

Basic Topics on Tropical Geometry and Singularities

(トロピカル幾何と特異性に関する基本的トピックス)

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Motivated from real algebraic geometry, Viro, Mikhalkin, Shustin, Itenberg, and other mathematicians have developed the “tropical (algebraic) geometry” [8]. Algebraic curves are tropicalized to piecewise-linear curves. The method was used to construct topological types of real algebraic curves in Hilbert’s 16th problem [24].

In this rough sketch, we present several basic topics of tropical geometry, in particular, the notion of hyperfields introduced by Viro recently [25].

【 Tropical Limits of Operations 】

Let $\mathbf{R}_+$ denote the set of non-negative real numbers.

We fix $h > 0$ and consider the bijection

$$h \log : \mathbf{R}_+ \rightarrow \mathbf{R} \cup \{-\infty\}$$

defined by

$$u \mapsto h \log u, \quad e^{\frac{x}{h}} \mapsto x.$$

On $\mathbf{R} \cup \{-\infty\}$, two operations

$$\begin{cases} x +_h y := h \log \left(e^{\frac{x}{h}} + e^{\frac{y}{h}} \right) \\ x \times_h y := h \log \left(e^{\frac{x}{h}} \cdot e^{\frac{y}{h}} \right) = x + y \end{cases}$$

are induced from the summation and the multiplication on $\mathbf{R}_+$.

Set $m = \max\{x, y\}$. Then we have

$$h \log \left(e^{\frac{m}{h}} \right) \leq x +_h y \leq h \log \left(e^{\frac{m}{h}} + e^{\frac{m}{h}} \right),$$

namely,

$$m \leq x +_h y \leq m + h \log 2.$$

Therefore we have that

$$\lim_{h \downarrow 0} (x +_h y) = \max\{x, y\}.$$

【 Tropical Semi-Ring 】

$\mathbf{R}_{\text{trop}} = \mathbf{R} \cup \{-\infty\}$ with the two operations

$$"x + y" := \max\{x, y\}, \quad "x \cdot y" := x + y,$$

is called the *tropical semi-ring* (or the *max-plus algebra*).

Moreover we set $"x/y" = x - y$ if $y \neq -\infty$. Note that there is no tropical subtraction. The tropical sum is idempotent :

$$"x + x" = x,$$

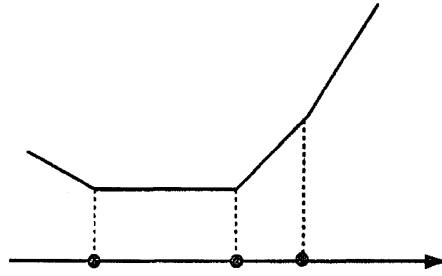
$-\infty$ being the tropical zero.

【 Tropical Polynomials 】

For a finite subset $A \subset \mathbf{Z}^n$, consider a "tropical" (Laurent) polynomial

$$F(x) = " \sum_{j \in A} c_j x^j " = \max\{c_j + j \cdot x \mid j \in A\},$$

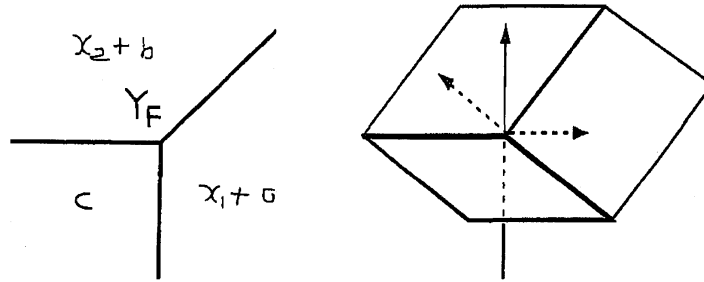
($c_j \in \mathbf{R}$), which is a PL-function on $\mathbf{R}^n$. Then the tropical hypersurface $Y_F \subset \mathbf{R}^n$ is defined by F as the *corner locus* of F .



