

On Separable Extensions of Noncommutative Rings

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In [2], K. Hirata and K. Sugano generalized the notion of separable algebras defining *separable* extensions of a ring. A ring extension T/S is called a *separable* extension, if the T - T -homomorphism of $T \otimes_S T$ onto T defined by $a \otimes b \rightarrow ab$ splits, and T/S is called an *H-separable* extension, if $T \otimes_S T$ is T - T -isomorphic to a direct summand of a finite direct sum of copies of T . As is well known every *H-separable* extension is a separable extension.

Throughout this paper, B will mean a ring with identity 1, ρ an automorphism of B , and Z the center of B . Let $B[X; \rho]$ be the skew polynomial ring in which the multiplication is given by $bX = X\rho(b)$ ($b \in B$). A monic polynomial f in $B[X; \rho]$ with $fB[X; \rho] = B[X; \rho]f$ is called a separable (resp. *H-separable*) polynomial if the factor ring $B[X; \rho]/fB[X; \rho]$ is a separable (resp. *H-separable*) extension of B .

Separable polynomials in skew polynomial rings are extensively studied by Kishimoto, Nagahara, Miyashita, Szeto, Xue and the author (see References). In [21, 22], Kishimoto studied some special type of separable polynomials in skew polynomial rings. In [27], Nagahara gave a thorough investigation of separable polynomial of degree 2. Miyashita [26] studied systematically separable polynomials and Frobenius polynomials. He give a characterization of a separable polynomial.

The following is a theorem of Y. Miyashita which characterizes separability of $X^n - u$ in $B[X; \rho]$.

Proposition 1. ([26, Theorem 3.1]) *Let $f = X^n - u$ be in $B[X; \rho]$. Then the following conditions are equivalent:*

- (1) *f is a separable polynomial in $B[X; \rho]$.*
- (2) (a) $\rho(u) = u$, and $\alpha u = u\rho^n(\alpha)$ for all $\alpha \in B$,
(b) *u is invertible in $U(B^\rho)$, and there exists an element $z \in Z$ such that $z + \rho(z) + \cdots + \rho^{n-1}(z) = 1$.*

Remark 0.1. The condition (2)(a) in the Proposition 1 is equivalent to $(X^n - u)B[X; \rho] = B[X; \rho](X^n - u)$.

In [9, 10, 11], the author has studied *H-separable* polynomials in skew polynomial rings. If the coefficient ring is commutative, the existence of *H-separable* polynomials in skew polynomial rings has been characterized in terms of Azumaya algebras and Galois extensions. In [10], the author proved that $B[X; \rho]$ contains an *H-separable polynomial* of prime degree if and only if the center Z of B is a Galois extension over

This is an abstract and the details will be published elsewhere.

Z^p . In [19], G. Szeto and L. Xue has succeeded in general degree case. The following is their theorem.

Proposition 2. ([19, Theorem 3.6]) *Let $f = X^n - u$ be in $B[X; \rho]$. Then the following conditions are equivalent:*

- (1) *f is an H -separable polynomial in $B[X; \rho]$.*
- (2) (a) *$\rho(u) = u$, and $\alpha u = u\rho^n(\alpha)$ for all $\alpha \in B$,*
 (b) *u is invertible in $U(B^p)$, and Z/Z^p is a G -Galois extension, where G is the group generated by $\rho|Z$ of degree n .*

The purpose of this paper is to generalize these results to the skew polynomial rings in several variables. We need some notations as in K. Kishimoto [21], S. Ikehata [5] and S. A. Amitsur and D. Saltman [1].

Let ρ_i ($1 \leq i \leq e$) be automorphisms of a ring B , and let u_{ij} ($1 \leq i, j \leq e$) be invertible elements in B such that

- (i) $u_{ij} = u_{ji}^{-1}$, and $u_{ii} = 1$,
- (ii) $\rho_i \rho_j \rho_i^{-1} \rho_j^{-1} = (u_{ij})_l (u_{ij}^{-1})_r$,
- (iii) $u_{ij} \rho_j(u_{ik}) u_{jk} = \rho_i(u_{jk}) u_{ik} \rho_k(u_{ij})$.

