

Some global topological properties of complex hypersurfaces.

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§1. Introduction.

Local topological properties of a complex hypersurface V in a compact complex $(n+1)$ -manifold M at a singular point of V have been studied by many authors during the last decade. In this note we shall study some global topological properties of V and a pair (M, V) focusing on the normal euler class $X(V, M)$ of V in M .

In §2, we shall recall a formula for the total Chern-Macpherson homology class of V with isolated singularity. Existence of a nice almost complex resolution of V in M was basic for us to obtain the formula. We shall formulate a generalization of this notion for V with possibly non-isolated singularity and give an example which admit no nice almost complex resolution in M .

In §3, micro-equivalence classification of PL embeddings of a compact oriented ordered m -subpolyhedron V into closed oriented manifolds of codimension two with isolated singularity will be given. The complete set of invariants consists of the normal euler class and the singularity. This is a generalization of Noguchi [7]. Our purpose^{here} is to prove

Theorem. Let V and V' be complex irreducible curves in a compact complex surface M . Then there are open neighborhoods U, U' of V, V' in M so that (U, V) and (U', V') are "PL" homeomorphic if V and V' represent the same integral homology class in M and have the same sets of link types at singular points.

Here we understand that a "PL" homeomorphism means a PL homeomorphism of compatible triangulations of (U, V) and (U', V') in the sense of [1].

§ 2. Normal euler classes and Chern-Macpherson classes.

Let V be a complex hypersurface in a compact complex $(n+1)$ -manifold M with singular set ΣV .

The complex structures of $V - \Sigma V$ and M determines

the fundamental classes $[V] \in H_{2n}(V; \mathbb{Z})$ and

$[M] \in H_{2n+2}(M; \mathbb{Z})$. We have the Poincaré duality

isomorphism $P = (\cap [M]) : H^*(M; \mathbb{Z}) \rightarrow H_{2n+2-*}(M; \mathbb{Z})$

and a homomorphism $\cap [V] : H^*(V; \mathbb{Z}) \rightarrow H_{2n-*}(V; \mathbb{Z})$,

(for some properties of $\cap [V]$, see [A]).

Definition. Normal euler class $X(V, M)$ of V in M is defined by $X(V, M) = i^* \cdot P^{-1} \cdot i_* [V]$, where $i : V \rightarrow M$ is an inclusion map.

We shall say that V admits a nice almost complex resolution in M , if for any open neighborhood U of the singular set ΣV of V in M , there exists a complex analytic Whitney stratification $\mathcal{S}$ with a tubular neighborhood system $\{\pi_A : T_A \rightarrow A \mid A \in \mathcal{S}\}$ of ΣV in M and a smooth $2n$ -submanifold $\hat{V}$, called a nice almost complex resolution of V in M , satisfying the following two conditions;

(1) the tangent bundle $\tau(\hat{V})$ of $\hat{V}$ and the normal bundle $\nu(\hat{V}, M)$ of $\hat{V}$ in M admit complex reductions $\tau_c(\hat{V})$ and $\nu_c(\hat{V}, M)$ as continuous vector bundles so that the Whitney sum

$$\tau_c(\hat{V}) \oplus \nu_c(\hat{V}, M) \cong \tau_c(M)|_{\hat{V}}$$

as continuous complex vector bundles], and (, where $\tau_c(M)$ is the complex tangent bundle of M)

(2) if we put $T = \bigcup_{A \in S} T_A$, then

$$T \subset \hat{V}, \quad V - T = \hat{V} - T \quad \text{and}$$

for each $A \in S$, if we put $T_{\geq A} = \bigcup_{B \leq A} T_B$,

then π_A restricted to $(T - T_{\geq A}) \cap \hat{V}$ is a locally trivial fibration over $A - T_{\geq A}$, and

for each point $x \in A - T_{\geq A}$, the fiber $\pi_A^{-1}(x) \cap \hat{V}$ has the same euler number as a Milnor fiber of $L \cap V$ in L at x for a local transversal plane L to A in M .

The condition (1) says that by definition $\hat{V}$ is an almost complex submanifold of M and the condition (2) says that $\hat{V}$ approximates V nicely.

The following theorem is proved in [2].

Theorem 1. Suppose that V has isolated singularity. Then V admits a nice almost complex resolution $\hat{V}$ in M .

The total Chern-Macpherson class $c_*(V)$ of V is defined as an element of $H_{2*}(V; \mathbb{Z})$ by Macpherson [6].

