

コホモロジー複素射影空間上の余次元 1 の 軌道を持ったコンパクト変換群について

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序. $2n$ 次元の連結可微分多様体 M は、有理数係数のコホモロジー環が $P_n(\mathbb{C})$ のコホモロジー環と同型であるとき、 n 次元有理コホモロジー複素射影空間であるという。この報告において、単連結な n 次元有理コホモロジー複素射影空間 M 上の、コンパクト連結リー群 G の可微分作用が $2n-1$ 次元の軌道 G/K を持つ場合について考察する。可微分スライス定理を使って、 G/K が主軌道であり、 M はちょうど二つの特異軌道 $G/K_1, G/K_2$ を持つことが分かり、さらに、

$$p_s: K_s \longrightarrow O(k_s), \quad s=1,2$$

をスライス表現とすると、 M は G 多様体として

$$(*) \quad G \times_{K_1} D_{k_1}^{\perp} \cup_R G \times_{K_2} D_{k_2}^{\perp}$$

と、同変微分同相になることが分かる。今、 $u \in H^2(M; \mathbb{Q})$

を生成元とし, $f_s : G/K_s \rightarrow M$ を包含写像とする. 整数 n_s ($s=1,2$) を

$$f_s^*(u^{n_s}) \neq 0, f_s^*(u^{n_s+1}) = 0$$

によって定めると,

$$n = n_1 + n_2 + 1,$$

$$2 \leq k_s \leq 2(n - n_s), (s=1,2)$$

の成り立つことが分かる. 更に, ポアンカレ双対定理, トム同型定理, その他のコホモロジー論的手段を駆使して, 次の定理を得る.

定理 0.

(A) $G/K_1, G/K_2$ が共に向きづけ可能であるとき.

(i) $k_1 \equiv k_2 \pmod{2}$ ならば, G/K_s は n_s 次元有理コホモロジー複素射影空間になり, $k_s = 2(n - n_s)$ が成り立つ.

(ii) $k_1 \equiv 0 \pmod{2}, k_2 \equiv 1 \pmod{2}$ ならば, $k_1 + k_2 = n + 2$ が成り立ち, 次の二つの場合が起る:

(a) $n_1 = n_2$ 且つ

$$P(G/K_1) = (1+t^{k_1-1})(1+t^2+\dots+t^{2n_1}),$$

$$P(G/K_2) = (1+t^{k_1-1})(1+t^2+\dots+t^{2n_2}).$$

(b) $k_1 = 2(n_2+1), k_2 = n_1 - n_2 + 1$ 且つ

$$P(G/K_1) = (1+t^{n_1-n_2})(1+t^2+\dots+t^{n_1+n_2}),$$

$$P(G/K_2) = (1+t^n)(1+t^2+\dots+t^{2n_2}).$$

- (B) G/K_1 が向きづけ可能で, G/K_2 が向きづけ不能のとき,
 $n \equiv 0 \pmod{2}$ となり, G/K_1 は $n-1$ 次元有理コホモロジー
 複素射影空間で, G/K_2^0 は n 次元有理コホモロジー球面である.
 (C) $G/K_1, G/K_2$ が共に向きづけ不能であれば, $n=3$ とな
 り, $s=1, 2$ に対して

$$P(G/K_s) = 1+t^2, \quad P(G/K_s^0) = (1+t^2)^2$$

が成り立つ。

§1 において, 若干の一般論を展開し, §2 において,
 定理0の詳細な証明を行う。 §3 において, 定理0の夫々
 の場合に対応する例を挙げる。 分類定理を完成するには,
 定理0の条件を満たす等質空間 G/K_s を数え挙げ, 夫々につ
 いて, ある条件を満たす表現 $\rho_s: K_s \rightarrow O(n_s)$ をすべて求
 め, (*) によって G 多様体 M' を構成して, M' が n 次元有
 理コホモロジー複素射影空間となるものをすべて数え挙げた
 いのであるが, これにはまだ日時を要する。 今のところ,
 定理0の (C) の場合が起り得ないことが分かっている。 更
 に, (A)-(i) の場合, および (B) の場合が殆んど分類できて,
 §3 に挙げる例に尽きそうである。 (A)-(ii) の場合が, 最
 も難かしく, 目下研究中である。

§ 1. Transformation group with codimension one orbit

1.1. Let us first recall some basic facts about differentiable transformation groups.

(1.1.1) Let G be a compact Lie group acting differentiably on a manifold M . Then by averaging an arbitrary given Riemannian metric on M , we may have a G -invariant Riemannian metric on M .

(1.1.2) Let $x \in M$. Then the isotropy subgroup G_x acts on a normal vector space N_x of the orbit $G(x)$ at x ; orthogonally we call it the slice representation of G_x at x and denote by $\rho_x : G_x \longrightarrow O(N_x)$, where $O(N_x)$ is the group of orthogonal transformations on N_x .

(1.1.3) (Differentiable slice theorem) Let $E(\nu)$ be the normal bundle of the orbit $G(x) = G/G_x$. Then

$$E(\nu) = G \times_{G_x} N_x$$

where G_x acts on N_x via ρ_x . We note that G acts naturally on $E(\nu)$ as bundle mappings and we may choose small positive real number ε such that the exponential mapping gives an equivariant diffeomorphism of the ε -disk bundle of $E(\nu)$ onto an invariant tubular neighborhood of $G(x)$. (cf. [3], Lemma 3.1)

(1.1.4) Let $H \subset G$ be a closed subgroup. Denote by (H) , the set of all subgroups of G which is conjugate to H in G . We introduce the following partial ordering relation " $<$ " by defining $(H_1) < (H_2)$ if and only if there exists $H_1 \in (H_1)$ and $H_2 \in (H_2)$ such that $H_1 \subset H_2$. If M is connected, then there

exists an absolute minimal (H) among the conjugate classes $\{ (G_x) \mid x \in M \}$, moreover the set

$$M_{(H)} = \left\{ x \in M \mid G_x \in (H) \right\}$$

is a dense open submanifold. The conjugate class (H) is called the type of principal isotropy subgroups, and the orbit G/H is called principal (cf. [3], (2.2) and (2.4)). An orbit $G(x)$ is called singular if $\dim G(x) < \dim G/H$.