Then the set of all polynomials in e indeterminates

$$\left\{ \sum X_1^{\nu_1} X_2^{\nu_2} \cdots X_e^{\nu_e} b_{\nu_1 \nu_2 \dots \nu_e} \mid b_{\nu_1 \nu_2 \dots \nu_e} \in B, \nu_k \geq 0 \right\}$$

forms an associative ring if we define the multiplication by the distributive law and the rules

$$aX_i = X_i \rho_i(a) \quad (a \in B) \quad \text{and} \quad X_i X_j = X_j X_i u_{ij} \quad (1 \leq i, j \leq e).$$

This ring is denoted by $\mathbf{R}_e = B[X_1, X_2, \dots, X_e; \rho_1, \rho_2, \dots, \rho_e; \{u_{ij}\}]$ and is called a skew polynomial ring of automorphism type. Moreover, by $\mathbf{R}_k$ ($0 \leq k \leq e$), we denote the skew polynomial ring $B[X_1, X_2, \dots, X_k; \rho_1, \rho_2, \dots, \rho_k; \{u_{ij}\}]$ which is a subring of $\mathbf{R}_e$, where $\mathbf{R}_0 = B$.

Remark 0.2. For a permutation π of $\{1, 2, \dots, k\}$ ($k \leq e$), we have a B -ring automorphism $\mathbf{R}_k \cong B[X_{\pi(1)}, X_{\pi(2)}, \dots, X_{\pi(k)}; \rho_{\pi(1)}, \rho_{\pi(2)}, \dots, \rho_{\pi(k)}; \{u_{\pi(i)\pi(j)}\}]$ which maps X_i to $X_{\pi(i)}$ ($1 \leq i \leq k$)

Remark 0.3. ρ_{k+1} can be extended to an automorphism ρ_{k+1}^* of $\mathbf{R}_k$ by $\rho_{k+1}^*(X_j) = X_j u_{jk+1}$ ($1 \leq j \leq k$) and $\rho_{k+1}^*|B = \rho_{k+1}$. Moreover there holds $\mathbf{R}_{k+1} \cong \mathbf{R}_k[X_{k+1}; \rho_{k+1}^*]$.

Now, assume further that there exist elements u_i ($1 \leq i \leq e$) in B such that

$$(iv) \quad bu_i = u_i \rho_i^{m_i}(b) \quad (b \in B)$$

and

$$(v) \quad \rho_j(u_i) u_{ji} \rho_i(u_{ji}) \cdots \rho_i^{m_i-1}(u_{ji}) = u_i \quad (1 \leq i \leq e).$$

Then we have,

$$a(X_i^{m_i} - u_i) = (X_i^{m_i} - u_i) \rho_i^{m_i}(a) \quad (a \in B)$$

and

$$X_j(X_i^{m_i} - u_i) = (X_i^{m_i} - u_i) X_j u_{ji} \rho_i(u_{ji}) \cdots \rho_i^{m_i-1}(u_{ji}) \quad (1 \leq i, j \leq e).$$

This means $(X_i^{m_i} - u_i) \mathbf{R}_k$ is a two-sided ideal of $\mathbf{R}_k$ for $i \leq k \leq e$. The mapping $\bar{\rho}_i : \mathbf{R}_e \rightarrow \mathbf{R}_e$ defined by

$$\bar{\rho}_i(\sum X_1^{\nu_1} X_2^{\nu_2} \cdots X_e^{\nu_e} b_{\nu_1 \nu_2 \cdots \nu_e}) = \sum (X_1 u_{1i})^{\nu_1} (X_2 u_{2i})^{\nu_2} \cdots (X_e u_{ei})^{\nu_e} \rho_i(b_{\nu_1 \nu_2 \cdots \nu_e})$$

is an automorphism of $\mathbf{R}_e$ which is an extension of ρ_i .

