so many different definitions is an obvious sign of the richness of the concept, but at the same time an amazing source of confusion. All of them are supposed to be equivalent in their common domains of application, but this is often a matter of folklore or belief, as a complete and precise dictionary is still lacking (especially in the sensitive case where the polar divisor has non-normal crossings). A strange situation, indeed, for a supposedly well-understood classical notion!

It is thus essential to keep in mind the definition which we have adopted right at the beginning in order to understand what we do in this paper: comparing, in a purely algebraic way, a few of the existing algebraic definitions of *regular* connections. The wit is that the core of the paper deals with *irregular* connections.

2. Reduction to the plane case.

2.1. In this section, we reduce 1.3.1 to the case where X is the projective plane[2].

We may assume $\dim X \geq 2$, otherwise the theorem is essentially trivial. Replacing X by a quasi-projective neighborhood of Q and taking the

[1] this statement appears in [2, I.5.4], but the argument given there works only in case the polar divisor has normal crossings. Indeed, as was pointed out by J. Bernstein, lemma 5.5 of op. cit. on which it relies does not hold in greater generality.

[2] the next two lemmas are joint work wih F. Baldassarri.

normalization of its Zariski closure in some projective embedding, we may assume that X is projective.

2.1.1**. Lemma.** *Let $X \subset \mathbf{P}^N_{\mathbf{C}}$ be a closed normal connected subvariety of dimension $d \geq 2$. Let U, Z_j, Q, $h : C \to X$, P be as in 1.3.1.*

For any sufficiently large integer δ, there exists an irreducible complete intersection $Y \subset \mathbf{P}^N_{\mathbf{C}}$ of dimension $N - d + 2$ and multidegree $(\delta, \ldots, \delta)$ such that:
(i) Y contains $h(C)$ (with reduced induced structure) and cuts U transversally at its generic point $\eta_{h(C)}$;
(ii) $Y \cap X$ is an normal connected surface and Y cuts $U \setminus (h(C) \setminus Q)$ transversally;
(iii) in a neighborhood of Q, Y cuts each $Z_j \setminus Q$ transversally, and does not cut any irreducible component of ∂X of codimension > 1 in X, nor the singular locus of $Z := \cup Z_j$, except in Q.

Proof. This is more or less standard, but, for lack of adequate reference, we give some detail. For short, we change a little notation and now write C for the closure of $h(C)$ in X, with reduced structure (an integral, possibly singular, curve). Let $\pi : \tilde{\mathbf{P}} \longrightarrow \mathbf{P}^N$ be the blow-up centered at C, and let $E \subset \tilde{\mathbf{P}}$ the exceptional divisor.

We set $\mathcal{I}_C = \mathrm{Ker}(\mathcal{O}_{\mathbf{P}^N} \longrightarrow \mathcal{O}_C)$. Then $\mathrm{Im}(\pi^*\mathcal{I}_C \to \mathcal{O}_{\tilde{\mathbf{P}}}) = \mathcal{O}(-E)$ and, for $\delta >> 0$, $\pi^*(\mathcal{O}_{\mathbf{P}^N}(\delta)) \otimes \mathcal{O}(-E)$ is very ample: a basis of sections defines an embedding of $\tilde{\mathbf{P}}$ into $\mathbf{P}^M$. On the other hand, since π is birational and $\mathbf{P}^N$ is normal, one has $\pi_*\mathcal{O}_{\tilde{\mathbf{P}}} = \mathcal{O}_{\mathbf{P}^N}$. It follows that the two natural arrows

$$\mathcal{I}_C \to \pi_*(\mathrm{Im}(\pi^*\mathcal{I}_C \to \mathcal{O}_{\tilde{\mathbf{P}}})) = \pi_*\mathcal{O}(-E) \to \pi_*\mathcal{O}_{\tilde{\mathbf{P}}} = \mathcal{O}_{\mathbf{P}^N}$$