Example 1. (tropical line). We consider

$$F(x_1, x_2) = "ax_1 + bx_2 + c" = \max\{x_1 + a, x_2 + b, c\}.$$

Then Y_F consists of three half-lines meeting at one point.



【 Tropical Hyperfields 】

We define a *multi-valued* addition γ on $\mathbf{R} \cup \{-\infty\}$: For $a, b \in \mathbf{R} \cup \{-\infty\}$, we set

$$a \gamma b := \begin{cases} \max\{a, b\}, & (a \neq b) \\ \{y \in \mathbf{R} \cup \{-\infty\} \mid y \leq a\}, & (a = b) \end{cases}$$

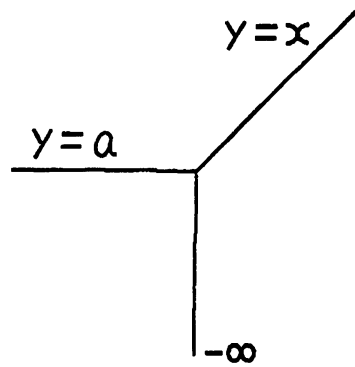
The multiplication is defined by the ordinary addition.

We set $\mathbb{Y} = (\mathbf{R} \cup \{-\infty\}, \gamma, +)$ and we call it the *tropical hyperfield*.

This implies the natural definition of "tropical zero".

Example 2. For $a \in \mathbf{R}$, we define the function $x \gamma a : \mathbb{Y} \rightarrow \mathbb{Y}$. Then we have

$$\text{graph}(x \gamma a) = \{y = a, x < a\} \cup \{y = x, a < x\} \cup \{y \leq a, x = a\}.$$



【 Definition of Hyperfields 】

Suppose there are given, on a set X , a multi-valued binary operation $\top$ and a single-valued binary operation $\cdot$.

Then $(X, \top, \cdot)$ is called a *hyperfield* if

- $a \top b = b \top a, \quad a \top (b \top c) = (a \top b) \top c$
- $\exists 0 \in X, \quad 0 \top a = a, \text{ for any } a \in X,$
- $\forall a \in X, \exists_1 -a \in X \text{ (minus } a) \text{ such that } 0 \in a \top (-a).$
- $c \in a \top b \iff (-c) \in (-a) \top (-b)$
- The operation $\cdot$ is commutative, associative and $0 \cdot a = 0$ holds for any $a \in X,$
- $(X \setminus \{0\}, \cdot)$ is a commutative group, which will be denoted by $X^\times,$
- the “distributive law” holds:

$$a \cdot (b \top c) = (a \cdot b) \top (a \cdot c), \quad (b \top c) \cdot a = (b \cdot a) \top (c \cdot a).$$

Lemma 3. *The tropical hyperfield $\mathbb{Y} = (\mathbf{R} \cup \{-\infty\}, \top, +)$ is a hyperfield.*

In fact we have

- The zero-element is $-\infty$.
- For $a \in \mathbb{Y}, -a$ equals a , since $-\infty \in a \top b \iff b = a.$
- The commutative group $\mathbb{Y}^\times = (\mathbf{R}, +),$ the unit being $0 \in \mathbf{R}.$

【Tropical Hypersurfaces and Newton Polyhedra】

For a tropical Laurent polynomial $F(x) = \sum_{j \in A} c_j x^j$, we define

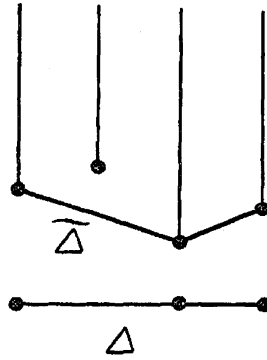
$$v = -c : A \rightarrow \mathbf{R}$$

by $v(j) := -c_j, (j \in A)$. Then we set

$$\sqcup(v) := \text{convex hull} \{ (j, y) \in \mathbf{R}^n \times \mathbf{R} \mid j \in A, y \geq v(j) \} \subset \mathbf{R}^{n+1}$$

We set $\Delta = \Delta_F = \text{convex hull}(A) \subset \mathbf{R}^n$, and $\tilde{\Delta}$ the union of compact faces of $\sqcup(v)$. We call $\Delta = \Delta_F$ the Newton polyhedra of F .

