

Church-Rosser Property and Unique Normal Form Property of Non-Duplicating Term Rewriting Systems

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Abstract

We propose a new type of conditional term rewriting systems: *left-right separated* conditional term rewriting systems, in which the left-hand side and the right-hand side of a rewrite rule have separate variables. By developing a concept of weight decreasing joinability we first present a sufficient condition for the Church-Rosser property of left-right separated conditional term rewriting systems which may have overlapping rewrite rules. We next apply this result to show sufficient conditions for the unique normal form property and the Church-Rosser property of unconditional term rewriting systems which are non-duplicating, non-left-linear, and overlapping.

1 Introduction

The original idea of the conditional linearization of non-left-linear term rewriting systems was introduced by De Vrijer (1990), Klop and De Vrijer (1989) for giving a simpler proof of Chew's theorem (Chew, 1981; Ogawa, 1992). They developed an interesting method for proving the unique normal form property for some non-Church-Rosser, non-left-linear term rewriting system R . The method is based on the fact that the unique normal form property of the original non-left-linear term rewriting system R follows the Church-Rosser property of an associated left-linear conditional term rewriting system R^L which is obtained from R by *linearizing* a non-left-linear rule, for example $Dxx \rightarrow x$, into a left-linear conditional rule $Dxy \rightarrow x \Leftarrow x = y$. Klop and Bergstra (1986) proved that non-overlapping left-linear semi-equational conditional term rewriting systems are Church-Rosser. Hence, combining these two results, Klop and De Vrijer (De Vrijer, 1990; Klop, 1992; Klop and De Vrijer, 1989) showed that the term rewriting system R has the unique normal form property if R^L is non-overlapping. However, as their conditional linearization technique is based on the Church-Rosser property for the traditional conditional term rewriting system R^L , its application is restricted in non-overlapping R^L (though this limitation may be slightly relaxed with R^L containing only trivial critical pairs).

In this paper, we introduce a new conditional linearization based on a *left-right separated* conditional term rewriting system R_L . The point of our linearization is that a non-left-linear rule $Dxx \rightarrow x$ is translated into a left-linear conditional rule $Dxy \rightarrow z \Leftarrow x = z, y = z$ in which the left-hand side and the right-hand side have separate variables. By considering this new system R_L instead of a traditional conditional system R^L we can easily relax the non-overlapping limitation of conditional systems originated from Klop and Bergstra (1986) if the original system R is non-duplicating. Here, R is non-duplicating if for any rewrite rule $l \rightarrow r$, no variable has more occurrences in r than it has in l .

By developing a new concept of weight decreasing joinability we first present a sufficient condition for the Church-Rosser property of a left-right separated conditional term rewriting system R_L .

which may have overlapping rewrite rules. We next apply this result to our conditional linearization, and show a sufficient condition for the unique normal form property of the original system R which is non-duplicating, non-left-linear, and overlapping.

Moreover, our result can be naturally applied to proving the Church-Rosser property of some non-duplicating non-left-linear overlapping term rewriting systems such as right-ground systems. More recently, Oyamaguchi and Ohta (1993) proved that non-E-overlapping right-ground term rewriting systems are Church-Rosser by using the joinability of E-graphs, and Oyamaguchi (1992) extended this result into some overlapping systems. The results by conditional linearization in this paper strengthen some part of the results by E-graphs in Oyamaguchi and Ohta (1993) and Oyamaguchi (1992), and vice versa.

In the next section we give a concise explanation of abstract reduction systems. In section 3 we introduce a notion of weight decreasing joinability, which is a main tool used throughout the paper to prove the Church-Rosser property of conditional term rewriting systems. Section 4 briefly explains the notions and definitions concerning term rewriting systems. In section 5 we define a notion of left-right separated conditional term rewriting systems and show a sufficient condition for the Church-Rosser property of the systems. Section 6 introduces a new conditional linearization based on left-right separated conditional term rewriting systems. By using the conditional linearization technique we give a sufficient condition for the unique normal form property of (unconditional) term rewriting systems which are non-duplicating, non-left-linear, and overlapping. In Section 7 we show that the conditional linearization proposed can be used as a useful method for proving the Church-Rosser property of some class of non-duplicating (unconditional) term rewriting systems.

2 Reduction Systems

Assuming that the reader is familiar with the basic concepts and notations concerning reduction systems in Klop (1992), we briefly explain notations and definitions.

A reduction system (or an abstract reduction system) is a structure $A = \langle D, \rightarrow \rangle$ consisting of some set D and some binary relation $\rightarrow$ on D (i.e., $\rightarrow \subseteq D \times D$), called a reduction relation. A reduction (starting with x_0) in A is a finite or infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \dots$. The identity of elements x, y of D is denoted by $x \equiv y$. $\leftrightarrow$ is the symmetric closure of $\rightarrow$, $\rightarrow^*$ is the transitive reflexive closure of $\rightarrow$, and $\leftrightarrow^*$ is the equivalence relation generated by $\rightarrow$ (i.e., the transitive reflexive symmetric closure of $\rightarrow$). We write $x \leftarrow y$ if $y \rightarrow x$; likewise $x \leftarrow^* y$.

If $x \in D$ is minimal with respect to $\rightarrow$, i.e., $\neg \exists y \in D[x \rightarrow y]$, then we say that x is a normal form; let NF be the set of normal forms. If $x \rightarrow^* y$ and $y \in NF$ then we say x has a normal form y and y is a normal form of x .

