

The syntomic realization functor for Shimura varieties of abelian type

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Abstract

This is a survey article on prismatic F -gauges and their application to Shimura varieties.

1 Introduction

Let (G, X) be a Shimura datum, and K a compact open subgroup of $G(\mathbb{A}_f)$. Let E be the reflex field of (G, X) . We write $\mathrm{Sh}_K(G, X)$ for the Shimura variety over E attached to (G, X) and K . This should be “a moduli of G -motives of type X with K -level structure”.

We define G^c as in [KSZ21, 1.5.8], which is a quotient of G by a subtorus of the center of G . We have $G^c = G$ if (G, X) is of Hodge type (*cf.* [IKY23, Remark 2.6]).

Let p be a prime number. Assume that $G_{\mathbb{Q}_p}$ has a reductive model $\mathcal{G}$ over $\mathbb{Z}_p$. We assume that $K_p = \mathcal{G}(\mathbb{Z}_p)$. Let $\mathcal{G}^c$ be an integral analogue of G^c define in [IKY23, §2.3]. Let v be a prime of E above p . We can construct a tensor functor

$$\omega_{\mathrm{et}}: \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c) \rightarrow \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_{K, E_v})$$

using a tower of Shimura varieties, where $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c)$ is the category of finite rank algebraic representations of $\mathcal{G}^c$ over $\mathbb{Z}_p$ and $\mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_{K, E_v})$ is the category of étale $\mathbb{Z}_p$ -local systems on Sh_{K, E_v} .

Assume (G, X) is of abelian type. Then there is a canonical integral model $\mathcal{S}_K$ of Sh_{K, E_v} constructed by Kisin in [Kis10]. In [Lov17], Lovering constructed a tensor functor

$$\omega_{\mathrm{crys}}: \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c) \rightarrow \mathrm{FilF}\text{-}\mathrm{Crys}(\widehat{\mathcal{S}}_K)$$

compatible with ω_{et} , where $\mathrm{FilF}\text{-}\mathrm{Crys}(\widehat{\mathcal{S}}_K)$ is a category of filtered F -crystals on the p -adic completion $\widehat{\mathcal{S}}_K$ of $\mathcal{S}_K$.

If (G, X) is of Hodge type, a shtuka realization compatible with ω_{et} is constructed in [PR24].

In [IKY23], we give a refinement of these realizations in prismatic F -gauges.

2 Prismatic F -crystal

We recall the notion of prism and prismatic F -crystal from [BS22] and [BS23].

Definition 2.1. A δ -ring is a commutative $\mathbb{Z}_{(p)}$ -algebra A with a map $\delta: A \rightarrow A$ such that $\delta(1) = 0$ and

$$\begin{aligned}\delta(x + y) &= \delta(x) + \delta(y) + \frac{x^p + y^p - (x + y)^p}{p}, \\ \delta(xy) &= x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)\end{aligned}$$

for any $x, y \in A$.

For a δ -ring A with δ , we define a Frobenius lift $\phi: A \rightarrow A$ by $\phi(x) = x^p + p\delta(x)$. A morphism of δ -rings is a ring homomorphism that intertwines the δ -structures.

Example 2.2 ([Joy85]). For a commutative ring R , let $W(R)$ be the ring of p -typical Witt vectors in R , and let

$$\delta_{W(R)}: W(R) \rightarrow W(R); (x_0, x_1, \dots) \mapsto (x_1, x_2, \dots).$$

Then $W(R)$ with $\delta_{W(R)}$ is a δ -ring. The functor $R \mapsto (W(R), \delta_{W(R)})$ gives the right adjoint of the forgetful functor from the category of δ -rings to the category of commutative rings.

Definition 2.3. A prism is a pair (A, I) where A is a δ -ring and $I \subset A$ is an invertible ideal such that A is derived (p, I) -adically complete, and $I + \phi(I)$ contains p .

For a prism (A, I) , we equip A with the (p, I) -adic topology.

Definition 2.4. We say that a commutative ring R has bounded p^∞ -torsion if there is some positive integer n such that $R[p^\infty] = R[p^n]$. We say that a prism (A, I) is bounded if A/I has bounded p^∞ -torsion.

Let $\mathfrak{X}$ be a p -adic formal scheme.

Definition 2.5. The absolute prismatic site $\mathfrak{X}_\Delta$ of $\mathfrak{X}$ is the opposite category of the category of bounded prisms (A, I) with $\mathrm{Spf}(A/I) \rightarrow \mathfrak{X}$, with topology given by flat covers of prisms.