Then $\tilde{\Delta}$ projects to Δ in bijection by $\pi : \tilde{\Delta} \rightarrow \mathbf{R}^n, \pi(j, y) := j$. An integral subdivision of Δ is induced from $\tilde{\Delta}$. We obtain the convex function $\bar{v} : \Delta \rightarrow \mathbf{R}$ having $\tilde{\Delta}$ as its graph.



The tropical hypersurface Y_F is an $(n - 1)$ -dimensional regular polyhedral complex. (Regularity condition: the boundary of each i -cell is a union of $(i - 1)$ -cells.)

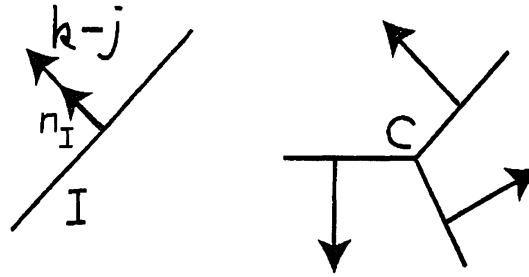
Along each $(n - 1)$ -cell I , two functions $c_j + j \cdot x, c_k + k \cdot x$ have the same value. From $c_j + j \cdot x = c_k + k \cdot x$, we have the equation

$$(k - j)x + (c_k - c_j) = 0$$

of the hyperplane containing I . Then the integer vector $k - j$ is orthogonal to I . Then there exist the unique positive integer w_I and the primitive integer vector n_I such that $k - j = w_I n_I$.

For each $(n - 2)$ -cell C , and $(n - 1)$ -cells $I_1, I_2, \dots, I_m$ adjacent to C , if we fix a co-orientation of C and take primitive orthogonal vectors n_{I_j} , then we have the *balanced condition*

$$w_{I_1} n_{I_1} + w_{I_2} n_{I_2} + \dots + w_{I_m} n_{I_m} = 0.$$



Thus the tropical hypersurface Y is an $(n - 1)$ -dimensional weighted rational polyhedral complex satisfying the regularity condition and the balanced condition.

Tropical hypersurface Y_F is invariant under the deformations, called the *fundamental deformations*, of the tropical Laurent polynomial F .

- (1) Replace c by $c' : A \rightarrow \mathbf{R}, c'(j) = c(j) + \text{const.}$
- (2) Replace A by $A' = A + j_0, j_0 \in \mathbf{Z}^n$ and c by $c' : A' \rightarrow \mathbf{R}, c'(j + j_0) = c(j)$.
- (3) Replace $c : A \rightarrow \mathbf{R}$ by $c' : A' \rightarrow \mathbf{R}$ such that convex hull $A' = \Delta$ and the convex function $\overline{-c'} = \overline{-c}$.

【 Legendre Transformations 】

Consider the contact manifold $M = \mathbf{R}^{2n+1}$ with coordinates

$$(x, y, p) = (x_1, \dots, x_n, y, p_1, \dots, p_n)$$

and with the contact form $\theta = dy - \sum_{i=1}^n p_i dx_i$.