are inclusions of ideals of $\mathcal{O}_{\mathbf{P}^N}$. Let $D \subset C$ be the closed subscheme corresponding to $\pi_*\mathcal{O}(-E)$. If $D \neq C$, D would be punctual and $1 \in \mathcal{O}_{\mathbf{P}^N \setminus D^{red}}$ would correspond to a function on $\tilde{\mathbf{P}} \setminus (\pi^{-1}(D))^{red}$ vanishing on $E \setminus (\pi^{-1}(D))^{red}$, a contradiction. Hence $D = C$ and therefore $\pi_*\mathcal{O}(-E) = \mathcal{I}_C$. From the projection formula, one deduces

$$\pi_*(\pi^*(\mathcal{O}_{\mathbf{P}^N}(\delta)) \otimes \mathcal{O}(-E)) \cong \mathcal{O}_{\mathbf{P}^N}(\delta) \otimes \mathcal{I}_C \cong \mathcal{I}_C(\delta),$$

whence

$$\begin{aligned}\mathrm{Ker}(H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(\delta)) \to H^0(C, \mathcal{O}_C(\delta))) &= H^0(\mathbf{P}^N, \mathcal{I}_C(\delta)) \\ &= H^0(\tilde{\mathbf{P}}, \pi^*(\mathcal{O}_{\mathbf{P}^N}(\delta)) \otimes \mathcal{O}(-E)),\end{aligned}$$

and the linear system of hypersurfaces of degree δ in $\mathbf{P}^N$ containing C gives rise to a locally closed embedding

$$\mathbf{P}^N \setminus C \hookrightarrow \mathbf{P}^M = \mathbf{P}(H^0(\mathbf{P}^N, \mathcal{I}_C(\delta))) \ ,$$

with Zariski closure $\tilde{\mathbf{P}}$. The canonical bijection between hyperplanes $\mathcal{H}$ of $\mathbf{P}^M$ and hypersurfaces H of degree δ in $\mathbf{P}^N$ containing C, is such that the intersection $\mathcal{H} \cap (\mathbf{P}^N \setminus C)$ (in $\mathbf{P}^M$) equals $H \setminus C$. So, the intersection of $X \setminus C$ with a general complete intersection Y of multidegree $(\delta, \ldots, \delta)$ ($1 \leq s \leq d-1$ entries) in $\mathbf{P}^N$ containing η_C, is the intersection of $X \setminus C$ with a general linear subvariety $\mathcal{Y}$ of codimension s in $\mathbf{P}^M$. By [3, Exp. XI, Thm. 2.1. (i)], Y cuts $X \setminus C$ (resp. the smooth part of $Z \setminus (C \cap Z)$) transversally and intersects properly any irreducible component of $\partial X \setminus (C \cap \partial X)$. Since $s < d$, Bertini's theorem shows that the intersection of $\mathcal{Y}$ with the strict transform of X in $\mathbf{P}^M$ is normal and connected. On the other hand, since η_C is a simple point of X, it is well-known that a general complete intersection of s hypersurfaces of degree δ in $\mathbf{P}^N$ containing η_C, intersects X transversally at this point.

Applying these considerations for $s = d-2$, one obtains (i), (ii) and (iii). □

2.1.2. **Corollary.** *There exists a normal quasi-projective irreducible neighborhood X' of Q in $Y \cap X$, containing an open subset of $h(C)$, such that $U' := X' \cap U$ is smooth, the distinct irreducible components of codimension one of $\partial X' = X' \setminus U'$ passing through Q, are precisely $Z_1 \cap X', \ldots, Z_t \cap X'$.* □

If $(\mathcal{E}, \nabla)$ is regular along each $Z_1, \ldots Z_t$, so is its pullback $(\mathcal{E}, \nabla)_{|U'}$ along $Z_1 \cap X', \ldots, Z_t \cap X'$ (*cf.* [2, I.3.4.4]). This reduces theorem 1.3.1 to the case where X is a normal surface, or even a projective normal surface (by the same argument as in the beginning of this section).