Definition 1 $A = \langle D, \rightarrow \rangle$ is Church-Rosser (or confluent) iff
 $\forall x, y, z \in D[x \rightarrow^* y \wedge x \rightarrow^* z \Rightarrow \exists w \in D, y \rightarrow^* w \wedge z \rightarrow^* w]$.

Definition 2 $A = \langle D, \rightarrow \rangle$ has unique normal forms iff
 $\forall x, y \in NF[x \leftrightarrow^* y \Rightarrow x \equiv y]$.

The following fact observed by Klop and De Vrijer (1989) plays an essential role in our linearization too.

Proposition 1 [Klop and De Vrijer] Let $A_0 = \langle D, \rightarrow_0 \rangle$ and $A_1 = \langle D, \rightarrow_1 \rangle$ be two reduction systems with the sets of normal forms NF_0 and NF_1 respectively. Then A_0 has unique normal forms if each of the following conditions holds:

- (i) $\rightarrow_1$ extends $\rightarrow_0$,
- (ii) A_1 is Church-Rosser,
- (iii) NF_1 contains NF_0 .

Proof. Easy. $\square$

3 Weight Decreasing Joinability

This section introduces the new concept of weight decreasing joinability. In the later sections this concept is used for analyzing the Church-Rosser property of conditional term rewriting systems with extra variables occurring in conditional parts of rewrite rules.

Let N^+ be the set of positive integers. $A = \langle D, \rightarrow \rangle$ is a weighted reduction system if $\rightarrow = \bigcup_{w \in N^+} \rightarrow_w$, that is, positive integers (weights w) are assigned to each reduction step to represent costs.

Definition 3 A proof of $x \xrightarrow{*} y$ is a sequence $\mathcal{P}: x_0 \xrightarrow{w_1} x_1 \xrightarrow{w_2} x_2 \cdots \xrightarrow{w_n} x_n$ ($n \geq 0$) such that $x \equiv x_0$ and $y \equiv x_n$. The weight $w(\mathcal{P})$ of the proof $\mathcal{P}$ is $\sum_{i=1}^n w_i$. If $\mathcal{P}$ is a 0 step sequence (i.e., $n = 0$), then $w(\mathcal{P}) = 0$.

We usually abbreviate a proof $\mathcal{P}$ of $x \xrightarrow{*} y$ by $\mathcal{P}: x \xrightarrow{*} y$. The form of a proof may be indicated by writing, for example, $\mathcal{P}: x \xrightarrow{*} \cdot \xrightarrow{*} y$, $\mathcal{P}': x \leftarrow \cdot \xrightarrow{*} \cdot \leftarrow y$, etc. We use the symbols $\mathcal{P}, \mathcal{Q}, \dots$ for proofs.

Definition 4 A weighted reduction system $A = \langle D, \rightarrow \rangle$ is weight decreasing joinable iff for all $x, y \in D$ and any proof $\mathcal{P}: x \xrightarrow{*} y$ there exists some proof $\mathcal{P}': x \xrightarrow{*} \cdot \xrightarrow{*} y$ such that $w(\mathcal{P}) \geq w(\mathcal{P}')$.

It is clear that if a weighted reduction system A is weight decreasing joinable then A is Church-Rosser. We will now show a sufficient condition for the weight decreasing joinability.

Lemma 1 Let A be a weighted reduction system. Then A is weight decreasing joinable if for any $x, y \in D$ and any proof $\mathcal{P}: x \leftarrow \cdot \rightarrow y$ one of the following conditions holds:

- (i) there exists a proof $\mathcal{P}': x \xrightarrow{*} y$ such that $w(\mathcal{P}) > w(\mathcal{P}')$, or
- (ii) there exist proofs $\mathcal{P}': x \rightarrow \cdot \xrightarrow{*} y$ and $\mathcal{P}'': x \xrightarrow{*} \cdot \leftarrow y$ such that $w(\mathcal{P}) \geq w(\mathcal{P}')$ and $w(\mathcal{P}) \geq w(\mathcal{P}'')$, or
- (iii) there exists a proof $\mathcal{P}': x \rightarrow y$ (or $x \leftarrow y$) such that $w(\mathcal{P}) \geq w(\mathcal{P}')$.

Proof. By induction on the weight $w(\mathcal{Q})$ of a proof $\mathcal{Q}: x \xrightarrow{*} y$, we prove that there exists a proof $\mathcal{Q}': x \xrightarrow{*} \cdot \xrightarrow{*} y$ such that $w(\mathcal{Q}) \geq w(\mathcal{Q}')$. *Base step* ($w(\mathcal{Q}) = 0$) is trivial. *Induction step:* Let $\mathcal{Q}: x \xrightarrow{*} x' \xrightarrow{*} y$ and let $\mathcal{S}: x' \xrightarrow{*} y$ be the subproof of $\mathcal{Q}$. From induction hypothesis, there exists a proof $\mathcal{S}': x' \xrightarrow{*} \cdot \xrightarrow{*} y$ such that $w(\mathcal{S}) \geq w(\mathcal{S}')$. Thus, if $x \rightarrow x'$ then we have $\mathcal{Q}': x \rightarrow x' \xrightarrow{*} \cdot \xrightarrow{*} y$ such that $w(\mathcal{Q}) \geq w(\mathcal{Q}')$. Otherwise we have a proof $\mathcal{Q}'': x \leftarrow x' \xrightarrow{n} \cdot \xrightarrow{*} y$ such that $w(\mathcal{Q}) \geq w(\mathcal{Q}'')$, where $\xrightarrow{n}$ denotes a reduction of n ($n \geq 0$) steps. By induction on n we will prove that $\mathcal{Q}'$ exists. The case $n = 0$ is trivial. Let $\mathcal{Q}'': x \leftarrow x' \rightarrow z \xrightarrow{n-1} \cdot \xrightarrow{*} y$ and let $\mathcal{P}: x \leftarrow x' \rightarrow z$ be the subproof of $\mathcal{Q}''$. Then $\mathcal{P}$ can be replaced with $\mathcal{P}'$ satisfying one of the above conditions (i), (ii), or (iii).