We define presheaves $\mathcal{O}_\Delta$ and $\mathcal{I}_\Delta$ on $\mathfrak{X}_\Delta$ by $\mathcal{O}_\Delta(A, I) = A$ and $\mathcal{I}_\Delta(A, I) = I$. These are sheaves by [BS22, Corollary 3.12].

Definition 2.6. A prismatic F -crystal on $\mathfrak{X}$ is a vector bundle $\mathcal{E}$ on $\mathfrak{X}_\Delta$ with an isomorphism $\varphi_{\mathcal{E}}: (\phi^*\mathcal{E})[\frac{1}{\mathcal{I}_\Delta}] \xrightarrow{\sim} \mathcal{E}[\frac{1}{\mathcal{I}_\Delta}]$.

Let $\mathrm{Vect}^\varphi(\mathfrak{X}_\Delta)$ be the category of prismatic F -crystals on $\mathfrak{X}$.

Example 2.7. If $\mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space over $\mathbb{F}_p$ with an untilt $\mathrm{Spa}(R^\sharp, R^{+\sharp})$ and a morphism $\mathrm{Spf}(R^{+\sharp}) \rightarrow \mathfrak{X}$, then $(W(R^+), \mathrm{Ker}(W(R^+) \rightarrow R^{+\sharp}))$ with $\mathrm{Spf}(R^{+\sharp}) \rightarrow \mathfrak{X}$ gives an object of $\mathfrak{X}_\Delta$. Using the value of $(\mathcal{E}, \varphi_{\mathcal{E}})$ at this object, we can obtain a shtuka with one leg at $\mathrm{Spa}(R^\sharp, R^{+\sharp})$.

The construction in Example 2.7 gives a connection between the prismatic theory and shtukas. See [IKY24, Theorem 3] for details.

3 Prismatic F -gauge

The notion of prismatic F -gauge is introduced by Bhatt–Lurie (*cf.* [Bha23]) based on previous works [Dri24], [BL22a], [BL22b] on the stacky approach to the integral p -adic Hodge theory.

Let W be the ring scheme of p -typical Witt vectors over $\mathbb{Z}_{(p)}$. We have the Frobenius morphism $F: W \rightarrow W$.

Definition 3.1. *Let R be a p -nilpotent ring. A Cartier–Witt divisor over R is a morphism $\alpha: I \rightarrow W(R)$ of $W(R)$ -modules, where I is an invertible $W(R)$ -module, such that the image of*

$$I \xrightarrow{\alpha} W(R) \xrightarrow{\gamma_0} R$$

is a nilpotent ideal, where γ_0 is the projection to the 0-th component, and the image of

$$I \xrightarrow{\alpha} W(R) \xrightarrow{\delta_{W(R)}} W(R)$$

generates the unit ideal.

We assume that $\mathfrak{X}$ is bounded in the sense that for any affine open U of $\mathfrak{X}$ the ring $\mathcal{O}_{\mathfrak{X}}(U)$ has bounded p^∞ -torsion.

Definition 3.2. *The prismaticization of $\mathfrak{X}$ is a formal stack $\mathfrak{X}^\Delta$ over $\mathbb{Z}_p$ defined as follows: For a p -nilpotent ring R , the groupoid $\mathfrak{X}^\Delta(R)$ consists of Cartier–Witt divisors $I \rightarrow W(R)$ with a morphism $\mathrm{Spec}(\mathrm{Cone}(I \rightarrow W(R))) \rightarrow \mathfrak{X}$ of derived formal schemes.*

For a p -nilpotent ring R and Cartier–Witt divisor $I \xrightarrow{\alpha} W(R)$ over R , the induced map $F^*I \xrightarrow{F^*\alpha} W(R)$ is also a Cartier–Witt divisor over R and we have the map

$$\mathrm{Cone}(I \xrightarrow{\alpha} W(R)) \rightarrow \mathrm{Cone}(F^*I \xrightarrow{F^*\alpha} W(R))$$

of animated rings induced from F . This gives a morphism $\phi_{\mathfrak{X}^\Delta}: \mathfrak{X}^\Delta \rightarrow \mathfrak{X}^\Delta$.