2.1.3. **Lemma.** *In the notation of lemma 2.1.1, let us further assume that $d = 2$. There exists a morphism $g : X \longrightarrow \mathbf{P}^2_{\mathbf{C}}$, which is finite in a neighborhood V of Q, such that $g(h(C) \cap V)$ is not contained in the branch locus, and such that, for any irreducible component T of ∂X of dimension 1 with $Q \notin T$, $g(Q) \notin g(T)$.*

Proof. Let $\mathbf{G}(N-3, \mathbf{P}^N)$ be the Grassmannian of linear subvarieties of $\mathbf{P}^N$ of codimension 3, and let G be its dense open subset consisting of linear subvarieties which do not intersect X.

For any $\gamma \in G$, X may be considered as a closed subvariety of the blow-up $\tilde{\mathbf{P}}_\gamma$ of $\mathbf{P}^N$ at γ, and the projection with center γ

$$p_{//\gamma} : \tilde{\mathbf{P}}_\gamma \to \mathbf{P}^2$$

induces a morphism

$$g_\gamma : X \to \mathbf{P}^2.$$

Let $\mathbf{\Lambda} \subset \mathbf{G}(N-2, \mathbf{P}^N) \times \mathbf{G}(N-3, \mathbf{P}^N)$ the incidence subvariety (locus of (λ, α) such that λ contains α), and let p_2, p_3 be the natural projections. Notice that p_3 is a fibration with fiber $\mathbf{P}^2$ and admits a section above G: there is a unique $\lambda_\gamma \in \mathbf{G}(N-2, \mathbf{P}^N)$ of $\mathbf{P}^N$ of codimension 2 passing through Q and containing γ. Then γ varies, the λ_γ form (via p_2) a dense open subset of $\mathbf{G}(N-2, \mathbf{P}^N)$.

On the other hand, λ_γ may be identified with the fiber of $p_{//\gamma}$ above $g_\gamma(Q) \in \mathbf{P}^2$, and $\lambda_\gamma \cap X$ with $g_\gamma^{-1}(g_\gamma(Q))$.

By Bertini, one deduces that there is an open dense subset $G' \subset G$ such that for any $\gamma \in G'$, g_γ is finite in a neighborhood of Q.

Moreover, if there exists $\lambda \in p_3^{-1}(\gamma)$ which intersects X transversally and cuts $h(C) \cap V$ (*resp.* which passes through Q and avoids the T's), then $g_\gamma(h(C) \cap V)$ is not contained in the branch locus of g_γ (*resp.* $g_\gamma(Q) \notin g_\gamma(T)$).

One deduces that there is an open dense subset $G'' \subset G'$ such that for any $\gamma \in G''$, $g = g_\gamma$ satisfies the conditions in the lemma. $\square$

In the situation of theorem 1.3.1 with X a projective surface, g induces an étale morphism

$$X' = V \setminus (g_{|V}^{-1}(B) \cup g_{|V}^{-1}(g(\partial X))) \longrightarrow \mathbf{P}^2_{\mathbf{C}} \setminus (B \cup g(\partial X))$$

which is finite over its image. Up to replacing C by a suitable neighborhood of P, $h(C \setminus P) \subset U \cap X'$.

Moreover, using [2, I.3.2.5] and 2.1.3, one sees that the push-forward $g_*((\mathcal{E}, \nabla)_{|U \cap X'})$ on $g(U \cap X')$ is regular along each 1-dimensional irreducible components of $\mathbf{P}^2_{\mathbf{C}} \setminus g(U \cap X')$ passing through $g(Q)$ (which are either the $g(Z_j)$'s or else are contained in B).

On the other hand, if the connection $(g \circ h)^* g_*((\mathcal{E}, \nabla)_{|U \cap X'})$ on $C \setminus P$ is regular at P, the same is true for its subconnection $h^*((\mathcal{E}, \nabla)_{|U \cap X'})$.

Therefore, in order to prove the theorem, one may substitute to X any Zariski neighborhood of $g(Q)$ in the projective plane.