Combining (1.1.3) and (1.1.4), we have the following result.

Lemma 1.1.5. If M is connected, then the slice representation of G_x at $x \in M$ is trivial if and only if G_x is a principal isotropy subgroup.

Corollary 1.1.6. If M and G are connected and $G(x)$ is an orbit of codimension one, then $G(x)$ is a principal orbit only when the normal line bundle of $G(x)$ in M is orientable.

1.2. Now we prove the following result.

Lemma 1.2.1. Let G be a compact connected Lie group. Let M be a compact connected manifold without boundary and assume

$$H^1(M; \mathbb{Z}_2) = 0.$$

Suppose that G acts differentiably on M with an orbit G/K of codimension one. Then G/K is a principal orbit, and M has just two singular orbits G/K_1 and G/K_2 (equivariantly diffeomorphic or not). Moreover there is a closed invariant tubular neighborhood X_s of G/K_s ($s = 1, 2$) such that

$$M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

Proof. Let N be a closed invariant tubular neighborhood of G/K in M . Consider the following commutative diagram :

$$\begin{array}{ccccc} H^0(G/K; \mathbb{Z}_2) & \xrightarrow{\phi} & H^1(N, \partial N; \mathbb{Z}_2) & \xleftarrow{\cong} & H^1(M, M - \text{int} N; \mathbb{Z}_2) \\ \downarrow \cdot w_1 & & \downarrow & & \downarrow \\ H^1(G/K; \mathbb{Z}_2) & \xleftarrow{s^*} & H^1(N; \mathbb{Z}_2) & \xleftarrow{i^*} & H^1(M; \mathbb{Z}_2). \end{array}$$

Here ϕ is a Thom isomorphism and w_1 is a first Stiefel-Whitney class of the normal line bundle of G/K in M . Then $H^1(M; \mathbb{Z}_2) = 0$ implies $w_1 = 0$, and hence G/K is a principal orbit by Corollary 1.1.2. Next, if M has no singular orbit, then M has just one isotropy type (K) , and hence there is a differentiable fibration

$$G/K \longrightarrow M \xrightarrow{p} M^*$$

where M^* is the orbit space which is a circle. Then the homomorphism

$$p_* : H_1(M; \mathbb{Z}) \longrightarrow H_1(M^*; \mathbb{Z}) \cong \mathbb{Z}$$

is surjective, because G/K is connected. This fact contradicts $H^1(M; \mathbb{Z}_2) = 0$. Therefore M has at least one singular orbit. Then we can easily see that M is a special G -manifold (in the sense of Hirzebruch-Mayer) with the orbit space $M^* = [1, 2]$ by the differentiable slice theorem (1.1.3). Let $p : M \longrightarrow M^*$ be

a natural projection. Then $p^{-1}(s)$ is a singular orbit for $s = 1, 2$ and $M_{(K)} = p^{-1}((1, 2))$. Moreover, let

$$X_1 = p^{-1}([1, 3/2]) \quad \text{and} \quad X_2 = p^{-1}([3/2, 2]).$$

Then X_s is a closed invariant tubular neighborhood of $G/K_s = p^{-1}(s)$ for $s = 1, 2$. q.e.d.

§ 2. Cohomological aspect

2.1. Let M be a $2n$ -dimensional compact connected orientable manifold without boundary and assume

$$H^*(M; \mathbb{Q}) = \mathbb{Q}[u]/(u^{n+1}), \quad \deg u = 2.$$

We call such a manifold M rational cohomology complex projective n -space. Let X_1, X_2 be $2n$ -dimensional compact connected submanifolds of M such that

$$M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

Let $f_s^* : H^*(M; \mathbb{Q}) \longrightarrow H^*(X_s; \mathbb{Q})$ be the homomorphism induced by the inclusion map $f_s : X_s \longrightarrow M$ for $s = 1, 2$. Then we have the following result.

Lemma 2.1.1. Let n_1, n_2 be non-negative integers such that

$$f_s^*(u^{n_s}) \neq 0 \quad \text{but} \quad f_s^*(u^{n_s+1}) = 0$$

for $s = 1, 2$. Then we have $n = n_1 + n_2 + 1$.

Proof. By the following exact sequence :

$$\begin{array}{c}
H^{k-1}(X_s; \mathbb{Q}) \xrightarrow{\delta_s} H^k(M, X_s; \mathbb{Q}) \longrightarrow H^k(M; \mathbb{Q}) \xrightarrow{f_s^*} H^k(X_s; \mathbb{Q}) \\
\parallel \\
H^k(X_{3-s}, \partial X_{3-s}; \mathbb{Q})
\end{array}$$

we have the following equations of Poincaré polynomials :

$$P(X_{3-s}, \partial X_{3-s}; t) = P(\ker f_s^*; t) + P(\operatorname{im} \delta_s; t),$$

$$P(X_s; t) = P(\operatorname{im} f_s^*; t) + t^{-1} P(\operatorname{im} \delta_s; t).$$

Thus we have

$$(2.1.2) \quad P(X_{3-s}, \partial X_{3-s}; t) - t P(X_s; t) = P(\ker f_s^*; t) - t P(\operatorname{im} f_s^*; t)$$

for $s = 1, 2$. By the Poincaré duality for X_s , we have

$$P(X_s, \partial X_s; t) = t^{2n} P(X_s; t^{-1}),$$

$$P(X_s; t) = t^{2n} P(X_s, \partial X_s; t^{-1}).$$

Then we have from (2.1.2)

$$P(\ker f_1^*; t) - t P(\operatorname{im} f_1^*; t) = t^{2n} (P(\operatorname{im} f_2^*; t^{-1}) - t P(\ker f_2^*; t^{-1})).$$

By the assumption on the integers n_1, n_2 we have

$$P(\operatorname{im} f_s^*; t) = 1 + t^2 + \dots + t^{2n_s},$$

$$P(\ker f_s^*; t) = t^{2n_s+2} + \dots + t^{2n}.$$

Therefore we have

$$\begin{aligned}
& t^{2n_1+2} + \dots + t^{2n} - t(1 + t^2 + \dots + t^{2n_1}) \\
&= t^{2n}(1 + t^{-2} + \dots + t^{-2n_2} - t(t^{-2n_2-2} + \dots + t^{-2n})).
\end{aligned}$$

Put $t = 1$. Then we have $n = n_1 + n_2 + 1$. q.e.d.