Corollary to Theorem 1. If V has isolated singularity, we have a formula

$$c_*(V) = \left(\frac{i^* c^*(M)}{c^*(V, M)} \right) \cap [V] + (-1)^{n-1} \mu(V, M),$$

where $c^*(M) \in H^{2*}(M; \mathbb{Z})$ is the total chern class of the complex $(n+1)$ -manifold M , $c^*(V, M)$ is the total normal chern class $1 + X(V, M) \in H^0(V; \mathbb{Z}) + H^2(V; \mathbb{Z})$ and $\mu(V, M)$ is the sum of all the Milnor numbers μ_x of V in M at $x \in \Sigma V$.

Example 1. If $M = \mathbb{P}^{n+1}$ (complex projective $(n+1)$ -space) and V is of degree d , then we have an explicit formula for $c_*(V)$ by means of homology classes of multiple hyperplane sections $V_m = V \cap L^{(m+1)}$,

($m = 0, 1, \dots, n$) in $\mathbb{P}^{n+1}$;

$$C_*(V) = \sum_{m=0}^n \left(\sum_{k=0}^{n-m} \binom{n+2}{n-m-k} \cdot (-d)^k \right) [V_m] + (-1)^{n-1} \mu(V, \mathbb{P}^{n+1})$$

Example 2. Let V be a complex ^(irreducible) curve in a compact complex surface M , and let $\tilde{V}$ be a normalization of V . Then the topological type of $\tilde{V}$ is completely determined by the homology class $i_*[V] \in H_2(M; \mathbb{Z})$ and the link types of V in M at singular points. In fact, we put

Γ_x = the number of connected components of a link K_x at x ,

μ_x = the Milnor number of V in M at x ,

we have that the euler number $\chi(\tilde{V})$ of $\tilde{V}$ is given by

$$\begin{aligned} \chi(\tilde{V}) &= \chi(V) + \sum_{x \in \Sigma V} (\Gamma_x - 1) \\ &= C^1(M) \cap i_*[V] - \langle i_*[V], i_*[V] \rangle \\ &\quad + \sum_{x \in \Sigma V} (\mu_x + \Gamma_x - 1) \end{aligned}$$

, since by Corollary to Theorem 1

$$\chi(V) = C_0(V) = (i^* C^1(M) - \chi(V, M)) \cap [V] + \sum_{x \in \Sigma V} \mu_x,$$

where $\langle \ , \ \rangle$ stands for the intersection number.

By the Milnor's formula $2\delta_x = \mu_x + \Gamma_x - 1$, the formula for $\chi(\tilde{V})$ above is a topological expression of the so-called adjunction formula;

$$2 - 2 \cdot g(\tilde{V}) = V \cdot (K + V) - 2 \cdot \sum_{x \in \Sigma V} \delta_x ,$$

where $g(\tilde{V})$ is the genus of $\tilde{V}$, V is a complex line bundle over M determined by the divisor V in M and K is the canonical line bundle of M , see Kodaira ([5], p. 30, (3.10)) or Serre ([8], p. 75, Proposition 5.).

At this point, we remark that not every hypersurface admits a nice almost complex resolution in M ;

Example 3. Let V be a singular Enriques surface in $\mathbb{P}^3$ defined by

$$(Z_0 Z_1 Z_2)^2 + (Z_1 Z_2 Z_3)^2 + (Z_2 Z_3 Z_0)^2 + (Z_1 Z_2 Z_3)^2 \\ + (Z_0 Z_1 Z_2 Z_3) \cdot (\text{generic quadratic form}) = 0 .$$

The singular surface V admits no nice almost complex resolution in $\mathbb{P}^3$.

In fact, $\Sigma V = 6$ lines $Z_i = Z_j = 0$ ($i \neq j$), the "minimal" complex analytic Whitney stratification of $\Sigma V = \{ \Sigma V - (4 \text{ triple points} \cup 24 \text{ cusp (or Whitney umbrella) points}) , \text{ these } 28 \text{ points} \}$. From this, if we suppose that there exists a nice almost

complex resolution $\hat{V}$ of V in $\mathbb{P}^3$, then we have that $\left\{ \begin{array}{l} \text{from the condition (2) for } \hat{V} \\ \chi(\hat{V}) = \chi(V) - \chi(\Sigma V) + 24 \cdot 2 \\ = \chi(V) - (6 \cdot 2 - 2 \cdot 4) + 24 \cdot 2 \\ = \chi(V) + 44. \end{array} \right.$

By the arguments of Kodaira ([5], pp.41-51, (4.13)), we have that $\chi(V) = 28$. Hence we have that

$$\chi(\hat{V}) = 72.$$

On the other hand, by the condition (1) for $\hat{V}$, $\chi(\hat{V}) =$ the euler number of a non-singular surface of degree 6 in $\mathbb{P}^3$. Thus we should have $\chi(\hat{V}) = 108 (\neq 72)$. This is a contradiction.

