

Topological groups, k -networks, and weak topology

广西大学 刘川 (Chuan Liu)¹

神奈川大学 酒井政美 (Masami Sakai)

東京学芸大学 田中祥雄 (Yoshio Tanaka)

Let G be a topological group. Then, we give affirmative answers to (Q1), and partial answers to (Q2) and (Q3) in the following questions.

(Q1) (A) Let G have a σ -hereditarily closure-preserving k -network. Is G an $\aleph$ -space?

(B) Let G be a k -space with a star-countable k -network. Is G an $\aleph$ -space?

(Q2) Let G be the quotient s -image of a metric space. Is G paracompact (or, meta-Lindelöf)?

(Q3) (A. V. Arhangel'skii). Let G be a sequential space. Does G contain no (closed) copy of S_{ω_1} ?

Let us recall some definitions which will be used in this paper.

A family $\{A_\alpha : \alpha \in I\}$ of subsets of a space X is *hereditarily closure-preserving* (simply, HCP) if $\bigcup\{cl B_\alpha : \alpha \in J\} = cl(\bigcup\{B_\alpha : \alpha \in J\})$, whenever $J \subset I$ and $B_\alpha \subset A_\alpha$ for each $\alpha \in J$.

Let $\mathcal{P}$ be a cover of a space X . Then, $\mathcal{P}$ is a k -network for X , if whenever $K \subset U$ with K compact and U open in X , $K \subset \bigcup \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. When a k -network $\mathcal{P}$ is a closed cover, then $\mathcal{P}$ is called a *closed k -network*.

Recall that a space is an $\aleph$ -space (resp. $\aleph_0$ -space) if it has a σ -locally finite k -network (resp. countable k -network).

Following [GMT], a space X is *determined by* a cover $\mathcal{C}$, if $F \subset X$ is closed in X iff $F \cap C$ is closed in C for every $C \in \mathcal{C}$. We use " X is determined by $\mathcal{C}$ " instead of the usual " X has the weak topology with respect to $\mathcal{C}$ ". Obviously, every space X is determined by any open cover, or any HCP closed cover of X .

¹Current address: Ohio University, Athens, Ohio 45701, U.S.A

A space is a *k-space* (resp. *sequential space*) if it is determined by a cover of compact subsets (resp. compact metric subsets). As is well-known, every *k-space* (resp. *sequential space*) is precisely the quotient image of a locally compact space (resp. (locally compact) metric space).

A space X has *countable tightness* ($= t(X) \leq \omega$) if, whenever $x \in clA$, then $x \in clB$ for some countable subset B with $B \subset A$. As is well-known, $t(X) \leq \omega$ iff X is determined by a cover of countable subsets.

Let us recall canonical quotient spaces S_α , and the *Arens' space* S_2 .

For an infinite cardinal α , S_α is the space obtained from the topological sum of α convergent sequences by identifying all the limit points to a single point. In particular, S_ω is called the *sequential fan*.

Let $L = \{a_n : n \in \omega\}$ be an infinite sequence with a limit point $\infty \notin L$. Let L_n ($n \in \omega$) be an infinite sequence with a limit point $a_n \notin L_n$. Then, S_2 is the space obtained from the topological sum of L and these L_n by identifying each $a_n \in L$ with the limit point a_n of L_n .

We assume that spaces are regular and T_1 , and maps are continuous and onto.

Results

Lemma 1. ([JZ]) Let X be a space with a σ -HCP k -network. Then, X is an $\aleph$ -space if and only if X contains no (closed) copy of S_{ω_1} .

Every Fréchet space X with a σ -HCP k -network (equivalently, X is a Lašnev space [F]) need not be an $\aleph$ -space; see Example 16(1). But, we have the following among topological groups.

Theorem 2. Let G be a topological group. If G has a σ -HCP k -network, then G is an $\aleph$ -space. (Affirmative answer to (A) in (Q1))

Corollary 3. Let G be a topological group which is the closed image of an $\aleph$ -space. Then, G is an $\aleph$ -space.

Remark 4. For a space X , the following decomposition theorems hold. (1) is due to [M] or [Ln], and (2) is due to [LT1].

(1) Let X be a space with a σ -HCP k -network. Then X , as well as every closed image of X , is decomposed into a σ -discrete space and an $\aleph$ -space.

(2) Let X be a Fréchet space with a star-countable k -network (more gen-

generally, point-countable k -network of separable subsets). Then X is decomposed into a closed discrete space and a space which is the topological sum of $\aleph_0$ -spaces. (The Fréchetness of X is essential; see Example 16(2)).

Let us consider topological groups having certain point-countable covers. The parenthetical part is due to [NT].

Lemma 5. Let $t(X) \leq \omega$. If X contains a copy of S_{ω_1} (resp. S_ω), then X contains a closed copy of S_{ω_1} (resp. S_ω).