Remark. Let $V = \bigoplus_{n \geq 0} V_n$ be a finitely generated graded module over the rational numbers $\mathbb{Q}$ and $b_n = \dim V_n$. Then the polynomial

$$P(V; t) = b_0 + b_1 t + b_2 t^2 + \dots$$

is called the Poincaré polynomial of V . If $V = H^*(X; \mathbb{Q})$ for a topological space X , then simply denote

$$P(X; t) = P(V; t).$$

2.2. From now on, we assume that M is a simply connected rational cohomology complex projective n -space and G is a compact connected Lie group which acts differentiably on M with a codimension one orbit G/K . Then by Lemma 1.2.1, there are just two singular orbits G/K_1 and G/K_2 (we can assume $K \subset K_s$ for $s = 1, 2$), moreover there is a closed invariant tubular neighborhood X_s of G/K_s ($s = 1, 2$) in M , such that

$$M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

Let n_1, n_2 be non-negative integers defined in Lemma 2.1.1, and let

$$k_s = 2n - \dim G/K_s$$

for $s = 1, 2$. Then it is clear that

$$(2.2:1) \quad 2 \leq k_s \leq 2(n - n_s), \quad (s = 1, 2).$$

Because $\mathcal{O}X_s = G/K$ as G -manifolds, the fibre bundle

$$K_s/K \longrightarrow G/K \xrightarrow{p_s} G/K_s$$

is a (k_s-1) -sphere bundle.

Lemma 2.2.2. If $k_2 > 2$, then G/K_1 is simply connected and hence K_1 is connected.

Proof. If $k_2 > 2$, then $\pi_1(G/K_1) = \pi_1(M)$ by the general position theorem. Thus G/K_1 is simply connected by the assumption that M is simply connected. Let K_1^0 be the identity component of K_1 . Then G/K_1^0 is a connected finite covering space over a simply connected space G/K_1 . Thus $K_1^0 = K_1$. q.e.d.

2.3. First we assume that G/K_1 and G/K_2 are orientable, and we have the following result.

Proposition 2.3.1. Assume that G/K_1 and G/K_2 are orientable.

- (i) If $k_1 \equiv k_2 \equiv 0 \pmod{2}$, then G/K_s is a rational cohomology complex projective n_s -space and $k_s = 2(n-n_s)$ for $s = 1, 2$.
- (ii) The case $k_1 \equiv k_2 \equiv 1 \pmod{2}$ does not occur.
- (iii) If $k_1 \equiv 0 \pmod{2}$ and $k_2 \equiv 1 \pmod{2}$, then $k_1 + k_2 = n + 2$ and there are two cases :

(a) $n_1 = n_2$ and

$$P(G/K_1; t) = (1 + t^{k_2-1}) (1 + t^2 + \dots + t^{2n_1}),$$

$$P(G/K_2; t) = (1 + t^{k_1-1}) (1 + t^2 + \dots + t^{2n_2}).$$

(b) $k_1 = 2(n_2 + 1)$, $k_2 = n_1 - n_2 + 1$ and

$$P(G/K_1; t) = (1 + t^{n_1 - n_2}) (1 + t^2 + \dots + t^{n_1 + n_2}),$$

$$P(G/K_2; t) = (1 + t^n) (1 + t^2 + \dots + t^{2n_2}).$$

Proof. We have

$$P(X_s, \partial X_s; t) = t^{k_s} P(G/K_s; t)$$

by Thom isomorphism and

$$P(X_s; t) = P(G/K_s; t).$$

Thus we have from (2.1.2),

$$(2.3.2)_1 \quad P(G/K_1; t) = t^{k_2 - 1} P(G/K_2; t) + (1 + t^2 + \dots + t^{2n_1}) \\ - t^{-1} (t^{2n_1 + 2} + \dots + t^{2n}),$$

$$(2.3.2)_2 \quad P(G/K_2; t) = t^{k_1 - 1} P(G/K_1; t) + (1 + t^2 + \dots + t^{2n_2}) \\ - t^{-1} (t^{2n_2 + 2} + \dots + t^{2n}).$$

Because $n = n_1 + n_2 + 1$, we have from (2.3.2),

$$(2.3.3)_1 \quad (1 - t^{k_1 + k_2 - 2}) P(G/K_1; t) = (1 - t^{k_2 + 2n_2}) (1 + t^2 + \dots + t^{2n_1}) \\ + (t^{k_2 - 1} - t^{2n_1 + 1}) (1 + t^2 + \dots + t^{2n_2}),$$

$$(2.3.3)_2 \quad (1 - t^{k_1 + k_2 - 2}) P(G/K_2; t) = (1 - t^{k_1 + 2n_1}) (1 + t^2 + \dots + t^{2n_2}) \\ + (t^{k_1 - 1} - t^{2n_2 + 1}) (1 + t^2 + \dots + t^{2n_1}).$$

Put $t = -1$ in (2.3.3). We have

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$$\begin{aligned}
 (2.3.4) \quad & (1 - (-1)^{k_1+k_2}) \chi(G/K_1) = (1 - (-1)^{k_2}) (n+1), \\
 & (1 - (-1)^{k_1+k_2}) \chi(G/K_2) = (1 - (-1)^{k_1}) (n+1)
 \end{aligned}$$

where $\chi(G/K_s) = P(G/K_s; -1)$ is the Euler characteristic of G/K_s . In particular, $k_1 \equiv k_2 \pmod{2}$ implies $k_1 \equiv k_2 \equiv 0 \pmod{2}$ by (2.3.4).