2.1.4. **Corollary.** *Statement 1.3.1 holds in general if it holds when X is an affine neighborhood of the origin Q in the complex plane and $\partial X = Z := Z_1 \cup \ldots \cup Z_t$.* $\square$

2.2. By the classical embedded resolution of plane curves, there is a sequence of blow-ups

$$\pi : X' = X_N \to \cdots \to X_0 = X$$

such that $\pi^{-1}(h(C) \cup Z)$ (with its reduced structure) has strict normal crossings. We denote by C', Z'_j the strict transforms of $h(C), Z_j$ respectively; they are smooth curves on X', and h lifts to a morphism $C \to C'$.

We set $U' = X' \setminus \pi^{-1}(Z)$. We denote by $E_i, i = 1, \dots, s$, the irreducible components of the exceptional divisor $\pi^{-1}(Q)$, and set $E_i^0 = E_i \cap U'$.

We denote by $(\mathcal{E}', \nabla')$ the inverse image of $(\mathcal{E}, \nabla)$ on U'.

2.2.1. **Proposition.** *$(\mathcal{E}', \nabla')$ is regular along every E_i.*

Via 2.1.4, it is clear that 1.3.1 follows from this assertion, whose proof will occupy the next three sections.

3. Poincaré-Katz rank and the Turrittin-Levelt decomposition.

3.1. Let K be a field of characteristic 0, and let M_K be a differential module over $K((x))$, *i.e.* a vector space over $K((x))$ of finite dimension μ endowed with an action $\nabla(\theta_x)$ of $\theta_x = x\frac{d}{dx}$ satisfying the Leibniz rule.

3.1.1. **Example.** $K = \mathbf{C}(E_i^0) \cong \mathbf{C}(y)$, x is a local coordinate on X' such that $x = 0$ defines E_i^0, y is a local coordinate on E_i, and M_K is the generic fiber of $\mathcal{E}'$ tensored with $\mathbf{C}(E_i^0)((x))$ over the subfield $\mathbf{C}(X')$, endowed with the induced action of $\nabla(\theta_x)$.

According to the Turrittin-Levelt theorem, there is a finite extension K'/K and a positive integer e such that, putting $x' = x^{1/e}$, one has a canonical decomposition of differential modules

$$M_K \otimes K'((x')) = \oplus L_{\varphi_j} \otimes_{K'((x'))} R_j \tag{3.1}$$

where
- R_j is regular, *i.e.* admits a basis in which the matrix of $\nabla(\theta_x)$ has entries in $K'[[x']]$,
- $L_{\varphi_j} = K'((x'))$ with $\nabla(\theta_x)(1) = \varphi_j \in \frac{1}{x'}K'[\frac{1}{x'}]$ (and the φ_j's are distinct).

We denote by μ_j the dimension of the differential module R_j.

Moreover, if one gathers together the summands $L_{\varphi_j} \otimes R_j$ according to the *slope*, *i.e.* the negative of the (fractional) degree of φ_j, the resulting coarser decomposition descends to $K((x))$ and gives the *slope decomposition*

$$M_K = \oplus_{\sigma \in \mathbf{Q}} M_{K,(\sigma)}. \tag{3.2}$$

We denote by $\mu_{(\sigma)}$ the dimension of the differential module $M_{K,(\sigma)}$, so that $\mu = \sum \mu_{(\sigma)}$.

3.2. The highest slope is the *Poincaré-Katz rank of* M, denoted by $\rho(M)$ or ρ for short. If φ_j is of slope ρ, we write it in the form

$$\varphi_j = \varphi_{j,-\rho}.x^{-\rho} + h.o.t., \quad \varphi_{j,-\rho} \in K'.$$

If $\rho > 0$, the coefficients $\varphi_{j,-\rho}$ may be calculated as follows: one takes a cyclic vector m for M_K. Then the matrix of $\nabla(\theta_x)$ in the basis $(m, x^\rho\nabla(\theta_x)(m), \ldots, x^{(\mu-1)\rho}\nabla(\theta_x)^{\mu-1}(m))$ of $M_K \otimes K'((x'))$ is of the form $x^{-\rho}H$ where $H(x) \in M_\mu(K[[x']])$ and the $\varphi_{j,-\rho}$'s are the non-zero eigenvalues of $H(0)$ (*cf.* e.g. [1, §2]).