We define a group scheme $\mathbb{G}_a^\sharp$ over $\mathbb{Z}_{(p)}$ by

$$\mathbb{G}_a^\sharp = \mathrm{Ker}(F: W \rightarrow F_*W).$$

Definition 3.3. *Let R be a p -nilpotent ring.*

- (1) *A $\sharp$ -invertible W -module over R is an affine W -module scheme over R that is fpqc locally isomorphic to $\mathbb{G}_a^\sharp$ as W -modules.*
- (2) *An admissible W -module over R is an affine W -module scheme over R that can be realized as an extension of an invertible F_*W -module over R by a $\sharp$ -invertible W -module over R .*

Proposition 3.4 ([Bha23, Remark 5.2.5]). *Let R be a p -nilpotent ring. Let M and N be admissible W -modules over R which sit in short exact sequences*

$$0 \rightarrow L_M \rightarrow M \rightarrow F_*I_M \rightarrow 0, \quad (*_M)$$

$$0 \rightarrow L_N \rightarrow N \rightarrow F_*I_N \rightarrow 0, \quad (*_N)$$

*where L_M, L_N are $\sharp$ -invertible W -modules over R , and I_M and I_N are invertible W -module over R . Then any morphism $M \rightarrow N$ of W -modules induces a morphism of short exact sequences from $(*_M)$ to $(*_N)$.*

*In particular, the short exact sequence $(*_M)$ is unique up to unique isomorphism.*

Definition 3.5. Let R be a p -nilpotent ring. A filtered Cartier–Witt divisor over R is a morphism $d: M \rightarrow W$ of W -modules over R , where M is an admissible W -module over R , such that the induced morphism

$$F_* I_M \rightarrow F_* W \quad (3.1)$$

comes from a Cartier–Witt divisor over R via F_* , where (3.1) is induced by d , $(*_M)$ and the sequence

$$0 \rightarrow \mathbb{G}_a^\sharp \rightarrow W \xrightarrow{F} F_* W \rightarrow 0$$

using Proposition 3.4.

Definition 3.6. The filtered prismatization of $\mathfrak{X}$ is a formal stack $\mathfrak{X}^\mathcal{N}$ over $\mathbb{Z}_p$ defined as follows: For a p -nilpotent ring R , the groupoid $\mathfrak{X}^\mathcal{N}(R)$ consists of filtered Cartier–Witt divisors $M \rightarrow W$ with a morphism $\mathrm{Spec}((W/M)(R)) \rightarrow \mathfrak{X}$ of derived formal schemes, where

$$(W/M)(R) = \mathrm{R}\Gamma(\mathrm{Spec}(R), \mathrm{Cone}(M \rightarrow W)).$$

Definition 3.7. Let R be a p -nilpotent ring.

Associating a filtered Cartier–Witt divisor $d: M \rightarrow W$ over R with the Cartier–Witt divisor over R giving (3.1), we obtain a morphism $\pi: \mathfrak{X}^\mathcal{N} \rightarrow \mathfrak{X}^\Delta$.

Associating a Cartier–Witt divisor $I \xrightarrow{\alpha} W(R)$ with the filtered Cartier–Witt divisor

$$I \otimes_{W(R)} W \xrightarrow{\alpha \otimes \mathrm{id}_W} W(R) \otimes_{W(R)} W \cong W,$$

we obtain a morphism $j_{\mathrm{HT}}: \mathfrak{X}^\Delta \rightarrow \mathfrak{X}^\mathcal{N}$.

Associating a Cartier–Witt divisor $I \xrightarrow{\alpha} W(R)$ with the filtered Cartier–Witt divisor M given by the following pullback

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{G}_a^\sharp & \longrightarrow & M & \longrightarrow & F_*(I \otimes_{W(R)} W) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow F_*(\alpha \otimes \mathrm{id}_W) \\ 0 & \longrightarrow & \mathbb{G}_a^\sharp & \longrightarrow & W & \xrightarrow{F} & F_* W \longrightarrow 0, \end{array}$$

we obtain a morphism $j_{\mathrm{dR}}: \mathfrak{X}^\Delta \rightarrow \mathfrak{X}^\mathcal{N}$.

By the construction, we have $\pi \circ j_{\mathrm{HT}} = \phi_{\mathfrak{X}^\Delta}$ and $\pi \circ j_{\mathrm{dR}} = \mathrm{id}_{\mathfrak{X}^\Delta}$. Further, j_{HT} and j_{dR} are open immersion with the disjoint images (cf. [Bha23, Remark 5.3.6]).