Note that $-\theta = d(\sum_{i=1}^n p_i x_i - y) - \sum_{i=1}^n x_i dp_i$. Then we have the *double Legendrian fibration*:

$$\mathbf{R}^{n+1} \xleftarrow{\pi_1} \mathbf{R}^{2n+1} \xrightarrow{\pi_2} \mathbf{R}^{n+1},$$

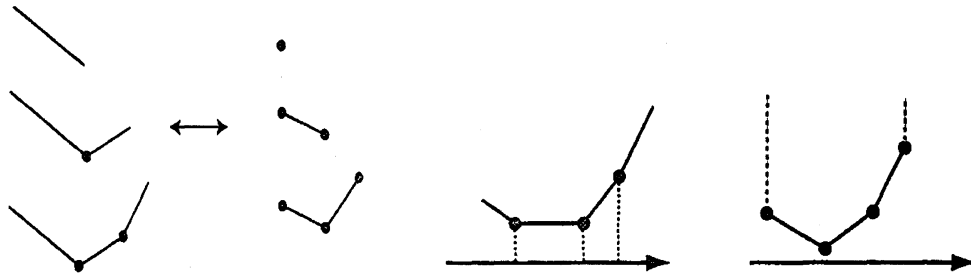
$$\pi_1(x, y, p) = (x, y), \quad \pi_2(x, y, p) = (\tilde{y}, p), \quad \tilde{y} = \sum_{i=1}^n p_i x_i - y.$$

For a function $h : \Delta \rightarrow \mathbf{R}$ on a convex set $\Delta \subset \mathbf{R}^n$, the *Legendre transformation* of h is defined as the set of *supporting hyperplanes* of the *epi-graph* of h .

Lemma 4. *The graph of tropical polynomial function*

$$F(x) = \max_{j \in A} c_j x^j$$

and the graph of the convex function $\overline{-c} : \mathbb{R}^n \rightarrow \mathbb{R}$ are the Legendre transformations to each other.



We consider the topological classification problem of tropical polynomial functions preserving corner loci.

Definition 5. Two tropical polynomials $F(x)$ and $G(x)$ are called *topologically equivalent* if there exist homeomorphisms $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\Psi(F(x)) = G(\Phi(x)), \quad \Phi(Y_F) = Y_G.$$

Proposition 6. *There exists a semialgebraic set $\Sigma \subset \mathbb{R}^A$ of codim > 0 such that, for any $c \in \mathbb{R}^A \setminus \Sigma$, the decomposition of Δ is simplicial.*

For each connected component U of $\mathbb{R}^A \setminus \Sigma$, the family $F_c(x), c \in U$ of tropical polynomial functions is topologically trivial.

【 Topological Bifurcations of Singularities 】

The topology of a tropical polynomial with a non-simplicial decomposition bifurcates into a generic tropical polynomial.

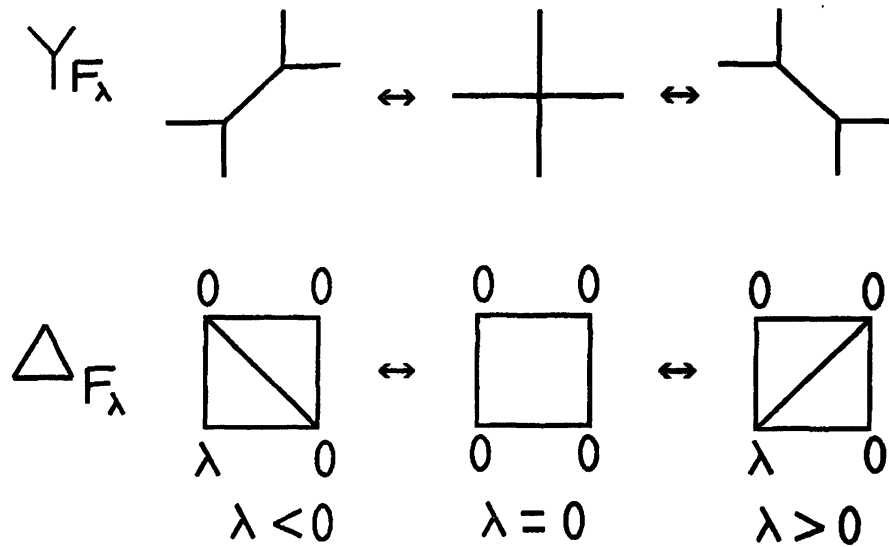
Example 7. Let us consider the tropical polynomial

$$F = "0 + 0x_1 + 0x_2 + 0x_1x_2" = \max\{0, x_1, x_2, x_1 + x_2\}.$$

Then F has the deformation:

$$F_\lambda = "\lambda + 0x_1 + 0x_2 + 0x_1x_2" = \max\{\lambda, x_1, x_2, x_1 + x_2\}, (\lambda \in \mathbf{R}, \lambda \neq 0).$$