(i) If $k_1 \equiv k_2 \equiv 0 \pmod{2}$, then

$$\chi(G/K_s) \neq 0$$

for $s = 1, 2$ from (2.3.3). Thus

$$\text{rank } K_s^0 = \text{rank } G$$

for $s = 1, 2$ and hence

$$H^{\text{odd}}(G/K_s^0; \mathbb{Q}) = \bigoplus_k H^{2k+1}(G/K_s^0; \mathbb{Q}) = 0$$

(cf. [1], Theorem 26.1), where K_s^0 is the identity component of K_s . Because the induced homomorphism

$$H^*(G/K_s; \mathbb{Q}) \longrightarrow H^*(G/K_s^0; \mathbb{Q})$$

is injective, the Poincaré polynomial $P(G/K_s; t)$ is an even function for $s = 1, 2$. Then we have from (2.3.2),

$$P(G/K_s; t) = 1 + t^2 + \dots + t^{2n_s}$$

for $s = 1, 2$. Therefore G/K_s is a rational cohomology complex projective n_s -space and $k_s = 2(n - n_s)$ for $s = 1, 2$.

(iii) Next, if $k_1 \equiv 0 \pmod{2}$ and $k_2 \equiv 1 \pmod{2}$, then

$$\chi(G/K_1) = n+1 \neq 0 \quad \text{and} \quad \chi(G/K_2) = 0$$

from (2.3.4). Thus $P(G/K_1; t)$ is an even function, and we have from (2.3.3),

$$(2.3.5) \quad \begin{aligned} P(G/K_1; t) &= 1 + t^2 + \dots + t^{2n_1} + t^{k_2-1} (1 + t^2 + \dots + t^{2n_2}), \\ t^{k_1+k_2-2} P(G/K_1; t) &= t^{k_2+2n_2} (1 + t^2 + \dots + t^{2n_1}) \\ &\quad + t^{2n_1+1} (1 + t^2 + \dots + t^{2n_2}). \end{aligned}$$

Thus we have

$$(2.3.6) \quad \begin{aligned} &(t^{k_1+k_2-2} - t^{k_2+2n_2}) (1 + t^2 + \dots + t^{2n_1}) \\ &= (t^{2n_1+1} - t^{k_1+2k_2-3}) (1 + t^2 + \dots + t^{2n_2}). \end{aligned}$$

Recall that $k_1 - 2 \leq 2n_2$ from (2.2.1) and Lemma 2.1.1.

(iii)_a First assume $k_1 - 2 < 2n_2$. Then we have

$$k_1 + k_2 - 2 = 2n_1 + 1$$

and

$$\begin{aligned} &(1 + t + \dots + t^{2n_2-k_1+1}) (1 + t^2 + \dots + t^{2n_1}) \downarrow \\ &= (1 + t + \dots + t^{k_2-2}) (1 + t^2 + \dots + t^{2n_2}) \end{aligned}$$

from (2.3.6). Put $t = 1$. Then we have

$$\begin{aligned} (2n_2 + 2 - k_1)(n_1 + 1) &= (k_2 - 1)(n_2 + 1) \\ &= (2n_1 + 2 - k_1)(n_2 + 1), \end{aligned}$$

and hence $n_1 = n_2$. Moreover

$$k_1 + k_2 = 2n_1 + 3 = n + 2.$$

(iii)_b Next assume $k_1 - 2 = 2n_2$. Then

$$2n_1 + 1 = k_1 + 2k_2 - 3$$

from (2.3.6), and hence

$$k_1 = 2(n_2 + 1) \quad \text{and} \quad k_2 = n_1 - n_2 + 1.$$

Moreover

$$k_1 + k_2 = n_1 + n_2 + 3 = n + 2.$$

The Poincaré polynomial $P(G/K_1; t)$ is obtained from (2.3.5), and $P(G/K_2; t)$ is obtained from (2.3.2) and the polynomial $P(G/K_1; t)$.

q.e.d.

2.4. Now we assume $k_1 = 2$ and consider certain relation between $H^*(G/K_S^0; \mathbb{Q})$ and $H^*(G/K_S; \mathbb{Q})$, where K_S^0 is the identity component of K_S . The following argument is essentially due to H.C.Wang [4].

Remark. If G/K_2 is non-orientable, then we have $k_1 = 2$ from (2.2.1) and Lemma 2.2.2.

Lemma 2.4.1. If $k_1 = 2$, then the induced homomorphism R_k^* is an identity on $H^*(G/K^0; \mathbb{Q})$ for every $k \in K$. Here the right translation R_k on G/K^0 is given by $R_k(gK^0) = gkK^0$.

Proof. (i) First assume $k_2 > 2$. Then K_1 is connected from Lemma 2.2.2 and the coset space K_1/K is a circle. Therefore there is a connected central subgroup T of K_1 such that

$$K \subset K_1 = T \cdot K^0.$$

Hence for each $k \in K$ there is $u \in T \cap K$ such that $R_k = R_u$ on G/K^0 . Because T is connected, there is a continuous mapping $u : [0,1] \longrightarrow T$ such that $u(0)$ is the identity element and $u(1) = u$. Because each $u(t)$ commutes with each element of K , a homotopy

$$H_t : G/K^0 \longrightarrow G/K^0$$

can be defined by $H_t(gK^0) = gu(t)K^0$, where H_0 is the identity and $H_1 = R_u = R_k$. Therefore R_k^* is an identity.

(ii) Next assume $k_2 = 2$. Let X_s be the invariant closed tubular neighborhood of G/K_s in M ($s = 1, 2$) such that

$$M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

Let $i_s : X_1 \cap X_2 \longrightarrow X_s$ be an inclusion mapping. Then the induced homomorphism

$$i_{s*} : \pi_1(X_1 \cap X_2) \longrightarrow \pi_1(X_s)$$

is surjective for $s = 1, 2$ from the general position theorem.