Example 3.8 ([Bha23, §5.5.1]). Assume that $\mathfrak{X} = \mathrm{Spf} R$ where R is a perfectoid ring in the sense of [BMS18, Definition 3.5]. Then the natural homomorphism

$$\theta: W(R^\flat) \rightarrow R$$

defined in [BMS18, §3.1] is surjective by [BMS18, Lemma 3.9]. We put $(\Delta_R, I) = (W(R^\flat), \ker \theta)$. Let ϕ act on $\Delta_R = W(R^\flat)$ via the p -th power map on $R^\flat$. We define a filtration $\mathrm{Fil}_\bullet^\mathcal{N} \Delta_R$ of Δ_R by

$$\mathrm{Fil}_\mathcal{N}^i \Delta_R = \begin{cases} \phi^{-1}(I^i) & \text{if } i \geq 0, \\ \Delta_R & \text{if } i < 0. \end{cases}$$

We put

$$\mathrm{Rees}(\mathrm{Fil}_{\mathcal{N}}^{\bullet} \Delta_R) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}_{\mathcal{N}}^i \Delta_R t^{-i} \subset \Delta_R[t, t^{-1}].$$

Let $\mathbb{G}_m$ act on this ring by i -th power on $\mathrm{Fil}_{\mathcal{N}}^i \Delta_R t^{-i}$. Then

$$\mathfrak{X}^{\Delta} = \mathrm{Spf} \Delta_R, \quad \mathfrak{X}^{\mathcal{N}} = [\mathrm{Spec}(\mathrm{Rees}(\mathrm{Fil}_{\mathcal{N}}^{\bullet} \Delta_R)) / \mathbb{G}_m] \times_{\mathrm{Spec} \Delta_R} \mathrm{Spf} \Delta_R,$$

and $\pi: \mathfrak{X}^{\mathcal{N}} \rightarrow \mathfrak{X}^{\Delta}$ is the natural projection. Further, j_{HT} and j_{dR} are induced from

$$\mathrm{Rees}(\mathrm{Fil}_{\mathcal{N}}^{\bullet} \Delta_R) \xrightarrow{\phi} \bigoplus_{i \in \mathbb{Z}} I^i t^{-i} \quad \text{and} \quad \mathrm{Rees}(\mathrm{Fil}_{\mathcal{N}}^{\bullet} \Delta_R) \hookrightarrow \Delta_R[t^{\pm 1}]$$

respectively.

Definition 3.9. We define the syntomification $\mathfrak{X}^{\mathrm{syn}}$ of $\mathfrak{X}$ by the cocartesian diagram

$$\begin{array}{ccc} \mathfrak{X}^{\Delta} \amalg \mathfrak{X}^{\Delta} & \xleftarrow{j_{\mathrm{HT}} \amalg j_{\mathrm{dR}}} & \mathfrak{X}^{\mathcal{N}} \\ \downarrow & & \downarrow \\ \mathfrak{X}^{\Delta} & \longrightarrow & \mathfrak{X}^{\mathrm{syn}}. \end{array}$$

The category of prismatic F -gauges in vector bundles is defined as the category of vector bundles on $\mathfrak{X}^{\mathrm{syn}}$, for which we write $\mathrm{Vect}(\mathfrak{X}^{\mathrm{syn}})$.

4 Tannakian framework

Let k be a perfect field of characteristic p . Assume that $\mathfrak{X}$ is a smooth p -adic formal scheme over $W(k)$.

For an object $(\mathcal{E}, \varphi_{\mathcal{E}})$ of $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$, we define the filtration $\mathrm{Fil}_{\mathrm{Nyg}}^{\bullet}(\phi^* \mathcal{E}) \subset \phi^* \mathcal{E}$ by $\mathcal{O}_{\Delta}$ -submodules, which we call the Nygaard filtration, so that

$$\mathrm{Fil}_{\mathrm{Nyg}}^r(\phi^* \mathcal{E})(A, I) = \{x \in \phi^* \mathcal{E}(A, I) \mid \varphi_{\mathcal{E}}(x) \in I^r \mathcal{E}(A, I)\}$$

for an object (A, I) of $\mathfrak{X}_{\Delta}$.

Let $\mathrm{Vect}^{\varphi, \mathrm{lff}}(\mathfrak{X}_{\Delta})$ be the category of prismatic F -crystal on $\mathfrak{X}$ with locally filtered free Nygaard filtration.