3.3. Let us now assume that K is the function field of a smooth affine curve $\operatorname{Spec} A$, and that M_K comes from a differential module M over $A((x))$ (as in example 3.1.1, where $A = \mathcal{O}(E_i^0)$ is a localization of $\mathbf{C}[y]$). Let A' be the normalization of A in K'.

It is not true in general that in the above decompositions, one may replace K and K' by A and A' respectively (this is the well-known problem of *turning points*). However, this becomes true if one restricts to a suitable dense open subset of the curve (see [1, §3] for a detailed analysis of this point).

More precisely, there exists $f \in A \setminus \{0\}$ such that $A'[\frac{1}{f}]$ is finite etale over $A[\frac{1}{f}]$, $\varphi_j \in A'[\frac{1}{f}]((x'))$, and the decompositions (3.1) and (3.2) come, respectively, from decompositions

$$M \otimes A'[\frac{1}{f}]((x')) = \oplus L_{\varphi_j} \otimes_{A'[\frac{1}{f}]((x'))} R_j \tag{3.3}$$

$$M \otimes A[\frac{1}{f}]((x)) = \oplus_{\sigma\in\mathbf{Q}} M_{(\sigma)}. \tag{3.4}$$

3.4. In general, $\varphi_j \notin A'((x'))$, but one always has $\varphi_{j,-\rho} \in A'$, due to the above interpretation of these coefficients (*cf.* 3.2).

Denoting by μ_j the rank of R_j[3], a simple Galois argument then shows that $\varphi(x) := \prod(x^\rho - \varphi_{j,-\rho})^{\mu_j}$ lies in $A[x]$. For $\rho > 0$, this allows to define the effective divisor $D = (\varphi(x))$ on $\operatorname{Spec} A[x]$, which is finite of degree $\mu_{(\rho)}.\rho$ over $\operatorname{Spec} A$. If $\rho = 0$, we set $D = 0$.

3.5. When M comes from an *integrable* connection, to the effect that there is an action $\nabla(\theta_y)$ of $\theta_y = y\frac{d}{dy}$ commuting with $\nabla(\theta_x)$, all the above decompositions are automatically stable under $\nabla(\theta_y)$, *i.e.* are decompositions of integrable connections.

3.5.1. **Example.** Let us come back to example 3.1.1, and take $M = \mathcal{E} \otimes \mathcal{O}(E_i^0)((x))$. In the sequel, we shall have to deal with all the E_i's simultaneously, so we emphasize the index i and write ρ_i for the

[3] not to be confused with the dimensions $\mu_{(\sigma)}$ introduced in 3.1. Notice that $\mu_{(\rho)} = \sum \mu_j$ where the sums runs over all j's such that L_{φ_j} is of slope ρ, the Poincaré-Katz rank of M.

Poincaré-Katz rank of $\mathcal{E}$ along E_i, and $\mu_{(\rho_i)}$, $\varphi_{i,j}$, $\mu_{i,j}$, $\varphi_{i,j,-\rho_i}$, D_i, instead of $\mu_{(\rho)}$, φ_j, μ_j, $\varphi_{j,-\rho}$, D.

Geometrically, for $\rho_i > 0$, the divisor

$$D_i = (\prod (x^{\rho_i} - \varphi_{i,j,-\rho_i})^{\mu_{i,j}}) \tag{3.5}$$

has an intrinsic meaning as a *divisor on the normal bundle* $N_{E_i^0}X' \cong \operatorname{Spec} A[x]$ (the point is that the term of lower degree of $\varphi_{i,j}$ has an intrinsic meaning independent of the choice of the transversal derivation at E_j^0 by means of which one identifies $N_{E_i^0}X'$ and $\operatorname{Spec} A[x]$). It is *finite of degree* $\mu_{(\rho_i)}.\rho_i$ *over* E_i^0.

4. Formal decompositions at crossings.

4.1. Let us now consider a crossing point Q' of E_i with another component of $\pi^{-1}(Z)$ (which is either another component E_j or one of the Z_j's). Let x, y be etale coordinates at Q' such that E_i is defined by $x = 0$ and the other component by $y = 0$.