Case (i). $\mathcal{P}': x \xrightarrow{*} z$ and $w(\mathcal{P}) > w(\mathcal{P}')$. Then we have $\hat{\mathcal{Q}}: x \xrightarrow{*} z \xrightarrow{n-1} \cdot \xrightarrow{*} y$ such that $w(\mathcal{Q}'') > w(\hat{\mathcal{Q}})$. Thus, by using induction hypothesis concerning the weight $w(\mathcal{Q})$, we obtain $\mathcal{Q}'$ from $\hat{\mathcal{Q}}$.

Case (ii). $\mathcal{P}': x \rightarrow z' \xrightarrow{*} z$ and $w(\mathcal{P}) \geq w(\mathcal{P}')$. Then we have $\hat{\mathcal{Q}}: x \rightarrow z' \xrightarrow{*} z \xrightarrow{n-1} \cdot \xrightarrow{*} y$ such that $w(\mathcal{Q}'') \geq w(\hat{\mathcal{Q}})$. Let $\hat{\mathcal{Q}}': z' \xrightarrow{*} z \xrightarrow{n-1} \cdot \xrightarrow{*} y$ be the subproof of $\hat{\mathcal{Q}}$. From induction hypothesis concerning the weight $w(\mathcal{Q})$ there exists a proof $\hat{\mathcal{Q}}'': z' \xrightarrow{*} \cdot \xrightarrow{*} y$ such that $w(\hat{\mathcal{Q}}') \geq w(\hat{\mathcal{Q}}'')$. Thus, by replacing $\hat{\mathcal{Q}}'$ of $\hat{\mathcal{Q}}$ with $\hat{\mathcal{Q}}''$, we have $\mathcal{Q}'$.

Case (iii). $\mathcal{P}': x \leftarrow z$ and $w(\mathcal{P}) \geq w(\mathcal{P}')$. (If $\mathcal{P}': x \rightarrow z$, the claim trivially holds.) Then we have $\hat{\mathcal{Q}}: x \leftarrow z \xrightarrow{n-1} \cdot \xrightarrow{*} y$ such that $w(\mathcal{Q}'') \geq w(\hat{\mathcal{Q}})$. From induction hypothesis concerning the number n of reduction steps, we have $\mathcal{Q}'$. $\square$

The following lemma is used to show the Church-Rosser property of non-left-linear systems in Section 7.

Lemma 2 Let $A_0 = \langle D, \rightarrow_0 \rangle$ and $A_1 = \langle D, \rightarrow_1 \rangle$. Let $\mathcal{P}_i: x_i \xrightarrow{*}_1 y$ ($i = 1, \dots, n$) and let $\rho = \sum_{i=1}^n w(\mathcal{P}_i)$. Assume that for any $a, b \in D$ and any proof $\mathcal{P}: a \xrightarrow{*}_1 b$ such that $w(\mathcal{P}) \leq \rho$ there exist proofs $\mathcal{P}': a \xrightarrow{*}_1 c \xrightarrow{*}_1 b$ with $w(\mathcal{P}') \leq w(\mathcal{P})$ and $a \xrightarrow{*}_0 c \xrightarrow{*}_0 b$ for some $c \in D$. Then, there exist proofs $\mathcal{P}'_i: x_i \xrightarrow{*}_0 z$ ($i = 1, \dots, n$) and $\mathcal{Q}: y \xrightarrow{*}_1 z$ with $w(\mathcal{Q}) \leq \rho$ for some z (Figure 3.1).