Remark. There is a simpler example of V in $\mathbb{P}^3$ admitting no nice almost complex resolution in $\mathbb{P}^3$; two planes $(x_0, x_1 = 0)$ in $\mathbb{P}^3$. But this is not irreducible.

§3. Normal euler classes, singular types and micro-equivalence.

Definition. A PL embedding $f: V \rightarrow M$ of

a polyhedron V of dimension m into a $PL(m+c)$ -manifold M has isolated singularity, if there is a 0-dim. sub-polyhedron R of V such that for each point x of $V - R$, there is an open neighborhood U_x of $f(x)$ in M such that $(U_x, U_x \cap V)$ is PL homeomorphic to $(\mathbb{R}^{m+c}, \mathbb{R}^m \times 0)$.

The minimal R is called the singular set of f and denoted by $\Sigma V (= \Sigma_f V)$.

Remark. A restriction $\overset{\text{of } f}{f|V - \Sigma V} : V - \Sigma V \rightarrow M - f(\Sigma V)$ is locally flat. If V is a PL m -manifold, and if f has isolated singularity, then f is called 1-flat by Noguchi [6].

$(f : V \rightarrow M \text{ has isolated singularity,})$

In the following we shall assume that V is a compact polyhedron of dim. m such that $V - \Sigma V$ is oriented and M is a closed oriented $\overset{(PL)}{(m+c)}$ -manifold, when otherwise stated.

Taking divisions L, K of M, V , we make $f : K \rightarrow L$ simplicial. For the second barycentric subdivisions L'', K'' of L, K , we put

$$D_x = st(f(x), L''), \quad st(x, K'') = C_x, \quad lk(f(x), L'') = S_x, \quad lk(x, K'') = K_x \quad \text{and}$$

$$f_x = f|_{C_x} : C_x \rightarrow D_x, \quad \dot{f}_x = f|_{K_x} : K_x \rightarrow S_x.$$

Then D_x, C_x are cones $f(x) * S_x, x * K_x$ and f_x

is a cone extension of a locally flat embedding

$\dot{f}_x : K_x \rightarrow S_x$ of the orientable closed $(m-1)$ -manifold K_x into the $(m+c-1)$ -sphere. We put

$$D_\Sigma = \bigcup_{x \in \Sigma V} D_x, \quad C_\Sigma = \bigcup_{x \in \Sigma V} C_x, \quad S_\Sigma = \bigcup_{x \in \Sigma V} S_x,$$

$$K_\Sigma = \bigcup_{x \in \Sigma V} K_x, \quad V_0 = (V - C_\Sigma) \cup K_\Sigma, \quad M_0 = (M - D_\Sigma) \cup S_\Sigma,$$

$$f_\Sigma = \bigcup_{x \in \Sigma V} f_x : C_\Sigma \rightarrow D_\Sigma, \quad \dot{f}_\Sigma = \bigcup_{x \in \Sigma V} \dot{f}_x : K_\Sigma \rightarrow S_\Sigma$$

and give orientations $[S_\Sigma, K_\Sigma]$ induced from

M_0, V_0 as their "boundaries". Let $V_{0,i}, i=1, \dots, r,$

be all connected components of V_0 and let

$$K_{\Sigma,i} = K_\Sigma \cap V_{0,i} \quad \text{and} \quad V_{\Sigma,i} = V_{0,i} \cup x * K_{\Sigma,i}$$

, $i=1, \dots, r$. Then we may regard of V as

an ordered union $V_1 \cup \dots \cup V_r$ of irreducible

components $V_1, \dots, V_r$ of V . In the same way,

K_Σ is regarded as an ordered union $K_{\Sigma,1} \cup \dots \cup K_{\Sigma,r}$.

Thus we regard of f and f_Σ as ordered oriented PL embeddings.

Definition. Two ordered oriented PL embeddings $\varphi : W \rightarrow N$ and $\varphi' : W \rightarrow N'$ are equivalent, if there is an orientation preserving PL homeomorphism $h : N \rightarrow N'$ ^{called an equivalence,} such that $h \circ \varphi|_{W_i} = \varphi'|_{W_i}$ for each i . Two ordered oriented PL embeddings

φ and φ' are micro-equivalent, if for some open neighborhoods U, U' of $\varphi(W), \varphi'(W)$ in N, N' ,

$\varphi : W \rightarrow U$ and $\varphi' : W \rightarrow U'$ are equivalent. The equivalence $g : U \rightarrow U'$ is called a micro-equivalence.