We put here

$$\mathbf{B}_i = B[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_e; \rho_1, \dots, \rho_{i-1}, \rho_{i+1}, \dots, \rho_e; \{u_{ij}\}].$$

Naturally, we have

$$\mathbf{R}_e = \mathbf{B}_i[X_i; \bar{\rho}_i], \quad \text{and}$$

$$\beta(X_i^{m_i} - u_i) = (X_i^{m_i} - u_i) \bar{\rho}_i^{m_i}(\beta) \quad (\beta \in \mathbf{B}_i) \quad \text{and} \quad \bar{\rho}_i(u_i) = u_i,$$

where $\bar{\rho}_i$ means $\bar{\rho}_i|_{\mathbf{B}_i}$.

Let $\mathbf{M} = (X_1^{m_1} - u_1, X_2^{m_2}, \dots, X_e^{m_e} - u_e)$ be the two sided ideal of $\mathbf{R}_e$ generated by $\{X_1^{m_1} - u_1, X_2^{m_2}, \dots, X_e^{m_e} - u_e\}$. Then the factor ring $\mathbf{R}_e/\mathbf{M}$ is a free ring extension over B with a basis

$$\{x_1^{\nu_1} x_2^{\nu_2} \cdots x_e^{\nu_e} \mid 0 \leq \nu_i < m_i, 1 \leq i \leq e\}, \quad \text{where} \quad x_i = X_i + \mathbf{M} \in \mathbf{R}_e/\mathbf{M}.$$

Under the above notations, we shall state the following theorem which is a generalization of Proposition 1.

Theorem 3. *The following are equivalent.*

- (1) $\mathbf{R}_e/\mathbf{M}$ is a separable extension of B .
- (2) (a) $u_i \in U(B^{\rho_i})$ ($1 \leq i \leq e$).
- (b) There exists an element $z \in Z$ such that

$$\sum_{i=1}^e \sum_{0 \leq \nu_i < m_i} \rho_1^{\nu_1} \rho_2^{\nu_2} \cdots \rho_e^{\nu_e}(z) = 1$$

- (3) $X_i^{m_i} - u_i$ is a separable polynomial in $\mathbf{B}_i[X_i; \bar{\rho}_i]$ for each i ($1 \leq i \leq e$).
- (4) (a) $u_i \in U(B^{\rho_i})$ ($1 \leq i \leq e$).
- (b) There exist elements $c_i \in Z^{\rho_1, \rho_2, \dots, \rho_{i-1}, \rho_{i+1}, \dots, \rho_e}$ such that

$$c_i + \rho_i(c_i) + \cdots + \rho_i^{m_i-1}(c_i) = 1$$

To state the result concerning H -separable extensions, we need some more notations.

$$S_0 = B \text{ and } S_1 = B[X_1; \rho_1]/(X_1^{m_1} - u_1)B[X_1; \rho_1].$$

For $1 \leq k \leq e$, We put here $S_k = S_{k-1}[X_k; \bar{\rho}_k]/(X_k^{m_k} - u_k)S_{k-1}[X_k; \bar{\rho}_k]$.
Naturally, we have

$$R_e/M = S_e \supset S_{e-1} \supset \cdots \supset S_1 \supset S_0 = B.$$

Under the above notations, we have the following:

Theorem 4. *The following are equivalent.*

- (1) R_e/M is an H -separable extension of B , and the centralizers of B in R_e/M , $V_{R_e/M}(B) = Z$.
- (2) $X_k^{m_k} - u_k$ is an H -separable polynomial in $S_{k-1}[X_k; \bar{\rho}_k]$ for each k ($1 \leq k \leq e$).
- (3) (a) $u_i \in U(B^{\rho_i})$ ($1 \leq i \leq e$)
(b) The order of $(\rho_i|Z) = m_i$ ($1 \leq i \leq e$), the set $\{\rho_i|Z \mid 1 \leq i \leq e\}$ generates an abelian group $\langle \rho_1|Z \rangle \times \langle \rho_2|Z \rangle \times \cdots \times \langle \rho_e|Z \rangle = G$, and Z/Z^G is a G -Galois extension.

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