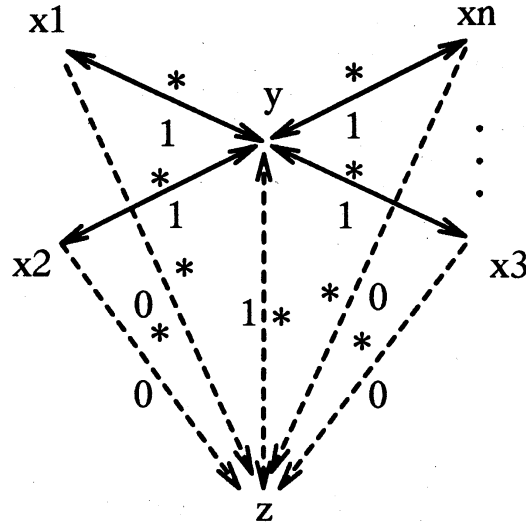


Figure 3.1

Proof. By induction on ρ . *Base step* ($\rho = 0$) is trivial. *Induction step:* From induction hypothesis, we have proofs $\tilde{P}_i: x_i \xrightarrow{*}_0 z'$ ($i = 1, \dots, n-1$) and $\tilde{Q}: y \xrightarrow{*}_1 z'$ for some z' such that $\sum_{i=1}^{n-1} w(\tilde{P}_i) \geq w(\tilde{Q})$. By connecting the proofs $\tilde{Q}$ and $\tilde{P}_n$ we have a proof $\hat{P}: z' \xrightarrow{*}_1 y \xrightarrow{*}_1 x_n$. Since $\sum_{i=1}^{n-1} w(\tilde{P}_i) \geq w(\tilde{Q})$ and $w(\hat{P}) = w(\tilde{Q}) + w(\tilde{P}_n)$, it follows that $\rho \geq w(\hat{P})$. By the assumption, we have proofs $\tilde{P}: z' \xrightarrow{*}_1 z \xrightarrow{*}_1 x_n$ with $\rho \geq w(\hat{P}) \geq w(\tilde{P})$ and $z' \xrightarrow{*}_0 z \xrightarrow{*}_0 x_n$ for some z . Thus we obtain proofs $\mathcal{P}'_i: x_i \xrightarrow{*}_0 z$ ($i = 1, \dots, n$).

By combining subproofs of $\tilde{P}: z' \xrightarrow{*}_1 y \xrightarrow{*}_1 x_n$ and $\tilde{P}: z' \xrightarrow{*}_1 z \xrightarrow{*}_1 x_n$, we can make $Q': y \xrightarrow{*}_1 z' \xrightarrow{*}_1 z$ and $Q'': y \xrightarrow{*}_1 x_n \xrightarrow{*}_1 z$. Note that $\rho + \rho \geq w(\hat{P}) + w(\tilde{P}) = w(Q') + w(Q'')$. Thus $\rho \geq w(Q')$ or $\rho \geq w(Q'')$. Take Q' as Q if $\rho \geq w(Q')$; otherwise, take Q'' as Q . $\square$

4 Term Rewriting Systems

In the following sections, we briefly explain the basic notions and definitions concerning term rewriting systems (Dershowitz and Jouannaud, 1990; Klop, 1992).

Let $\mathcal{F}$ be an enumerable set of function symbols denoted by $f, g, h, \dots$, and let $\mathcal{V}$ be an enumerable set of variable symbols denoted by $x, y, z, \dots$ where $\mathcal{F} \cap \mathcal{V} = \emptyset$. By $T(\mathcal{F}, \mathcal{V})$, we denote the set of terms constructed from $\mathcal{F}$ and $\mathcal{V}$. $V(t)$ denotes the set of variables occurring in a term t .

A substitution θ is a mapping from a term set $T(\mathcal{F}, \mathcal{V})$ to $T(\mathcal{F}, \mathcal{V})$ such that for a term t , $\theta(t)$ is completely determined by its values on the variable symbols occurring in t . Following common usage, we write this as $t\theta$ instead of $\theta(t)$.

Consider an extra constant $\square$ called a hole and the set $T(\mathcal{F} \cup \{\square\}, \mathcal{V})$. Then $C \in T(\mathcal{F} \cup \{\square\}, \mathcal{V})$ is called a context on $\mathcal{F}$. We use the notation $C[\dots]$ for the context containing n holes ($n \geq 0$), and if $t_1, \dots, t_n \in T(\mathcal{F}, \mathcal{V})$, then $C[t_1, \dots, t_n]$ denotes the result of placing $t_1, \dots, t_n$ in the holes of $C[\dots]$ from left to right. In particular, $C[\]$ denotes a context containing precisely one hole. s is called a subterm of t if $t \equiv C[s]$. If s is a subterm occurrence of t , then we write $s \subseteq t$. If a term t has an occurrence of some (function or variable) symbol e , we write $e \in t$. The variable occurrences $z_1, \dots, z_n$ of $C[z_1, \dots, z_n]$ are fresh if $z_1, \dots, z_n \notin C[\dots]$ and $z_i \neq z_j$ ($i \neq j$).

A rewrite rule is a pair $\langle l, r \rangle$ of terms such that $l \notin \mathcal{V}$ and any variable in r also occurs in l . We write $l \rightarrow r$ for $\langle l, r \rangle$. A redex is a term $l\theta$, where $l \rightarrow r$. In this case $r\theta$ is called a contractum of $l\theta$. The set of rewrite rules defines a reduction relation $\rightarrow$ on T as follows:

$t \rightarrow s$ iff $t \equiv C[l\theta]$, $s \equiv C[r\theta]$ for some rule $l \rightarrow r$, and some $C[\], \theta$.

When we want to specify the redex occurrence $\Delta \equiv l\theta$ of t in this reduction, we write $t \xrightarrow{\Delta} s$.

Definition 5 A term rewriting system R is a reduction system $R = \langle T(\mathcal{F}, \mathcal{V}), \rightarrow \rangle$ such that the reduction relation $\rightarrow$ on $T(\mathcal{F}, \mathcal{V})$ is defined by a set of rewrite rules. When we want to specify the term rewriting system R in the reduction relation $\rightarrow$, we write $\xrightarrow{R}$. If R has $l \rightarrow r$ as a rewrite rule, we write $l \rightarrow r \in R$.

We say that R is left-linear if for any $l \rightarrow r \in R$, l is linear (i.e., every variable in l occurs only once). If R has a critical pair then we say that R is overlapping: otherwise non-overlapping (Dershowitz and Jouannaud, 1990; Klop, 1992). A rewrite rule $l \rightarrow r$ is duplicating if r contains more occurrences of some variable than l ; otherwise, $l \rightarrow r$ is non-duplicating. We say that R is non-duplicating if every $l \rightarrow r \in R$ is non-duplicating.