In case $c \leq 2$, we shall call the equivalence class $\langle f \rangle$ of an ordered oriented locally flat PL embedding $f_{\Sigma} : K_{\Sigma} \rightarrow S_{\Sigma}$ as to be the (ordered oriented) singularity of f and denote it by $\sigma(f)$.

(Ordered) ^(normal) Euler class $X(f)$ of $f : V \rightarrow M$ is defined by $X(f) = f^* \circ P^{-1} \circ f_* [V] = f^* \circ P^{-1} \circ f_* \left(\sum_{i=1}^r [V_i] \right) \in H^c(V; \mathbb{Z}) \cong \sum_{i=1}^r H^c(V_i; \mathbb{Z})$ ($m \geq 2$), where P is the Poincaré duality isomorphism $\cap[M] : H^*(M; \mathbb{Z}) \rightarrow H_{m-*}(M; \mathbb{Z})$ and $[V], [M]$ are the fundamental classes of V, M determined by the orientations of $V - \Sigma V, M$.

Theorem 2. Let two PL embeddings $f: V \rightarrow M$, $f': V \rightarrow M'$ be with isolated singularity from a compact oriented ordered m -polyhedron V into closed oriented $(m+c)$ -manifolds M, M' .

- (1) If $c=1$, then f and f' are micro-equivalent if and only if $\sigma(f) = \sigma(f')$.
- (2) If $c=2$, then f and f' are micro-equivalent if and only if $\sigma(f) = \sigma(f')$ and $X(f) = X(f')$.

(Compare with Noguchi [7]).

The proof of Theorem 2 will be given in the forthcoming paper [3].

Let V be a compact oriented ordered m -subpolyhedron of a closed oriented PL $(m+c)$ -manifold M such that an inclusion map $i: V \rightarrow M$ has isolated singularity. We define $\sigma(V, M) = \sigma(i)$ and $X(V, M) = X(i)$. Two such pairs (M, V) and (M', V') are micro-equivalent, if there is a PL homeomorphism $h: V \rightarrow V'$ such that i and $i' \circ h$ are micro-equivalent. In this case, putting $h^* \sigma(V', M') = \sigma(i' \circ h)$, we have that

$$\sigma(V, M) = h^* \sigma(V', M') \quad \text{and} \quad X(V, M) = h^* X(V', M').$$

Hence we have by Theorem 2

Corollary to Theorem 2. An orientation ^(and order) preserving

PL homeomorphism $h: V \rightarrow V'$ extends to an orientation preserving PL homeomorphism of ^{some} neighborhoods U and U' of V and V' in M and M' (, called a micro-equivalence,) if and only if $h^* \sigma(V', M') = \sigma(V, M)$ and $h^* X(V', M') = X(V, M)$.

Now we are ready to prove Theorem stated in the introduction.

Proof of Theorem. We regard of (M, V) and (M, V') as to be compatibly triangulated.

The only if part is immediate from Theorem 2.

We are going to prove that if there is a bijection $\alpha: \Sigma V \rightarrow \Sigma V'$ such that two oriented pairs (S_α, K_α) and $(S'_{\alpha(x)}, K'_{\alpha(x)})$ are PL homeomorphic (preserving orientations) and if V and V' represent the same homology class in M , i.e., $i_\alpha[V] = i'_\alpha[V']$, then (M, V) and (M, V') are micro-equivalent. Let $H_\Sigma: (S_\Sigma, K_\Sigma) \rightarrow (S'_\Sigma, K'_\Sigma)$ be an orientation

preserving) PL homeomorphism such that $H_\Sigma(S_x, K_x) = (S'_{x(x)}, K_{x(x)})$ for each $x \in \Sigma V$. By §2, Example 2, normalizations $\tilde{V}$ and $\tilde{V}'$ are PL homeomorphic.

Since V_0 and V_0' are obtained from $\tilde{V}$ and $\tilde{V}'$ by deleting the interiors of disks on $\tilde{V}$ and $\tilde{V}'$, it follows from the homogeneity of disks on a surface that H_Σ / K_Σ can be extended to an orientation

preserving PL homeomorphism $h_0: V_0 \rightarrow V_0'$.

By cone extensions, $h_0: V_0 \rightarrow V_0'$ can be extended to a PL homeomorphism $h: V \rightarrow V'$.

From the construction, we have that $h^* \sigma(V', M) = \sigma(V, M)$. Moreover, V and V' represent the same homology class in M . (and $\hat{h}_* [V] = [V']$) Hence $h^* X(V, M) = X(V', M)$.

It follows from Corollary to Theorem 2 that

(M, V) and (M, V') are micro-equivalent, completing the proof.

Remark. Theorem still holds without hypothesis of irreducibility of V , if we consider "orders" of V and link types.

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