Thus there is a natural surjection

$$h_s : \pi_1(X_s) \longrightarrow \pi_1(X_1 \cap X_2) / (\ker i_{1*}) \cdot (\ker i_{2*})$$

for $s = 1, 2$ such that the following diagram is commutative :

$$\begin{array}{ccc} \pi_1(X_1 \cap X_2) & \xrightarrow{i_{1*}} & \pi_1(X_1) \\ \downarrow i_{2*} & & \downarrow h_1 \\ \pi_1(X_2) & \xrightarrow{h_2} & \pi_1(X_1 \cap X_2) / (\ker i_{1*}) \cdot (\ker i_{2*}). \end{array}$$

Then there is a surjection

$$\pi_1(X_1 \cup X_2) \longrightarrow \pi_1(X_1 \cap X_2) / (\ker i_{1*}) \cdot (\ker i_{2*})$$

by van Kampen's theorem. But $M = X_1 \cup X_2$ is simply connected and hence

$$\pi_1(X_1 \cap X_2) = (\ker i_{1*}) \cdot (\ker i_{2*}).$$

On the other hand, the inclusion $i_s : X_1 \cap X_2 \longrightarrow X_s$ is homotopy equivalent to the projection $p_s : G/K \longrightarrow G/K_s$. Thus we have

$$(2.4.2) \quad \pi_1(G/K) = (\ker p_{1*}) \cdot (\ker p_{2*}).$$

From homotopy exact sequences for the principal bundles

$$G \longrightarrow G/K \quad \text{and} \quad G \longrightarrow G/K_s$$

we have a commutative diagram :

$$\begin{array}{ccccc} \pi_1(G) & \longrightarrow & \pi_1(G/K) & \xrightarrow{\theta} & K/K^0 \\ \downarrow \text{id} & & \downarrow p_{s*} & & \downarrow \wr_s \\ \pi_1(G) & \longrightarrow & \pi_1(G/K_s) & \xrightarrow{\theta_s} & K_s/K_s^0 \end{array}$$

where θ and θ_s ($s = 1, 2$) are surjective. Thus we have from (2.4.2),

$$\begin{aligned} K/K^0 &= \theta(\pi_1(G/K)) = \theta((\ker p_{1*}) \cdot (\ker p_{2*})) \\ &= \theta(\ker p_{1*}) \cdot \theta(\ker p_{2*}) \subset (\ker \wr_1) \cdot (\ker \wr_2) \subset K/K^0. \end{aligned}$$

Therefore

$$(2.4.3) \quad K/K^0 = (K_1^0 \cap K/K^0) \cdot (K_2^0 \cap K/K^0) \subset (K_1^0/K^0) \cdot (K_2^0/K^0),$$

because $\ker \lambda_s = K_s^0 \cap K/K^0$. Then the proof of Lemma 2.4.1 for $k_2 = 2$ is done similarly as for $k_2 > 2$. q.e.d.

Now we consider a commutative diagram of natural projections :

$$(2.4.4) \quad \begin{array}{ccc} G/K^0 & \xrightarrow{q} & G/K \\ \downarrow p_s^0 & & \downarrow p_s \\ G/K_s^0 & \xrightarrow{q_s} & G/K_s \end{array}$$

for $s = 1, 2$.

Lemma 2.4.5. If $k_1 = 2$, then

$$H^*(G/K_s^0; \mathbb{Q}) = q_s^* H^*(G/K_s; \mathbb{Q}) + (\ker p_s^{0*})$$

for $s = 1, 2$ (direct sum or not).

Proof. Because K_s/K is a $(k_s - 1)$ -sphere, K_s/K is connected and hence the natural mapping $K_s^0/K^0 \rightarrow K_s/K$ is surjective. Thus

$$(2.4.6) \quad K_s = K_s^0 \cdot K \quad (s = 1, 2).$$

Hence for each $a \in K_s$ there is $k \in K$ such that $R_a^* = R_k^*$ on $H^*(G/K_s^0; \mathbb{Q})$. By Lemma 2.4.1 and a commutative diagram :

$$\begin{array}{ccc} H^*(G/K_s^0; \mathbb{Q}) & \xrightarrow{p_s^{0*}} & H^*(G/K^0; \mathbb{Q}) \\ \downarrow R_a^* = R_k^* & & \downarrow R_k^* = \text{id} \\ H^*(G/K_s^0; \mathbb{Q}) & \xrightarrow{p_s^{0*}} & H^*(G/K^0; \mathbb{Q}), \end{array}$$

we have

$$(2.4.7) \quad p_s^{0*}(u) = p_s^{0*}(R_a^*(u))$$

for each $a \in K_s$ and each $u \in H^*(G/K_s^0; \mathbb{Q})$. By averaging (2.4.7) on a finite group K_s/K_s^0 , we have

$$p_s^{0*} H^*(G/K_s^0; \mathbb{Q}) = p_s^{0*} q_s^* H^*(G/K_s; \mathbb{Q}),$$

because

$$q_s^* H^*(G/K_s; \mathbb{Q}) = H^*(G/K_s^0; \mathbb{Q})^{K_s/K_s^0}.$$

Thus we have

$$H^*(G/K_s^0; \mathbb{Q}) = q_s^* H^*(G/K_s; \mathbb{Q}) + (\ker p_s^{0*}). \quad \text{q.e.d.}$$

2.5. Denote by $J = \bigoplus_k J_k$, $J_k = q_2^* H^k(G/K_2; \mathbb{Q})$. Then J is a graded subalgebra of $H^*(G/K_2^0; \mathbb{Q})$. Because

$$K_2^0/K^0 \longrightarrow G/K^0 \xrightarrow{p_2^0} G/K_2^0$$

is an orientable $(k_2 - 1)$ -sphere bundle, its rational Euler class $e(p_2^0)$ can be determined up to sign. Then

$$\text{Lemma 2.5.1.} \quad \ker p_2^{0*} = J \cdot e(p_2^0) + J \cdot e(p_2^0)^2.$$

Proof. From a Gysin sequence for a sphere bundle and Lemma 2.4.5, we have

$$\ker p_2^{0*} = H^*(G/K_2^0; \mathbb{Q}) \cdot e(p_2^0) = J \cdot e(p_2^0) + (\ker p_2^{0*}) \cdot e(p_2^0).$$