5 Left-Right Separated Conditional Systems

In this section we introduce a new conditional term rewriting system R in which l and r of any rewrite rule $l \rightarrow r \Leftarrow x_1 = y_1, \dots, x_n = y_n$ do not share the same variable; every variable y_i in r is connected to some variable x_i in l through the equational condition $x_1 = y_1, \dots, x_n = y_n$. A decidable sufficient condition for the Church-Rosser property of R is presented.

Definition 6 A left-right separated conditional term rewriting system is a conditional term rewriting system with extra variables in which every conditional rewrite rule has the form:

$l \rightarrow r \Leftarrow x_1 = y_1, \dots, x_n = y_n$
with $l, r \in T(\mathcal{F}, \mathcal{V})$, $V(l) = \{x_1, \dots, x_n\}$ and $V(r) \subseteq \{y_1, \dots, y_n\}$ ($n \geq 0$) such that:

- (i) $l \notin V$ is linear,
- (ii) $\{x_1, \dots, x_n\} \cap \{y_1, \dots, y_n\} = \emptyset$,
- (iii) $x_i \neq x_j$ if $i \neq j$,
- (iv) no variable has more occurrences in r than it has in the conditional part " $x_1 = y_1, \dots, x_n = y_n$ ".

Note. In the above conditional rewrite rule, the left-hand side l and the right-hand side r have separate variables, i.e., $V(l) \cap V(r) = \emptyset$, because of (ii). Since every variable y_i in r is connected to some variable x_i in l through the equational condition, it holds that $V(r\theta) \subseteq V(l\theta)$ for the substitution $\theta = [x_1 := y_1, \dots, x_n := y_n]$. Thus, $l\theta \rightarrow r\theta$ is an unconditional rewrite rule, and it is non-duplicating due to (iv).

Example 1 The following R is a left-right separated conditional term rewriting system:

$$R \quad \left\{ \begin{array}{l} f(x, x') \rightarrow g(y, y) \Leftarrow x = y, x' = y \\ h(x, x', x'') \rightarrow c \Leftarrow x = y, x' = y, x'' = y \end{array} \right.$$

The following R' is however not a left-right separated conditional term rewriting system since the condition (iv) does not hold:

$$R' \quad \left\{ f(x, x') \rightarrow h(y, y, y) \Leftarrow x = y, x' = y \right.$$

Definition 7 Let R be a left-right separated conditional term rewriting system. We inductively define reduction relations $\xrightarrow{R_i}$ for $i \geq 0$ as follows:

- (i) $\xrightarrow{R_0} = \emptyset$,
- (ii) $\xrightarrow{R_{i+1}} = \{ \langle C[l\theta], C[r\theta] \rangle \mid l \rightarrow r \Leftarrow x_1 = y_1, \dots, x_n = y_n \in R \text{ and } x_j\theta \xrightarrow{R_i}^* y_j\theta \ (j = 1, \dots, n) \}$.
In the reduction $t \equiv C[l\theta] \xrightarrow{R_{i+1}} s \equiv C[r\theta]$, the redex occurrence $\Delta \equiv l\theta$ is specified by writing $t \xrightarrow{\Delta} s$, if necessary.

Note that $\xrightarrow{R_i} \subseteq \xrightarrow{R_{i+1}}$ for all $i \geq 0$. $s \rightarrow t$ iff $s \xrightarrow{R_i} t$ for some i .

The weight $w(\mathcal{P})$ of a proof $\mathcal{P}$ of a left-right separated conditional term rewriting system R is defined as the total reduction steps appearing in the recursive structure of $\mathcal{P}$.

Definition 8 A proof $\mathcal{P}$ and its weight $w(\mathcal{P})$ are inductively defined as follows:

- (i) The empty sequence λ is a (0 step) proof of $t \xrightarrow{R_n}^* t$ ($n \geq 0$) and $w(\lambda) = 0$.
- (ii) An expression $\mathcal{P} : s \xrightarrow{[r, C[\], \theta, \mathcal{P}_1, \dots, \mathcal{P}_m]} t$ (resp. $t \xleftarrow{[r, C[\], \theta, \mathcal{P}_1, \dots, \mathcal{P}_m]} s$) is a proof of $s \xrightarrow{R_n} t$ (resp. $t \xleftarrow{R_n} s$) ($n \geq 1$), where r is a rewrite rule $l \rightarrow r \Leftarrow x_1 = y_1, \dots, x_m = y_m \in R$, $C[\]$ is a context, and θ is a substitution such that $t \equiv C[l\theta]$, $s \equiv C[r\theta]$, and $\mathcal{P}_i$ is a proof of $x_i\theta \xrightarrow{R_{n-1}}^* y_i\theta$ ($i = 1, \dots, m$). $w(\mathcal{P}) = 1 + \sum_{i=1}^m w(\mathcal{P}_i)$. $\mathcal{P}_1, \dots, \mathcal{P}_m$ are subproofs associated with the proof $\mathcal{P}$.
- (iii) A finite sequence $\mathcal{P} : \mathcal{P}_1 \dots \mathcal{P}_m$ ($m \geq 1$) of proofs is a proof of $t_0 \xrightarrow{R_n}^* t_m$ ($n \geq 1$), where $\mathcal{P}_i$ ($i = 1, \dots, m$) is a proof of $t_{i-1} \xrightarrow{R_n} t_i$ or $t_{i-1} \xleftarrow{R_n} t_i$. $w(\mathcal{P}) = \sum_{i=1}^m w(\mathcal{P}_i)$.