Let $\mathcal{G}$ be a smooth group scheme over $\mathbb{Z}_p$. For a $\mathbb{Z}_p$ -linear tensor category $\mathcal{C}$, let $\mathcal{G}\text{-}\mathcal{C}$ denote the category of $\mathbb{Z}_p$ -linear exact tensor functors $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \mathcal{C}$. Using [GL23, Theorem 2.31], we can show the following:

Proposition 4.1 ([IKY23, §1.3]). *There is a bi-exact $\mathbb{Z}_p$ -linear tensor-equivalence*

$$\mathrm{Vect}(\mathfrak{X}^{\mathrm{syn}}) \cong \mathrm{Vect}^{\varphi, \mathrm{lff}}(\mathfrak{X}_{\Delta}).$$

In particular, we have an equivalence

$$\mathcal{G}\text{-}\mathrm{Vect}(\mathfrak{X}^{\mathrm{syn}}) \cong \mathcal{G}\text{-}\mathrm{Vect}^{\varphi, \mathrm{lff}}(\mathfrak{X}_{\Delta}).$$

Definition 4.2. *An analytic prismatic F -crystal on $\mathfrak{X}$ is a compatible system of pairs $(\mathcal{E}_{(A,I)}, \varphi_{(A,I)})$ of a vector bundle $\mathcal{E}_{(A,I)}$ on $\mathrm{Spec}(A) \setminus V(p, I)$ and $\varphi_{(A,I)}: (\phi^* \mathcal{E}_{(A,I)})[\frac{1}{I}] \xrightarrow{\sim} \mathcal{E}_{(A,I)}[\frac{1}{I}]$ for $(A, I) \in \mathfrak{X}_\Delta$, where $V(p, I)$ is the closed subscheme of $\mathrm{Spec} A$ defined by the ideal (p, I) .*

Let $\mathrm{Vect}^{\mathrm{an}, \varphi}(\mathfrak{X}_\Delta)$ be the category of analytic prismatic F -crystals on $\mathfrak{X}$. We write $\mathfrak{X}_\eta$ for the generic fiber of $\mathfrak{X}$. Let $\mathrm{Loc}_{\mathbb{Z}_p}^{\mathrm{crys}}(\mathfrak{X}_\eta)$ be the category of crystalline $\mathbb{Z}_p$ -local systems on $\mathfrak{X}_\eta$.

Theorem 4.3 ([DLMS24, Theorem 3.46], [GR24, Theorem A]). *There is an equivalence*

$$\mathrm{Vect}^{\mathrm{an}, \varphi}(\mathfrak{X}_\Delta) \xrightarrow{\sim} \mathrm{Loc}_{\mathbb{Z}_p}^{\mathrm{crys}}(\mathfrak{X}_\eta)$$

of categories.

The restriction functor $\mathrm{Vect}^\varphi(\mathfrak{X}_\Delta) \rightarrow \mathrm{Vect}^{\mathrm{an}, \varphi}(\mathfrak{X}_\Delta)$ is fully faithful by [GR24, Proposition 3.7].

We say that an object of $\mathrm{Loc}_{\mathbb{Z}_p}^{\mathrm{crys}}(\mathfrak{X}_\eta)$ is prismatically good reduction if it comes from $\mathrm{Vect}^\varphi(\mathfrak{X}_\Delta)$ under the equivalence in Theorem 4.3. Let $\mathrm{Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}(\mathfrak{X}_\eta) \subset \mathrm{Loc}_{\mathbb{Z}_p}^{\mathrm{crys}}(\mathfrak{X}_\eta)$ be the exact full subcategory of prismatically good reduction objects. By the definition, we have an equivalence

$$\mathrm{Vect}^\varphi(\mathfrak{X}_\Delta) \xrightarrow{\sim} \mathrm{Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}(\mathfrak{X}_\eta) \quad (4.1)$$

of categories.

Remark 4.4. *The equivalence in Theorem 4.3 is bi-exact, but the equivalence (4.1) may not be bi-exact. More precisely, a quasi-inverse of (4.1) may not be exact. The problem is that there is a non-exact sequence in $\mathrm{Vect}^\varphi(\mathfrak{X}_\Delta)$ whose image in $\mathrm{Vect}^{\mathrm{an}, \varphi}(\mathfrak{X}_\Delta)$ is exact.*

In spite of Remark 4.4, we still can show the following:

Theorem 4.5 ([IKY24, §2.4]). *Assume that $\mathcal{G}$ is a reductive group scheme over $\mathbb{Z}_p$. Then there is an equivalence*

$$\mathcal{G}\text{-Vect}^\varphi(\mathfrak{X}_\Delta) \xrightarrow{\sim} \mathcal{G}\text{-Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}(\mathfrak{X}_\eta)$$

of categories.