The tropical curve Y_F bifurcates into Y_{F_λ} ($\lambda > 0, \lambda < 0$). The decomposition of Newton polyhedron Δ_F bifurcates into Δ_{F_λ} ($\lambda > 0, \lambda < 0$).



【 Amoeba and Patchworking 】

For a complex Laurent polynomial

$$f(z) = \sum_{j \in A} b_j z^j \in \mathbf{C}[z_1^\pm, \dots, z_n^\pm], \quad b_j \in \mathbf{C}^\times,$$

we have a hypersurface

$$Z_f = \{z \in (\mathbf{C}^\times)^n \mid f(z) = 0\} \subset (\mathbf{C}^\times)^n$$

in the complex torus $(\mathbf{C}^\times)^n$.

For a given function $v : A \rightarrow \mathbf{R}$, consider the family of polynomials,

$$f_t = f_t^v(z) := \sum_{j \in A} b_j t^{-v(j)} z^j, \quad (t > 0).$$

We call it the *patchworking polynomial* induced by f and v . Note that $f_1 = f$.

Let us define $\text{Log}_t : \mathbb{C}^n \rightarrow (\mathbb{R} \cup \{-\infty\})^n$ by

$$\text{Log}_t(z_1, \dots, z_n) = (\log_t |z_1|, \dots, \log_t |z_n|).$$

We set $\mathcal{A}_f = \text{Log}(Z_f) \subset \mathbb{R}^n$ and we call it the *amoeba* of Z_f .

Proposition 8. (Viro, Kapranov)

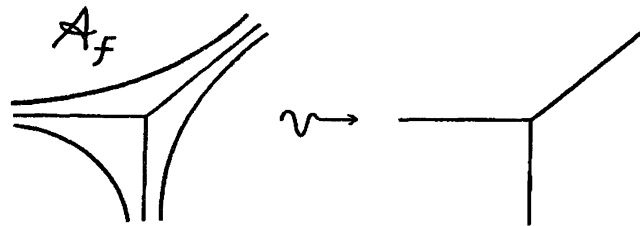
$$\lim_{t \rightarrow \infty} \text{Hausdorff-dist}(\text{Log}_t(Z_{f_t}), Y_{f_{\text{trop}}^v}) = 0$$

where

$$f_{\text{trop}}^v(x) := \sum_{j \in A} (-v(j))x^j = \max_{j \in A} (j \cdot x - v(j))$$

(Legendre transformation of v).

Example 9. Amoeba of $f(z_1, z_2) = z_1 + z_2 + 1$.



【Puisseux Series and Non-Archimedean Amoeba】

Let us denote by $\mathbb{C}[\mathbb{R}]$ the group algebra of the additive group $\mathbb{R}$ over $\mathbb{C}$. We consider its formal version:

A *Puisseux-Laurent series* of real power (Hahn series[4]) is given by

$$a := a(s) = \sum_{p \in I} \alpha_p s^p$$

where $\alpha_p \in \mathbb{C}^\times$ and the support $I = I_a \subset \mathbb{R}$ of a is a well-ordered subset.

We set

$$\mathbb{C}((\mathbb{R})) := \{a(s) \mid a(s) : \text{Puisseux-Laurent series of real power}\} \cup \{0\}.$$

Lemma 10. $\mathbf{K} = \mathbf{C}((\mathbf{R}))$ is an algebraically closed field.