$\mathcal{P}$ is a proof of $s \xleftrightarrow{R_n}^* t$ if it is a proof of $s \xrightarrow{R_n}^* t$ for some n . For convenience, we often use the abbreviations introduced in Section 3; i.e., we abbreviate a proof $\mathcal{P}$ of $s \xleftrightarrow{R_n}^* t$ by $\mathcal{P} : s \xleftrightarrow{R_n}^* t$, and the form of a proof is indicated by writing, for example, $\mathcal{P} : s \xrightarrow{*} \cdot \xleftarrow{*} t$, $\mathcal{P}' : s \xleftarrow{*} \cdot \xrightarrow{*} t$, $\mathcal{P}'' : s \xrightarrow{\Delta} \cdot \xrightarrow{\Delta} t$, etc.

Let $l \rightarrow r \Leftarrow x_1 = y_1, \dots, x_m = y_m$ and $l' \rightarrow r' \Leftarrow x'_1 = y'_1, \dots, x'_n = y'_n$ be two rules in a left-right separated conditional term rewriting system R . Assume that we have renamed the variables appropriately, so that two rules share no variables. Assume that $s \notin V$ is a subterm occurrence in l , i.e., $l \equiv C[s]$, such that s and l' are unifiable, i.e., $s\theta \equiv l'\theta$, with the most general unifier θ . Note that $r\theta \equiv r$, $r'\theta \equiv r'$, $y_i\theta \equiv y_i$ ($i = 1, \dots, m$) and $y'_j\theta \equiv y'_j$ ($j = 1, \dots, n$) as $\{x_1, \dots, x_m\} \cap \{y_1, \dots, y_m\} = \emptyset$ and $\{x'_1, \dots, x'_n\} \cap \{y'_1, \dots, y'_n\} = \emptyset$. Since $l \equiv C[s]$ is linear and the domain of θ is contained in $V(s)$, $C[s]\theta \equiv C[s\theta]$. Thus, from $l\theta \equiv C[s]\theta \equiv C[l'\theta]$, two reductions starting with $l\theta$, i.e., $l\theta \rightarrow C[r']$ and $l\theta \rightarrow r$, can be obtained by using $l \rightarrow r \Leftarrow x_1 = y_1, \dots, x_m = y_m$ and $l' \rightarrow r' \Leftarrow x'_1 = y'_1, \dots, x'_n = y'_n$ if we assume the equations $x_1\theta = y_1, \dots, x_m\theta = y_m$ and $x'_1\theta = y'_1, \dots, x'_n\theta = y'_n$. Then we say that $l \rightarrow r \Leftarrow x_1 = y_1, \dots, x_m = y_m$ and $l' \rightarrow r' \Leftarrow x'_1 = y'_1, \dots, x'_n = y'_n$ are overlapping, and $E \vdash \langle C[r'], r \rangle$ is a conditional critical pair associated with the multiset of equations $E = [x_1\theta = y_1, \dots, x_m\theta = y_m, x'_1\theta = y'_1, \dots, x'_n\theta = y'_n]$ in R . We may choose $l \rightarrow r \Leftarrow x_1 = y_1, \dots, x_m = y_m$ and $l' \rightarrow r' \Leftarrow x'_1 = y'_1, \dots, x'_n = y'_n$ to be the same rule, but in this case we shall not consider the case $s \equiv l$. If R has no critical pair, then we say that R is non-overlapping.

Example 2 Let R be the left-right separated conditional term rewriting system with the following rewrite rules:

$$R \quad \begin{cases} f(x', x'') \rightarrow g(x) \Leftarrow x' = x, x'' = x \\ f(y', h(y'')) \rightarrow g(y) \Leftarrow y' = y, y'' = y \end{cases}$$

Let $\theta = [x' := y', x'' := h(y'')]$ be the most general unifier of $f(x', x'')$ and $f(y', h(y''))$. By applying the substitution θ to the conditional parts " $x' = x, x'' = x$ " and " $y' = y, y'' = y$ " we have the multiset of equations $E = [y' = x, h(y'') = x, y' = y, y'' = y]$. Then, assuming the equations in E , $g(x) \leftarrow g(x)\theta \leftarrow f(x', x'')\theta \equiv f(y', h(y'')) \rightarrow g(y)$. Thus, we have a conditional critical pair $E \vdash \langle g(x), g(y) \rangle$.

Note that in a left-right separated conditional term rewriting system the application of the same rule at the same position does not imply the same result as the variables occurring in the left-hand side of a rule do not cover that in the right-hand side: See the following example.

Example 3 Let R be the left-right separated conditional term rewriting system with the following rewrite rules:

$$R \quad \begin{cases} f(x) \rightarrow g(y) \Leftarrow x = y \\ a \rightarrow c \\ b \rightarrow c \end{cases}$$

It is obvious that R is non-overlapping. We have however two reductions $f(c) \rightarrow g(a)$ and $f(c) \rightarrow g(b)$, as $c \xrightarrow{*} a$ and $c \xrightarrow{*} b$. Thus the application of the first rule at the root position of $f(c)$ does not guarantee a unique result.

We next discuss how to compare the weights of *abstract* proofs including the assumed equations of E . $E \sqcup E'$ denotes the union of multisets E and E' . We write $E \sqsubseteq E'$ if no elements in E occur more than E' .