The formalization of $(\mathcal{E}', \nabla')$ at Q' then gives rise to a differential module $\tilde{M}$ over $\mathbf{C}[[x, y]][\frac{1}{xy}]$ (with mutually commuting actions of θ_x and θ_y).

We shall say that $(\mathcal{E}', \nabla')$ has *nice formal structure at* Q'[4] if, after ramification along $xy = 0$, there is a decomposition

$$\tilde{M} = \oplus \tilde{L}_{\tilde{\varphi}_k, \tilde{\psi}_k} \otimes \tilde{R}_k \tag{4.1}$$

where

- $\tilde{R}_j$ is regular, *i.e.* admits a basis in which the matrix of $\tilde{\nabla}(\theta_x)$ and $\tilde{\nabla}(\theta_y)$ have entries in $\mathbf{C}[[x, y]]$,
- $\tilde{L}_{\tilde{\varphi}_k, \tilde{\psi}_k} = \mathbf{C}[[x, y]][\frac{1}{xy}]$ with

$$\tilde{\nabla}(\theta_x)(1) = \tilde{\varphi}_k,\ \tilde{\nabla}(\theta_y)(1) = \tilde{\psi}_k \ \in \ \mathbf{C}[[x, y]][\frac{1}{xy}],\ \ \theta_x(\tilde{\psi}_k) = \theta_y(\tilde{\varphi}_k).$$

Without loss of generality, one may then assume that the pairs $(\tilde{\varphi}_k, \tilde{\psi}_k)$ are distinct modulo $\mathbf{C}[[x, y]]$ (which ensures unicity of the decomposition).

4.2. For $K = \mathbf{C}((y))$, let us compare a decomposition (4.1) with the Turritin-Levelt decomposition of the differential module

$$M_K := \tilde{M} \otimes_{\mathbf{C}[[x,y]][\frac{1}{xy}]} K((x)).$$

One has a canonical decomposition

$$\frac{\mathbf{C}[[x, y]][\frac{1}{xy}]}{\mathbf{C}[[x, y]]} \cong \frac{1}{x}\mathbf{C}[[y]][\frac{1}{x}] \oplus \frac{1}{xy}\mathbf{C}[\frac{1}{x}, \frac{1}{y}] \oplus \frac{1}{y}\mathbf{C}[[x]][\frac{1}{y}]$$

[4]this is equivalent to saying that Q' is semi-stable for $(\mathcal{E}', \nabla')$ in the sense of [1], but weaker than saying that $(\mathcal{E}', \nabla')$ has a good formal structure in the sense of [9].

and the projection onto the first two terms can be written

$$\varpi : \frac{\mathbf{C}[[x,y]][\frac{1}{xy}]}{\mathbf{C}[[x,y]]} \to \frac{1}{x}K[\frac{1}{x}].$$

The decomposition (4.1) of $\tilde{M}$ thus gives rise, by tensorisation with $K((x))$, to a decomposition of M_K which refines (3.1) (taking into account the unicity of the latter):

$$L_{\varphi_j} \otimes R_j = \bigoplus_{k,\, \varpi(\tilde{\varphi}_k)=\varphi_j} (\tilde{L}_{\tilde{\varphi}_k,\tilde{\psi}_k} \otimes \tilde{R}_k) \otimes K((x)). \tag{4.2}$$

4.3. Whereas it is not always the case that $(\mathcal{E}', \nabla')$ has nice formal structure at Q', this can be fixed by blowing up:

4.3.1. **Proposition.** *After blowing up finitely many times some crossing points, $(\mathcal{E}', \nabla')$ acquires a nice formal structure at every crossing point of the inverse image of Z.*

This was first proved by C. Sabbah [9, III, 4.3.1], using a generalization in dimension 2 of the nilpotent orbit method of Babbitt-Varadarajan. In [1, 5.4.1], we have given a simpler and more straightforward proof (which is nevertheless too long to be repeated here).