Define the valuation $\text{val} : \mathbf{C}((\mathbf{R})) \rightarrow \mathbf{R} \cup \{\infty\}$ on $\mathbf{C}((\mathbf{R}))$ by

$$\text{val}(a) := \min I_a \in \mathbf{R}, \quad (a \in \mathbf{C}((\mathbf{R})) \setminus \{0\}), \quad \text{val}(0) = \infty,$$

Then we have that $\text{val}(a) = \infty$ if and only if $a = 0$, and that

$$\text{val}(ab) = \text{val}(a) + \text{val}(b), \quad \text{val}(a + b) \geq \min\{\text{val}(a), \text{val}(b)\}.$$

We define the *non-Archimedes norm* on $\mathbf{C}((\mathbf{R}))$ by

$$\|a\| := e^{-\text{val}(a)} \quad (a \in \mathbf{C}((\mathbf{R}))^\times), \quad \|0\| = 0.$$

Then we have the *tropical triangular inequality*

$$\|a + b\| \leq \max\{\|a\|, \|b\|\} = " \|a\| + \|b\| "$$

Define $\text{Log} : \mathbf{C}((\mathbf{R}))^n \rightarrow (\mathbf{R} \cup \{-\infty\})^n$ by

$$\begin{aligned} \text{Log}(a_1, \dots, a_n) &:= (\log \|a_1\|, \dots, \log \|a_n\|) \\ &= (-\text{val}(a_1), \dots, -\text{val}(a_n)). \end{aligned}$$

Given a Laurent polynomial $f(z) = \sum_j a_j z^j \in \mathbf{K}[z, z^{-1}]$, we define

$$Z_f := \{z \in (\mathbf{K}^\times)^n \mid f(z) = 0\} \subset (\mathbf{K}^\times)^n.$$

Its Log-image $\mathbf{A}_f := \text{Log}(Z_f) \subset (\mathbf{R} \cup \{-\infty\})^n$ is called the *non-Archimedean amoeba* of Z_f .

Define a tropical Laurent polynomial

$$\begin{aligned} f_{\text{trop}}(x) &:= " \sum_{j \in A} \log \|a_j\| x^j " = " \sum_{j \in A} (-\text{val}(a_j)) x^j " \\ &= \max_{j \in A} (j \cdot x - \text{val}(a_j)). \end{aligned}$$

We call $f_{\text{trop}}(x)$ the *tropicalization* of $f(z)$.

Proposition 11. (Kapranov) *Non-Archimedean amoeba is a tropical hypersurface: We have $\mathbf{A}_f = Y_{f_{\text{trop}}}$.*

【 Triangle hyperfield 】

On $\mathbf{R}_+$, define the multi-valued addition

$$\begin{aligned} a \nabla b &:= \{c \in \mathbf{R}_+ \mid |a - b| \leq c \leq a + b\} \\ &= \{|z + w| \mid |z| = a, |w| = b\}. \end{aligned}$$

This reminds us the superposition of waves.

Then $\mathbf{R}_+^{\text{tri}} = (\mathbf{R}_+, \nabla, \cdot)$ is a hyperfield.

【 Amoeba hyperfield 】

By the bijection $\log : \mathbf{R}_+ \rightarrow \mathbf{R} \cup \{-\infty\}$, we have the hyperfield

$$\log(\mathbf{R}_+^{\text{tri}}) := (\mathbf{R} \cup \{-\infty\}, \curlyvee, +),$$

which is called the *amoeba hyperfield*:

$$a \curlyvee b := \{c \in \mathbf{R} \cup \{-\infty\} \mid \log(|e^a - e^b|) \leq c \leq \log(e^a + e^b)\}.$$

【 Tropical Limits of Amoeba Hyperfield 】

Define, on $\mathbf{R} \cup \{-\infty\}$,

$$\begin{aligned} a \curlyvee_h b &:= h(\frac{a}{h} \curlyvee \frac{b}{h}) \\ &= \{c \in \mathbf{R} \cup \{-\infty\} \mid \\ &\quad h \log(|e^{\frac{a}{h}} - e^{\frac{b}{h}}|) \leq c \leq h \log(e^{\frac{a}{h}} + e^{\frac{b}{h}})\}, \\ a \curlyvee_h b &= \{c \in \mathbf{R} \cup \{-\infty\} \mid -\infty \leq c \leq a + h \log 2\} \\ &= [-\infty, a] =: a \curlyvee a. \end{aligned}$$