Hence

$$\ker p_2^{0*} = J \cdot e(p_2^0) + J \cdot e(p_2^0)^2 + \dots + J \cdot e(p_2^0)^N$$

for sufficiently large N , as submodules of $H^*(G/K_2^0; \mathbb{Q})$. For each $k \in K$, we have a commutative diagram :

$$\begin{array}{ccc} G/K^0 & \xrightarrow{R_k} & G/K^0 \\ \downarrow p_2^0 & & \downarrow p_2^0 \\ G/K_2^0 & \xrightarrow{R_k} & G/K_2^0 \end{array}$$

which is a bundle mapping. Thus we have

$$R_k^* e(p_2^0) = e(p_2^0) \text{ or } -e(p_2^0).$$

Here $R_k^* e(p_2^0) = -e(p_2^0)$ occurs when R_k reverses an orientation of the sphere bundle. Therefore

$$R_k^* (e(p_2^0)^2) = e(p_2^0)^2$$

for each $k \in K$. Because

$$J = q_2^* H^*(G/K_2^0; \mathbb{Q}) = H^*(G/K_2^0; \mathbb{Q})^{K_2} = H^*(G/K_2^0; \mathbb{Q})^K$$

by (2.4.6), we have

$$(2.5.2) \quad e(p_2^0)^2 \in J$$

and hence

$$\ker p_2^{0*} = J \cdot e(p_2^0) + J \cdot e(p_2^0)^2. \quad \text{q.e.d.}$$

$$\text{Lemma 2.5.3.} \quad \dim(\ker p_2^{0*}) \leq \dim J + \dim(J \cap \ker p_2^{0*}).$$

Here the equality occurs if and only if

$$J \cdot e(p_2^0) \cap J \cdot e(p_2^0)^2 = 0, \quad J \cdot e(p_2^0)^2 = J \cap \ker p_2^{0*}$$

and $E : J \longrightarrow \ker p_2^{0*}$ is injective, where E is defined by
 $E(x) = x \cdot e(p_2^0).$

Proof. By (2.5.2), we have

$$J \cdot e(p_2^0)^2 \subset J \cap \ker p_2^{0*}$$

and hence we have from Lemma 2.5.1

$$\dim(\ker p_2^{0*}) \leq \dim J + \dim(J \cap \ker p_2^{0*}).$$

Moreover we have the condition on which the equality occurs. q.e.d.

2.6. Now we assume that G/K_2 is non-orientable. Then we have $k_1 = 2$ from (2.2.1) and Lemma 2.2.2.

Lemma 2.6.1. If G/K_2 is non-orientable, then

$$P(G/K_2^0; t) = (1 + t^{k_2}) P(G/K_2; t),$$

$$P(G/K^0; t) = (1 + t^{2k_2-1}) P(G/K_2; t) - P(n_1, n_2; t).$$

Here $P(n_1, n_2; t) = 0$ for $n_1 \geq n_2$ and

$$P(n_1, n_2; t) = t^{2n_1+1} + t^{2n_1+2} + \dots + t^{2n_2}$$

for $n_1 < n_2.$

Proof. From a Gysin sequence :

$$H^{k-k_2}(G/K_2^0; \mathbb{Q}) \xrightarrow{e(p_2^0)} H^k(G/K_2^0; \mathbb{Q}) \xrightarrow{p_2^{0*}} H^k(G/K^0; \mathbb{Q}) \xrightarrow{\delta} H^{k+1-k_2}(G/K_2^0; \mathbb{Q}),$$

we have

$$\begin{aligned}
 P(G/K_2^0; t) &= P(\operatorname{im} p_2^{0*}; t) + P(\ker p_2^{0*}; t), \\
 (2.6.2) \quad P(G/K_2^0; t) &= t^{-k_2} P(\ker p_2^{0*}; t) + P(\operatorname{im} \mathcal{S}; t), \\
 P(G/K_2^0; t) &= t^{k_2-1} P(\operatorname{im} \mathcal{S}; t) + P(\operatorname{im} p_2^{0*}; t).
 \end{aligned}$$

By Lemma 2.4.5 and the definition $J = q_2^* H^*(G/K_2; \mathbb{Q})$,

$$\begin{aligned}
 P(\operatorname{im} p_2^{0*}; t) &= P(p_2^{0*}(J); t) \\
 &= P(J; t) - P(J \cap \ker p_2^{0*}; t),
 \end{aligned}$$

and hence

$$(2.6.3) \quad P(\operatorname{im} p_2^{0*}; t) = P(G/K_2^0; t) - P(J \cap \ker p_2^{0*}; t).$$

Because G/K_2 is non-orientable, there is $k \in K_2$ such that the right translation R_k on G/K_2^0 reverses an orientation of G/K_2^0 . Then

$$(2.6.4) \quad 2 \cdot \dim H^*(G/K_2; \mathbb{Q}) \leq \dim H^*(G/K_2^0; \mathbb{Q})$$

by Poincaré duality (cf. [2]). By Lemma 2.4.5, we have

$$(2.6.5) \quad \dim H^*(G/K_2^0; \mathbb{Q}) = \dim J + \dim(\ker p_2^{0*}) - \dim(J \cap \ker p_2^{0*}).$$

Then we have

$$\dim J \leq \dim(\ker p_2^{0*}) - \dim(J \cap \ker p_2^{0*})$$

from (2.6.4), (2.6.5) and $\dim J = \dim H^*(G/K_2; \mathbb{Q})$. Thus we have

$$\dim J = \dim(\ker p_2^{0*}) - \dim(J \cap \ker p_2^{0*})$$

by Lemma 2.5.3. Moreover we have

$$(2.6.6) \quad P(\ker p_2^{0*}; t) = t^{k_2} P(J; t) + P(J \cap \ker p_2^{0*}; t)$$

from Lemma 2.5.1 and Lemma 2.5.3. Combining (2.6.2), (2.6.3) and (2.6.6), we have

$$\begin{aligned} P(G/K_2^0; t) &= (1 + t^{k_2}) P(G/K_2; t), \\ P(G/K^0; t) &= (1 + t^{2k_2-1}) P(G/K_2; t) - (1 + t^{-1}) P(J \cap \ker p_2^{0*}; t). \end{aligned}$$

It remains to show

$$(1 + t^{-1}) P(J \cap \ker p_2^{0*}; t) = P(n_1, n_2; t).$$

Consider the following commutative diagram :

$$\begin{array}{ccc} H^*(G/K_2; Q) & \xrightarrow{p_2^*} & H^*(G/K; Q) \\ \downarrow q_2^* & & \downarrow q^* \\ H^*(G/K_2^0; Q) & \xrightarrow{p_2^{0*}} & H^*(G/K^0; Q). \end{array}$$