5. Proof of 2.2.1.

We assume from now on, as we may by 4.3.1, that $(\mathcal{E}', \nabla')$ *has a nice formal structure at all crossing points of $\pi^{-1}(Z)$ lying on the exceptional divisor.*

5.1. Let $P(N_{E_i}X')$ be the projectivization of the normal bundle $N_{E_i}X'$, and let (∞) be the section at infinity over E_i. Taking Zariski closure, the effective divisor D_i on $N_{E_i^0}X'$ gives rise to an effective divisor $\bar{D}_i$ on $P(N_{E_i}X')$, which is finite over E_i (of degree $\mu_{(\rho_i)}.\rho_i$). It is clear from equation (3.5) that $\bar{D}_i$ does not meet (∞) above E_i^0.

5.1.1. **Lemma.** *$i)$ Above $E_i \cap Z'_j$, $\bar{D}_i$ does not meet (∞).*
$ii)$ Above $Q' = E_i \cap E_{i'}$, the intersection multiplicity of $\bar{D}_i$ and (∞) is $\leq \mu_{(\rho_i)}.\rho_{i'}$.

Proof. Let us use coordinates x, y as in 4.1. Replacing them by $x^{1/e}, y^{1/e}$ if necessary, one may assume that one has a decomposition (4.1). One reads on equation (3.5) that the intersection mutiplicity of $\bar{D}_i$ and (∞) above Q' is

$$\max(0, -\sum_{j,\, \mathrm{ord}_x \varphi_j = -\rho_i} \mu_{i,j}.\mathrm{ord}_y\, \varphi_j).$$

It thus suffices to show that $-\mathrm{ord}_y\, \varphi_j \le \rho_{i'}$. Coming back to 4.1, it is clear that

$$-\mathrm{ord}_x \tilde{\varphi}_k \le \rho_i, \; -\mathrm{ord}_y \tilde{\psi}_k \le \rho_{i'},$$

and by the integrability condition $\theta_x(\tilde{\psi}_k) = \theta_y(\tilde{\varphi}_k)$, one gets

$$-\mathrm{ord}_y \tilde{\varphi}_k \le \rho_{i'}, \; -\mathrm{ord}_x \tilde{\psi}_k \le \rho_i,$$

whence (via ϖ) $-\mathrm{ord}_y\, \varphi_j \le \rho_{i'}$ (which is 0 in case i) of the lemma, since $(\mathcal{E}, \nabla)$ is assumed to be regular along $Z_{i'}$). □

5.1.2. **Lemma.** *For every i,*

$$(E_i, E_i).\rho_i \;\ge\; - \sum_{j,\, E_j \cap E_i \neq \emptyset} \rho_j \tag{5.1}$$

Proof. This follows from the general formula

$$\delta \; \deg(N_{E_i} X') \;=\; \bar{C}.(0) - \bar{C}.(\infty)$$

which holds for any curve (or 1-cycle) $\bar{C} \subset P(N_{E_i} X')$ whose projection to E_i is finite of degree δ. One applies this to $\bar{C} = \bar{D}_i$, taking into account the previous lemma, noting that $\delta = \mu_{(\rho_i)}.\rho_i$ in that case, and that $deg(N_{E_i} X')$ is the intersection number $(E_i, E_i) \le 0$. □

5.2. To finish, let us indicate how 5.1.2 implies 2.2.1. Let A be the matrix with entries $A_{ij} = (E_i, E_j)$. We may rewrite inequality (5.1) in the form

$$\sum_j A_{ij}.\rho_j \;\ge 0. \tag{5.2}$$

Now, it is well-known that A is a negative definitive symmetric matrix. Since $\rho_j \ge 0$, this together with (5.2) imply that $\rho_j = 0$, *i.e.* $(\mathcal{E}', \nabla')$ is regular along E_j. □

6. EXPONENTS.

6.1. Let us come back to the situation of 1.3.1, assuming X smooth. There is a maximal open subset $V \subset X$ containing U such that the complement of U in V is a smooth divisor (whose components are open subsets of the Z_j's).