For an infinite cardinal α , a space X is α -compact if every subset of cardinality α has an accumulation point in X . Clearly, Lindelöf spaces (resp. countably compact spaces) are ω_1 -compact (resp. ω -compact).

Corollary 6. Let $t(X) \leq \omega$. If X is determined by a point-countable (resp. point-finite) cover of ω_1 -compact (resp. ω -compact) subsets, then X contains no copy of S_{ω_1} (resp. S_ω).

In particular, S_{ω_1} (resp. S_ω) can not be embedded into any ω_1 -compact (resp. ω -compact) space of countable tightness.

Let us say that a cover $\mathcal{P}$ of X is a *cs-cover* of X if, for every infinite convergent sequence C in X , some $P \in \mathcal{P}$ contains at least two points of C . We note that S_{ω_1} has a point-countable *cs-cover* of two-point sets.

Theorem 7. Let G be a sequential group with a point-countable *cs-cover* of ω_1 -compact subsets. Then, G contains no copy of S_{ω_1} . (Partial answer to (Q3)).

Corollary 8. Let G be sequential group with a point-countable k -network of ω_1 -compact subsets. Then, G contains no copy of S_{ω_1} .

Lemma 9. Let G be a sequential topological group satisfying (a) and (b) below. Then, G is the topological sum of ω_1 -compact subsets.

In particular, if each element of $\mathcal{F}$ is cosmic (resp. compact), then G is the topological sum of cosmic subspaces (resp. σ -compact subspaces). Here, a space is *cosmic* if it has a countable network.

(a) G contains no (closed) copy of S_{ω_1} .

(b) G has a point-countable cover $\mathcal{F}$ such that $\mathcal{F}^* = \{\cup \mathcal{F}' : \mathcal{F}' \subset \mathcal{F}, \mathcal{F}' \text{ is finite}\}$ determines G ; and, any finite product of elements of $\mathcal{F}$ is ω_1 -compact.

Theorem 10. Let G be a topological group. If G is a k -space with a

point-countable k -network $\mathcal{P}$ of cosmic subspaces, then G is the topological sum of cosmic subspaces.

In particular, if G is a k -space with a star-countable k -network, then G is the topological sum of $\aleph_0$ -subspaces. (Affirmative answer to (B) in (Q1)).

Remark 11. In the previous theorem, the property " G is a k -space" is essential. According to [Tk2], under (CH) there exists a countably compact topological group G in which every compact set is finite, but G is not metrizable (cf. [Tk1]). Hence, the topological group G has a star-countable k -network of singletons, but not even a σ -space.

Let us recall that every CW-complex, more generally, every space *dominated* by k -and- $\aleph_0$ -subspaces is a k -space with a star-countable k -network ([IT]). (Conversely, every k -space with a star-countable k -network is a space *dominated* by k -and- $\aleph_0$ -subspaces ([S])). Then, the following holds by Theorem 7 and [T3; Corollary 6].

Corollary 12. Let K be a topological group. If K is a CW-complex, then K is the topological sum of countable CW-subcomplexes.

In the previous corollary, " K is a topological group" is essential, and the topological group K need not be metrizable; see Example 16.

Now, every quotient finite-to-one image of a locally compact metric space need not be paracompact, nor even meta-Lindelöf; see [GMT; Example 9.3]. But, we have the following among topological groups.

Theorem 13. Let $f : X \rightarrow G$ be a quotient s -map such that X is a locally separable metric space. If G is a topological group, then G is a paracompact space (actually, G is the topological sum of cosmic subspaces). (Partial answer to (Q2)).

In the previous theorem, the topological group G need not be metrizable by Example 16(3).

Similarly, we have the following since G is determined by a point-countable cover of compact subsets.

Theorem 14. Let $f : X \rightarrow G$ be a quotient s -map such that X is a locally compact paracompact space. If G is a sequential topological group, then G is a paracompact space (actually, G is the topological sum of σ -compact subspaces).

Remark 15. Let G be a topological group. Then, G is metrizable if the following (a), (b), or (c) holds. (Cf. [LST]).

(a) G is a k -space with a point-countable k -network, and G contains no closed copy of S_ω , or no S_2 .

(b) G is the quotient compact image of a metric space.

(c) G is a Fréchet space with a point-countable k -network. In particular, G is a Lašnev space, or a Fréchet space which is the quotient s -image of a metric space.

Example 16. (1) A Lašnev CW-complex K , but K is not an $\aleph$ -space.

(2) A CW-complex K which is not Fréchet, and K has the following properties. (Cf. [LT1]).

(a) K contains no copy of S_ω .

(b) K has a point-countable *closed* k -network.

(c) K has a star-countable k -network of separable metric subsets.

(d) K can not be decomposed into a σ -discrete space and a space with a σ -HCP k -network, or star-countable *closed* k -network.

(3) A topological group G which is a countable CW-complex (hence, an $\aleph_0$ -space), and G is the quotient countable-to-one image of a locally compact, separable metric space. But, G is not metrizable, not even Fréchet.

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