5 Integral crystalline functor

In this section, we construct the integral crystalline functor

$$\mathbb{D}_{\mathrm{crys}}: \mathrm{Vect}^\varphi(\mathfrak{X}_\Delta) \rightarrow \mathrm{VectNF}^\varphi(\mathfrak{X}_{\mathrm{crys}}),$$

where $\mathrm{VectNF}^\varphi(\mathfrak{X}_{\mathrm{crys}})$ is the category of naive filtered F -crystals on $\mathfrak{X}$ in the sense of [IKY25, §2.1.1]. In [BS23, Construction 4.12], Bhatt–Scholze constructed

$$\mathrm{Vect}^\varphi(\mathfrak{X}_\Delta) \rightarrow \mathrm{Vect}^\varphi(\mathfrak{X}_{\mathrm{crys}}); \mathcal{E} \mapsto \mathcal{E}^{\mathrm{crys}}.$$

Therefore it is enough to define a filtration on $\mathcal{E}^{\mathrm{crys}}$. Let

$$\mathbb{D}_{\mathrm{dR}}^+: \mathrm{Vect}^\varphi(\mathfrak{X}_\Delta) \rightarrow \mathrm{FilVect}(\mathfrak{X})$$

be the de Rham realization functor defined in [IKY25, §1.2]. As explained there, the filtration of $\mathbb{D}_{\mathrm{dR}}^+(\mathcal{E})$ can be described using the Nygaard filtration of $\phi^* \mathcal{E}$.

Theorem 5.1 ([IKY25, §1.3]). *For $\mathcal{E} \in \mathrm{Vect}^\varphi(\mathfrak{X}_\Delta)$, there is a canonical crystalline-de Rham comparison isomorphism $\mathcal{E}^{\mathrm{crys}} \cong \mathbb{D}_{\mathrm{dR}}^+(\mathcal{E})$.*

For $\mathcal{E} \in \mathrm{Vect}^\varphi(\mathfrak{X}_\Delta)$, we put $\mathbb{D}_{\mathrm{crys}}(\mathcal{E}) = \mathcal{E}^{\mathrm{crys}}$ and define the filtration on $\mathbb{D}_{\mathrm{crys}}(\mathcal{E})$ by the filtration on $\mathbb{D}_{\mathrm{dR}}^+(\mathcal{E})$ using the crystalline-de Rham comparison isomorphism in Theorem 5.1.

The functor $\mathbb{D}_{\mathrm{crys}}$ induces an equivalence between the category of prismatic F -crystals on $\mathfrak{X}$ with Hodge–Tate weights in $[0, p-2]$ and the category of Fontaine–Laffaille modules on $\mathfrak{X}$. See [IKY25, §3] for details.

6 Syntomic realization

Let $\mu_h: \mathbb{G}_{m, \mathcal{O}_{E_v}} \rightarrow \mathcal{G}_{\mathcal{O}_{E_v}}$ be the extension of a Hodge cocharacter of (G, X) . Let $\mu_h^c: \mathbb{G}_{m, \mathcal{O}_{E_v}} \rightarrow \mathcal{G}_{\mathcal{O}_{E_v}}^c$ be the cocharacter induced by μ_h . We assume that p is odd.

Theorem 6.1 ([IKY23, §2.4 and §2.6]). *There is a tensor functor*

$$\omega_{\mathrm{syn}}: \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c) \rightarrow \mathrm{Vect}(\widehat{\mathcal{S}}_K^{\mathrm{syn}})$$

of type μ_h^c that is compatible with ω_{crys} and ω_{et} .

For construction of ω_{syn} , first we show that ω_{et} belongs to $\mathcal{G}^c\text{-Loc}_{\mathbb{Z}_p}^{\Delta\text{-gr}}(\widehat{\mathcal{S}}_{K, \eta})$. Then we obtain an object of $\mathcal{G}^c\text{-Vect}^\varphi(\widehat{\mathcal{S}}_{K, \Delta})$ by Theorem 4.5. Next we show that the obtained object belongs to $\mathcal{G}^c\text{-Vect}^{\varphi, \mathrm{lf}}(\widehat{\mathcal{S}}_{K, \Delta})$. Then we can obtain ω_{syn} using Proposition 4.1.