If $a \neq b$, then $a \curlyvee_h b \rightarrow \{\max\{a, b\}\}$.

$$\lim_{h \rightarrow 0} a \curlyvee_h b = a \curlyvee b,$$

$$\lim_{h \rightarrow 0} \log(\mathbf{R}_+^{\text{tri}})_h \rightarrow \mathbb{Y} \text{ (: tropical hyperfield).}$$

【 Complex Tropical hyperfield 】

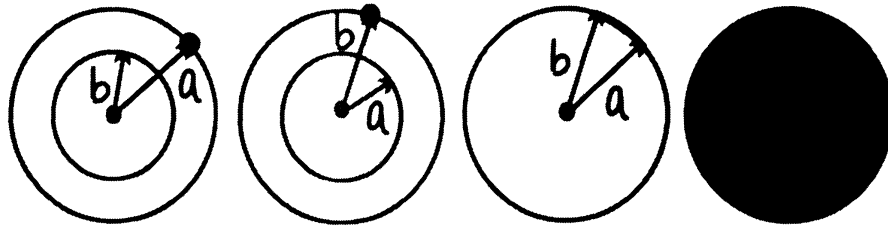
We define a multi-valued addition $\smile$ on $\mathbf{C}$: Let $a, b \in \mathbf{C}$. If $|a| \neq |b|$, then we set $a \smile b := a$ if $|a| > |b|$, and $a \smile b := b$ if $|a| < |b|$.

Suppose $|a| = |b|$. If $b \neq -a$, then

$a \smile b :=$ [the shortest arc connecting a and b
on the circle $\{z \in \mathbf{C} \mid |z| = |a|\}$].

If $b = -a$, then set

$$a \smile b := \{z \in \mathbf{C} \mid |z| \leq |a|\}.$$



We define the *complex tropical hyperfield* by

$$\mathcal{TC} := (\mathbf{C}, \smile, \text{the usual multiplication}).$$

On $\mathbf{C}$, we consider the bijection $S_h : \mathbf{C} \rightarrow \mathbf{C}$ defined by

$$S_h(z) := \begin{cases} |z|^{\frac{1}{h}} \frac{z}{|z|} & (z \neq 0), \\ 0 & (z = 0). \end{cases}$$

and we define

$$z +_h w := S_h^{-1}(S_h(z) + S_h(w)).$$

Then we have a family of fields $(\mathbf{C}, +_h, \times)$, $h > 0$.

Theorem 12. (Viro [25]) *Let*

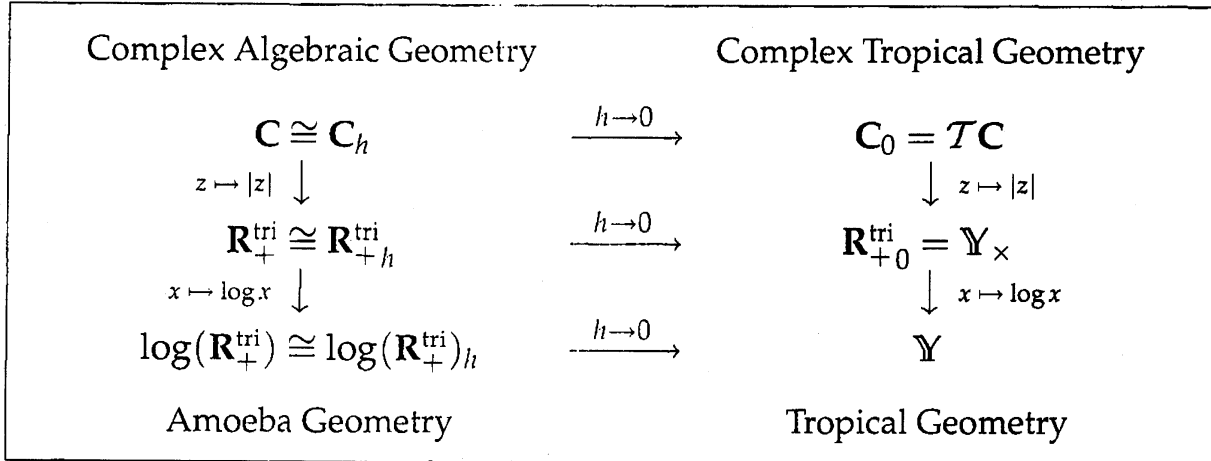
$$\Gamma = \{(z, w, z +_h w, h) \in \mathbf{C}^3 \times \mathbf{R}_+ \mid (z, w, h) \in \mathbf{C}^2 \times \mathbf{R}_+\}.$$