Because q^* is an isomorphism from Lemma 2.4.1, we have

$$P(J \cap \ker p_2^{0*}; t) = P(\ker p_2^*; t).$$

Recall that $p_2 : G/K \rightarrow G/K_2$ is homotopy equivalent to $i_2 : X_1 \cap X_2 \rightarrow X_2$, and consider the following commutative diagram :

$$\begin{array}{ccccc} H^*(M, X_1; Q) & \longrightarrow & H^*(M; Q) & \xrightarrow{f_1^*} & H^*(X_1; Q) \\ \downarrow \cong & & \downarrow f_2^* & & \downarrow i_1^* \\ H^*(X_2, X_1 \cap X_2; Q) & \longrightarrow & H^*(X_2; Q) & \xrightarrow{i_2^*} & H^*(X_1 \cap X_2; Q). \end{array}$$

Then we have

$$P(\ker p_2^{0*}; t) = P(\ker i_2^*; t) = \begin{cases} t^{2n_1+2} + \dots + t^{2n_2} & (\text{if } n_1 < n_2) \\ 0 & (\text{if } n_1 \geq n_2). \end{cases}$$

Thus we have

$$(1 + t^{-1})P(J \cap \ker p_2^{0*}; t) = \begin{cases} t^{2n_1+1} + \dots + t^{2n_2} & (\text{if } n_1 < n_2) \\ 0 & (\text{if } n_1 \geq n_2). \end{cases}$$

q.e.d.

2.7. Now we can prove the following result.

n is even,

Proposition 2.7.1. Assume that G/K_2 is non-orientable.
 (i) If G/K_1 is orientable, then G/K_1 is a rational cohomology complex projective $(n-1)$ -space and G/K_2^0 is a rational cohomology n -sphere.

(ii) If G/K_1 is non-orientable, then $n = 3$ and

$$P(G/K_s; t) = 1 + t^2, \quad P(G/K_s^0; t) = (1 + t^2)^2$$

for $s = 1, 2$.

Proof. Because G/K_2 is non-orientable, we have

$$k_1 = 2 \quad \text{and} \quad \dim G/K_1 = 2n - 2.$$

(i) First assume that G/K_1 is orientable. Then by the Poincaré duality for G/K_1 , we have from (2.3.2),

$$(2.7.2) \quad t^{2n-1} P(G/K_2; t^{-1}) = P(G/K_1; t) + t^{2n_1+1} (1 + t^2 + \dots + t^{2n_2}) - (1 + t^2 + \dots + t^{2n_1}).$$

By the Poincaré duality for G/K_2^0 , we have from Lemma 2.6.1,

$$(2.7.3) \quad t^{2n} P(G/K_2; t^{-1}) = t^{2k_2} P(G/K_2; t).$$

Combining (2.7.2), (2.7.3) and (2.3.2) with $k_1 = 2$, we have

$$(2.7.4) \quad (1 - t^{2k_2}) P(G/K_1; t) = (1 - t^{2k_2+2n_2}) (1 + t^2 + \dots + t^{2n_1}) \\ + (t^{2k_2-1} - t^{2n_1+1}) (1 + t^2 + \dots + t^{2n_2}).$$

In particular we have

$$\chi(G/K_1) = P(G/K_1; -1) \neq 0.$$

Therefore $P(G/K_1; t)$ is an even function by the same argument as in the proof of Proposition 2.3.1 (i). Hence we have from (2.7.4),

$$(2.7.5) \quad k_2 = n_1 + 1 \quad \text{and} \quad P(G/K_1; t) = 1 + t^2 + \dots + t^{2n-2}.$$

Then we have from (2.3.2) and (2.7.5),

$$(2.7.6) \quad P(G/K_2; t) = 1 + t + t^2 + \dots + t^{2n_2}.$$

Thus $\chi(G/K_2) \neq 0$ and hence $P(G/K_2; t)$ is an even function.

Therefore

$$n_2 = 0 \quad \text{and} \quad P(G/K_2; t) = 1$$

from (2.7.6), and hence $n_1 = n - 1$ by Lemma 2.1.1. Then

$$P(G/K_2^0; t) = 1 + t^n$$

from Lemma 2.6.1. Consequently G/K_1 is a rational cohomology complex projective $(n - 1)$ -space and G/K_2^0 is a rational cohomology n -sphere. Moreover $\chi(G/K_2) \neq 0$ implies $n \equiv 0 \pmod{2}$.

(ii) Next assume that G/K_1 is non-orientable. Then

$$k_1 = k_2 = 2.$$

From Lemma 2.6.1, we have

$$(2.7.7) \quad \begin{aligned} P(G/K_2^0; t) &= (1 + t^2)P(G/K_2; t), \\ P(G/K^0; t) &= (1 + t^3)P(G/K_2; t) - P(n_1, n_2; t). \end{aligned}$$

Similarly we have

$$(2.7.8) \quad \begin{aligned} P(G/K_1^0; t) &= (1 + t^2)P(G/K_1; t), \\ P(G/K^0; t) &= (1 + t^3)P(G/K_1; t) - P(n_2, n_1; t). \end{aligned}$$

Here $P(a, b; t) = 0$ for $a \geq b$ and

$$P(a, b; t) = t^{2a+1} + t^{2a+2} + \dots + t^{2b}$$

for $a < b$.

If $n_1 < n_2$, then we have from (2.7.7) and (2.7.8),

$$t^{2n_1+1} + t^{2n_1+2} + \dots + t^{2n_2} \equiv 0 \pmod{1 + t^3}.$$

Thus $n_1 \equiv n_2 \pmod{3}$ and

$$(2.7.9) \quad P(G/K_2; t) - P(G/K_1; t) = t^{2n_1+1} (1+t+t^2) (1+t^6+t^{12}+\dots+t^{2(n_2-n_1-3)}),$$

Then

$$(2.7.10) \quad \chi(G/K_2) - \chi(G/K_1) = (n_1 - n_2)/3 < 0.$$

If $\chi(G/K_s) \neq 0$ for $s = 1, 2$ then $P(G/K_s; t)$ is an even function for $s = 1, 2$ and this contradicts (2.7.9). Thus

$$\chi(G/K_1) \neq 0 \quad \text{and} \quad \chi(G/K_2) = 0$$

from (2.7.10), and hence

$$(2.7.11) \quad \text{rank } K_1^0 = \text{rank } G \neq \text{rank } K_2^0.$$

On the other hand,

$$\text{rank } K_s^0 = \text{rank } K^0 + 1$$

for $s = 1, 2$ because $K_s/K = S^1$. This contradicts (2.7.11).