We note that to make sense of the compatibility with ω_{crys} in Theorem 6.1, we need the functor $\mathbb{D}_{\mathrm{crys}}$ constructed in §5.

See [IKY23, §2.6.2] for cohomological consequences of Theorem 6.1.

7 Characterization of the integral model

Definition 7.1. *A potentially crystalline locus of $\mathrm{Sh}_{K, E_v}^{\mathrm{an}}$ is a quasi-compact open subspace U of $\mathrm{Sh}_{K, E_v}^{\mathrm{an}}$ uniquely determined by the following condition:*

For any classical point x of $\mathrm{Sh}_{K, E_v}^{\mathrm{an}}$, the point x is in U if and only if $\omega_{\mathrm{et}}(V)_x$ is potentially crystalline for any $V \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c)$.

Let $\mathrm{Sh}_{K, E_v}^{\mathrm{pc}} \subset \mathrm{Sh}_{K, E_v}^{\mathrm{an}}$ be the potentially crystalline locus constructed in [IM20, Theorem 5.17] (cf. [IM20, Remark 2.12] and [LZ17, Theorem 1.2]). We write ω_{pc} for the restriction of $\omega_{\mathrm{et}}^{\mathrm{an}}$ to $\mathrm{Sh}_{K, E_v}^{\mathrm{pc}}$.

We write $\mathrm{BT}^{\mathcal{G}^c, -\mu_h^c}$ for the moduli stack of prismatic F -gauges with $\mathcal{G}^c$ -structure of type $-\mu_h^c$ constructed by [GM24], confirming a conjecture in [Dri23]. Then we have $\widehat{\mathcal{S}}_K \rightarrow \mathrm{BT}^{\mathcal{G}^c, -\mu_h^c}$ given by ω_{syn} .

Definition 7.2. *A syntomic integral canonical model of Sh_{K, E_v} is a smooth, separated integral model $\mathcal{X}$ of Sh_{K, E_v} over $\mathcal{O}_{E_v}$ satisfying the following conditions:*

- (1) $\widehat{\mathcal{X}}_\eta = \mathrm{Sh}_{K, E_v}^{\mathrm{pc}}$.

(2) *There is a tensor functor*

$$\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^c) \rightarrow \mathrm{Vect}(\widehat{\mathcal{X}}^{\mathrm{syn}})$$

of type $-\mu_h^c$ whose étale realization is equal to ω_{pc} such that the resulting morphism $\widehat{\mathcal{X}} \rightarrow \mathrm{BT}^{\mathcal{G}^c, -\mu_h^c}$ is formally étale.

Remark 7.3. *To make sense of a condition like Definition 7.2 (2), the moduli of shtukas is not enough because the theory of v -sheaves can not distinguish nilpotent thickenings.*

Theorem 7.4 ([IKY23, §3.4]). *The scheme $\mathcal{S}_K$ over $\mathcal{O}_{E_v}$ is the unique syntomic integral canonical model.*

In the proof of Theorem 7.4, confirming the condition (1) in Definition 7.2 generalizes a result in [IM13, §7] for the PEL type case, and confirming the condition (2) in Definition 7.2 amounts to proving a Serre–Tate theorem for Shimura varieties of abelian type.

Remark 7.5. *The characterization in Theorem 7.4 works for each level K contrary to previously known characterization in [Kis10].*

We write $\overline{\mathcal{S}}_K$ and $\overline{\mathcal{G}}^c$ for the special fibers of $\mathcal{S}_K$ and $\mathcal{G}^c$ respectively. Let $\overline{\mathcal{G}}^c\text{-Zip}^{-\mu_h^c}$ be the moduli stack of $\overline{\mathcal{G}}^c$ -zips of type $-\mu_h^c$. As an application of the syntomic realization functor, we can obtain the following theorem:

Theorem 7.6 ([IKY23, §3.5]). *There is a natural smooth morphism*

$$\overline{\mathcal{S}}_K \rightarrow \overline{\mathcal{G}}^c\text{-Zip}^{-\mu_h^c}.$$

Theorem 7.6 generalizes [Zha18, Theorem 4.12] in the Hodge type case to the abelian type case. This allows us to apply results in [IK24] to Shimura varieties of abelian type as well.

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