Definition 9 Let E be a multiset of equations $t' = s'$ and a fresh constant $\bullet$. Then relations $t \sim_E s$ and $t \sim_{\triangleright_E} s$ on terms is inductively defined as follows:

- (i) $t \sim_{[t=s]} s$.
- (ii) If $t \sim_E s$ then $s \sim_E t$.
- (iii) If $t \sim_E r$ and $r \sim_{E'} s$ then $t \sim_{E \sqcup E'} s$.
- (iv) If $t \sim_E s$ then $C[t] \sim_E C[s]$.
- (v) If $l \rightarrow r \Leftarrow x_1 = y_1, \dots, x_n = y_n \in R$ and $x_i \theta \sim_{E_i} y_i \theta$ ($i = 1, \dots, n$) then $C[l\theta] \sim_{\triangleright_E} C[r\theta]$ where $E = E_1 \sqcup \dots \sqcup E_n$.
- (vi) If $t \sim_{\triangleright_E} s$ then $t \sim_{E \sqcup [\bullet]} s$.

In the above definition the fresh constant $\bullet$ keeps in E the number of concrete rewriting steps appearing in an *abstract* proof. We write $t \prec_{\sim_E} s$ if $s \sim_{\triangleright_E} t$.

Lemma 3 Let $E = [p_1 = q_1, \dots, p_m = q_m, \bullet, \dots, \bullet]$ be a multiset in which $\bullet$ occurs k times ($k \geq 0$), and let $\mathcal{P}_i: p_i \theta \xrightarrow{*} q_i \theta$ ($i = 1, \dots, m$).

- (1) If $t \sim_E s$ then there exists a proof $Q: t\theta \xrightarrow{*} s\theta$ with $w(Q) \leq \sum_{i=1}^m w(\mathcal{P}_i) + k$.
- (2) If $t \sim_{\triangleright_E} s$ then there exists a proof $Q': t\theta \rightarrow s\theta$ with $w(Q') \leq \sum_{i=1}^m w(\mathcal{P}_i) + k + 1$.

Proof. By induction on the construction of $t \sim_E s$ and $t \sim_{\triangleright_E} s$ in Definition 9, we prove (1) and (2) simultaneously. *Base Step:* Trivial as (i) $t \sim_{[t=s]} s$ of Definition 9. *Induction Step:* If we have $t \sim_E s$ by (ii) (iii) (iv) and $t \sim_{\triangleright_E} s$ by (vi) of Definition 9, then from the induction hypothesis (1) and (2) clearly follow. Assume that $t \sim_{\triangleright_E} s$ by (v) of Definition 9. Then we have a rule $l \rightarrow r \Leftarrow x_1 = y_1, \dots, x_n = y_n$ such that $t \equiv C[l\theta']$, $s \equiv C[r\theta']$, $x_i \theta' \sim_{E_i} y_i \theta'$ ($i = 1, \dots, n$) for some θ' and $E = E_1 \sqcup \dots \sqcup E_n$. From the induction hypothesis and $E = E_1 \sqcup \dots \sqcup E_n$, it can be easily shown that there exist proofs $Q_i: x_i \theta' \theta \xrightarrow{*} y_i \theta' \theta$ ($i = 1, \dots, n$) and $\sum_{i=1}^n w(Q_i) \leq \sum_{i=1}^n w(\mathcal{P}_i) + k$. Therefore we have a proof $Q': t\theta \rightarrow s\theta$ with $w(Q') \leq \sum_{i=1}^n w(\mathcal{P}_i) + k + 1$. $\square$

Theorem 1 Let R be a left-right separated conditional term rewriting system. Then R is weight decreasing joinable if for any conditional critical pair $E \vdash \langle q, q' \rangle$ one of the following conditions holds:

- (i) $q \sim_{E'} q'$ for some E' such that $E' \sqsubseteq E \sqcup [\bullet]$, or
- (ii) $q \sim_{E_1} \dots \sim_{E_2} q'$ and $q \sim_{E'_1} \dots \sim_{E'_2} q'$ for some E_1, E_2, E'_1 , and E'_2 such that $E_1 \sqcup E_2 \sqsubseteq E \sqcup [\bullet]$ and $E'_1 \sqcup E'_2 \sqsubseteq E \sqcup [\bullet]$, or
- (iii) $q \sim_{\triangleright_E} q'$ (or $q \prec_{\sim_E} q'$) for some E' such that $E' \sqsubseteq E \sqcup [\bullet]$.

Note. If R has finitely many rewrite rules then R has finitely many conditional critical pairs. For each $E \vdash \langle q, q' \rangle$, it is decidable whether one of the above conditions (i), (ii), or (iii) holds since each relation between q and q' is restricted by an upper bound $E \sqcup [\bullet]$. Thus, the theorem presents a decidable sufficient condition for guaranteeing the Church-Rosser property of R having finite rewrite rules.

Proof. The theorem follows from Lemma 1 if for any $\mathcal{P}: t \leftarrow p \rightarrow s$ ($t \neq s$) one of the following conditions holds: (i) there exists a proof $\mathcal{Q}: t \xrightarrow{*} s$ with $w(\mathcal{P}) > w(\mathcal{Q})$, or (ii) there exist proofs $\mathcal{Q}_1: t \rightarrow \cdot \xrightarrow{*} s$ and $\mathcal{Q}_2: t \xrightarrow{*} \cdot \leftarrow s$ such that $w(\mathcal{P}) \geq w(\mathcal{Q}_1)$ and $w(\mathcal{P}) \geq w(\mathcal{Q}_2)$, or (iii) there exists a proof $\mathcal{Q}: t \rightarrow s$ (or $t \leftarrow s$) such that $w(\mathcal{P}) \geq w(\mathcal{Q})$. Hence we will show that one of (i), (ii), or (iii) holds for a given proof $\mathcal{P}: t \leftarrow p \rightarrow s$.