Therefore the case $n_1 < n_2$ does not occur. Similarly the case $n_2 < n_1$ does not occur.

Finally if $n_1 = n_2$, then $n = 2n_1 + 1$ and we have from (2.7.7) and (2.7.8)

$$(2.7.12) \quad \begin{aligned} P(G/K_s^0; t) &= (1 + t^2)P(G/K_s; t), \\ P(G/K^0; t) &= (1 + t^3)P(G/K_s; t) \end{aligned}$$

for $s = 1, 2$. Let X_s be the invariant closed tubular neighborhood of G/K_s such that

$$M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

Consider the Mayer-Vietoris cohomology sequence of a triad

$(M; X_1, X_2)$. Then we have

$$P(G/K_1; t) + P(G/K_2; t) - P(G/K; t) = (1 - t^{2n_1+1})(1 + t^2 + \dots + t^{2n_1})$$

Because $P(G/K; t) = P(G/K^0; t)$ from Lemma 2.4.1, we have from (2.7.12),

$$(2.7.13) \quad (1 - t^3)P(G/K_1; t) = (1 - t^{2n_1+1})(1 + t^2 + \dots + t^{2n_1}).$$

Thus $\chi(G/K_1) = n_1 + 1 \neq 0$ and hence $P(G/K_1; t)$ is an even function. Therefore we have from (2.7.13),

$$n_1 = 1 \quad \text{and} \quad P(G/K_1; t) = 1 + t^2.$$

Consequently, $n = 3$ and

$$P(G/K_s; t) = 1 + t^2,$$

$$P(G/K_s^0; t) = (1 + t^2)^2$$

for $s = 1, 2$ from (2.7.12). q.e.d.

§3. Examples

(例1) $n = n_1 + n_2 + 1$ として, $P_n(\mathbb{C}) = P(\mathbb{C}^{n_1+1} \oplus \mathbb{C}^{n_2+1})$ 上の $U(n_1+1) \times U(n_2+1)$ の自然な作用を考える。この作用は、余次元1の多様体

$$X = \{(u_0, \dots, u_{n_1}, v_0, \dots, v_{n_2}) \mid |u_0|^2 + \dots + |u_{n_1}|^2 = |v_0|^2 + \dots + |v_{n_2}|^2\}$$

に推移的に働き、特異軌道

$$P_{n_1}(\mathbb{C}) = \{(u_0, \dots, u_{n_1}, 0, \dots, 0)\},$$

$$P_{n_2}(\mathbb{C}) = \{(0, \dots, 0, v_0, \dots, v_{n_2})\}$$

を持つ。 $U(n_1+1) \times U(n_2+1)$ の部分群 G で, X に推移的に作用するものについても同様である。これは、定理0の (A)-(i) の場合である。

(例2) $P_n(\mathbb{C}) = P(\mathbb{R}^{n+1} \otimes_{\mathbb{R}} \mathbb{C})$ 上の $SO(n+1)$ の自然な作用を考える。点 $(0, \dots, 0, t, \sqrt{1-t^2})$ におけるイソトロピー群を H_t とおけば,

$$H_0 = S(O(n) \times O(1)), \quad H_1 = SO(n-1) \times SO(2),$$

$$H_t = SO(n-1) \times \mathbb{Z}_2, \quad (0 < t < 1)$$

となり、余次元1の軌道を持ち、 $SO(n+1)/H_0 = P_n(\mathbb{R})$ である。従って、 $n \equiv 1 \pmod{2}$ のとき、定理0の (A)-(ii)-(c) の場合で、 $n \equiv 0 \pmod{2}$ のとき、(B) の場合である。どちらの場合も、 $k_1 = 2$, $k_2 = n = n_1 + 1$, $n_2 = 0$ となっている。

(例 3) $n = 2p+1$ とし, $P_n(\mathbb{C}) = P(\mathbb{C}^{p+1} \otimes \mathbb{C}^2)$ 上の $SU(p+1) \times SU(2)$ のテンソル積による自然な作用を考える.
 $t \cdot e_p \otimes e_1 + e_{p+1} \otimes e_2$ が表わす軌のイソトロピー群を H_t とおけば,

$$H_0 = S(U(p) \times U(1)) \times S(U(1) \times U(1)),$$

$$H_1 = \left\{ \begin{pmatrix} * & 0 \\ 0 & \lambda \bar{A} \end{pmatrix} \times A; |\lambda| = 1 \right\}, \quad H_t = \left\{ \begin{pmatrix} * & 0 \\ 0 & \lambda \bar{A} \end{pmatrix} \times A; |\lambda| = 1, A \in S(U(1) \times U(1)) \right\}$$

$0 < t < 1$

となり, 余次元 1 の軌道を持つ. これは, 定理 0 の (A)-(ii)-(a) の場合であり, $k_1 = 2p$, $k_2 = 3$, $n = 2p+1$, $n_1 = n_2 = p$ となっている.

(例 4) $Q_n = SO(n+2)/SO(n) \times SO(2)$ 上に, 左移動によって $SO(n+1)$ を作用させる.

$$A_\theta = \begin{pmatrix} I_{n-1} & & \\ & \cos \theta & 0 & \sin \theta \\ & 0 & 1 & 0 \\ & -\sin \theta & 0 & \cos \theta \end{pmatrix} \in SO(n+2)$$

が表わす剰余類におけるイソトロピー群を H_θ とおけば,

$$H_0 = SO(n), \quad H_{\frac{\pi}{2}} = SO(n-1) \times SO(2),$$

$$H_\theta = SO(n-1), \quad 0 < \theta < \frac{\pi}{2}$$

となり, 余次元 1 の軌道を持つ. n が奇数で, $n \neq 1$ のとき, Q_n は単連結な n 次元有理コホモロジー複素射影空間であり, $\pi_n(Q_n) = \mathbb{Z}_2$ である. これは, 定理 0 の (A)-(ii)-(c) の場

合であり, $k_1=2$, $k_2=n=n_1+1$, $n_2=0$ となっている。

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