There is a purely algebraic construction of a logarithmic lattice, *i.e.* a locally free extension $\mathcal{E}_{\log}$ of $\mathcal{E}$ from U to V endowed with a logarithmic connection $\nabla_{\log}$ extending ∇ with poles at $Z \cap V$, *cf.* [2, I].

6.1.1. **Remark.** Let $j : V \hookrightarrow X$ be the open embedding. Since the complement of V in X is of codimension ≥ 2, $j_*\mathcal{E}_{\log}$ is a coherent reflexive module on X, hence locally free if X is a surface. The difficulty of 1.3.1 is related to the fact that around P, $h^*\nabla_{\log}$ is not a logarithmic connection on $h^*(j_*\mathcal{E}_{\log})$ in general (there is a counterexample by J. Bernstein in the case where Z is the union of 3 lines in the plane meeting at the origin Q, *cf.* [4]).

The *exponents* of ∇ at Z_j are the eigenvalues of the residues of $\nabla_{\log}$ at $Z_j \cap V$ modulo $\mathbf{Z}$ (the images of these eigenvalues in $\mathbf{C}/\mathbf{Z}$ do not depend on the logarithmic lattice). The following theorem is due to M. Kashiwara.

6.1.2. **Theorem.** [6][7] *The exponents modulo* $\mathbf{Q}$ *of* $h^*\nabla$ *at* P *belong to the* $\mathbf{Q}$*-subspace of* $\mathbf{C}/\mathbf{Q}$ *generated by the exponents of* ∇ *at the* Z_j*'s.* [5]

This is easily seen to be true if the polar divisor has normal crossings, but this is more difficult beyond this case. One can reduce via 2.1.1 to the case where X is a surface, or even an affine neighborhood of the origin in the complex plane. Actually, [6] deals only with the case of rational exponents, but a straightforward modification of Gabber's proof in [7] gives 6.1.2 as stated.

Here is a brief sketch of Gabber's argument, coming back to the notation of 2.2. One fixes a $\mathbf{Q}$-linear map $\kappa : \mathbf{C}/\mathbf{Q} \to \mathbf{Q}$ which sends the exponents of ∇ at the Z_j's to 0. One has to show that κ sends the exponents of ∇' at the E_k's to 0.

Let U_k be a tubular neighborhood of E_k in $X'(\mathbf{C})$ and set $U_k^0 = U_i \cap U'(\mathbf{C})$. Then $\pi_1(U_k^0)$ is generated by elements α_i (turning around E_i, $i = 1, \ldots, s$) and elements β_j (turning around Z'_j, $j = 1, \ldots, t$), such that α_k is central and

$$\gamma_k := \alpha_1^{(E_1.E_k)} \ldots \alpha_i^{(E_s.E_k)} \beta_1^{(Z'_1.E_k)} \ldots \beta_i^{(Z'_t.E_k)}$$

is a commutator. The monodromy representation of $\pi_1(U_k^0)$ attached to $(\mathcal{E}', \nabla')$ admits subrepresentations V_k and v_k where the function

$$\chi := \kappa \circ (\frac{1}{2\pi i} \log)$$

applied to the eigenvalues of α_k takes its maximal value M_k (*resp.* minimal value m_k). One has to show that $M_k = m_k = 0$. One has

$$\chi(det(\gamma_{k|V_k})) = \chi(det(\gamma_{k|v_k})) = 0$$

(since γ_k is a commutator) and

$$\chi(det(\beta_{j|V_k})) = \chi(det(\beta_{j|v_k})) = 0$$

[5] again, this statement appears in [2, I.6.5], but the argument given there works only in case the polar divisor has normal crossings, for the same reason as in footnote 1.

(by definition of χ). Using the definition of (M_k, m_k), this gives

$$\sum_j A_{ij} M_j \geq 0, \quad \sum_j A_{ij} m_j \leq 0.$$

Since the intersection matrix A is negative definite, one concludes that $M_j = m_j = 0$ for every j.

6.1.3. **Remark.** It would be interesting to give an algebraic version of this argument in the spirit of the previous section, using residues instead of monodromy.

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