Then

$$\overline{\Gamma} \cap (\mathbf{C}^3 \times \{0\}) := \{(a, b, a \smile b, 0) \mid (a, b) \in \mathbf{C}^2\}.$$

【 Viro's Diagram 2010】

Thus we have the diagram:

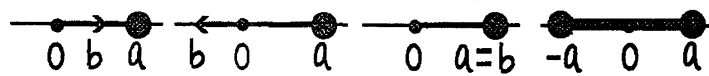


【 Real Tropical Hyperfield 】

Question: What is the real counterpart of the complex tropical hyperfield ?

We are naturally led to define the multi-valued addition $\smile_{\mathbb{R}}$ on $\mathbb{R}$ induced from $\smile$ on $\mathbb{C}$: For $a, b \in \mathbb{R}$, we set

$$\begin{cases} a \smile_{\mathbb{R}} b & := a & \text{if } |a| > |b|, \\ a \smile_{\mathbb{R}} b & := b & \text{if } |a| < |b|, \\ a \smile_{\mathbb{R}} a & := a, \\ a \smile_{\mathbb{R}} (-a) & := [-a, a]. \end{cases}$$



Theorem 13. $(\mathbb{R}, \smile_{\mathbb{R}}, \times)$ is a hyperfield. Moreover let

$$\Gamma_{\mathbb{R}} = \{(a, b, a \smile_{\mathbb{R}} b, h) \in \mathbb{R}^3 \times \mathbb{R}_+ \mid (a, b, h) \in \mathbb{R}^2 \times \mathbb{R}_+\}.$$

Then we have

$$\overline{\Gamma_{\mathbb{R}}} \cap (\mathbb{R}^3 \times \{0\}) = \{(a, b, a \smile_{\mathbb{R}} b, 0) \mid (a, b) \in \mathbb{R}^2\}.$$

The real tropical hyperfield is, in some sense, a “double covering” of the tropical hyperfield via $x \mapsto \log |x|$. Therefore, “real tropical geometry” can be constructed as a “double covering” of tropical geometry.

【 Several Questions 】

Question: Is the complex tropical hyperfield $\mathcal{TC}$ algebraically closed, in an appropriate sense ?

Question: Are the real tropical hyperfield and the tropical hyperfield $\mathbb{Y}$ real closed ?

Question: What is the real tropical algebraic geometry ?

In Amoeba geometry, it is known the *Ronkin function*

$$N_f(x) := \frac{1}{(2\pi\sqrt{-1})^n} \int_{\text{Log}^{-1}(x)} \log |f(z)| \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$$

is linear on each connected component of $\mathbb{R}^n \setminus \mathcal{A}_f$. We have $\text{grad} N_f : \mathbb{R}^n \setminus \mathcal{A}_f \rightarrow \Delta \cap \mathbb{Z}^n$ and $\text{grad} N_f$ separates every connected components of $\mathbb{R}^n \setminus \mathcal{A}_f$.

Question: Can the Ronkin function be described in terms of the amoeba hyperfield ?

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