Let $\mathcal{P}: t \xrightarrow{\Delta} p \xrightarrow{\Delta'} s$ where two redexes $\Delta \equiv l\theta$ and $\Delta' \equiv l'\theta'$ are associated with two rules $r_1: l \rightarrow r \Leftarrow x_1 = y_1, \dots, x_m = y_m$ and $r_2: l' \rightarrow r' \Leftarrow x'_1 = y'_1, \dots, x'_m = y'_m$, respectively.

Case 1. Δ and Δ' are disjoint. Then $p \equiv C[\Delta, \Delta']$ for some context $C[_, _]$ and $\mathcal{P}: t \equiv C[t', \Delta'] \xrightarrow{\Delta} C[\Delta, \Delta'] \xrightarrow{\Delta'} C[\Delta, s'] \equiv s$ for some t' and s' . Since we can take $\mathcal{Q}_1 = \mathcal{Q}_2: t \equiv C[t', \Delta'] \xrightarrow{\Delta'} C[t', s'] \xrightarrow{\Delta} C[\Delta, s'] \equiv s$ with $w(\mathcal{Q}_1) = w(\mathcal{Q}_2) = w(\mathcal{P})$, (ii) holds.

Case 2. Δ' occurs in θ of $\Delta \equiv l\theta$ (i.e., Δ' occurs below the pattern l). Without loss of generality we may assume that $r_1: C_L[x_1, \dots, x_m] \rightarrow C_R[y_1, \dots, y_n] \Leftarrow x_1 = y_1, \dots, x_m = y_m$ (all the variable occurrences are displayed), $\mathcal{P}: p \equiv C[C_L[p_1, \dots, p_m]] \xrightarrow{\Delta} t \equiv C[C_R[t_1, \dots, t_n]]$ with subproofs $\mathcal{P}_i: p_i \xrightarrow{*} t_i$ ($i = 1, \dots, m$), and $\mathcal{P}'': p \equiv C[C_L[p_1, p_2, \dots, p_m]] \xrightarrow{\Delta'} s \equiv C[C_L[p'_1, p_2, \dots, p_m]]$ by $p_1 \xrightarrow{\Delta'} p'_1$. Thus $w(\mathcal{P}) = w(\mathcal{P}') + w(\mathcal{P}'')$ and $w(\mathcal{P}') = 1 + \sum_{i=1}^m w(\mathcal{P}_i)$. Since we have a proof $\mathcal{Q}': p'_1 \xrightarrow{\Delta'} p_1 \xrightarrow{*} t_1$ with $w(\mathcal{Q}') = w(\mathcal{P}'') + w(\mathcal{P}_1)$, we can apply r_1 to $s \equiv C[C_L[p'_1, p_2, \dots, p_m]]$ too. Then, we have a proof $\mathcal{Q}: s \equiv C[C_L[p'_1, \dots, p_m]] \rightarrow t \equiv C[C_R[t_1, \dots, t_n]]$ with $w(\mathcal{Q}) = 1 + w(\mathcal{Q}') + \sum_{i=2}^m w(\mathcal{P}_i) = w(\mathcal{P})$. Thus, (iii) follows.

Case 3. Δ and Δ' coincide by the application of the same rule, i.e., $r = r_1 = r_2$. (We mentioned in Example 3 that in a left-right separated conditional term rewriting system the application of the same rule at the same position does not imply the same result as the variables occurring in the left-hand side of a rule do not cover that in the right-hand side. Thus this case is necessary even if the system is non-overlapping.) Let the rule applied to Δ and Δ' be $r: C_L[x_1, \dots, x_m] \rightarrow C_R[y_1, \dots, y_n] \Leftarrow x_1 = y_1, \dots, x_m = y_m$ (all the variable occurrences are displayed, and $m \geq n$ by the condition (iv) of Definition 6), and let $\mathcal{P}: p \equiv C[C_L[p_1, \dots, p_m]] \xrightarrow{\Delta} t \equiv C[C_R[t_1, \dots, t_n]]$ with subproofs $\mathcal{P}'_i: p_i \xrightarrow{*} t_i$ ($i = 1, \dots, m$) and $\mathcal{P}'': p \equiv C[C_L[p_1, \dots, p_m]] \xrightarrow{\Delta'} s \equiv C[C_R[s_1, \dots, s_n]]$ with subproofs $\mathcal{P}''_i: p_i \xrightarrow{*} s_i$ ($i = 1, \dots, m$). Here $w(\mathcal{P}) = w(\mathcal{P}') + w(\mathcal{P}'') = 2 + \sum_{i=1}^m w(\mathcal{P}'_i) + \sum_{i=1}^m w(\mathcal{P}''_i)$. Then, we have a proof $\mathcal{Q}: t \equiv C[C_R[t_1, \dots, t_n]] \xrightarrow{*} C[C_R[p_1, \dots, p_n]] \xrightarrow{*} C[C_R[s_1, \dots, s_n]] \equiv s$ with $w(\mathcal{Q}) = \sum_{i=1}^n w(\mathcal{P}'_i) + \sum_{i=1}^n w(\mathcal{P}''_i) < 2 + \sum_{i=1}^m w(\mathcal{P}'_i) + \sum_{i=1}^m w(\mathcal{P}''_i) = w(\mathcal{P})$. (Note that $m \geq n$ is necessary to guarantee $w(\mathcal{Q}) < w(\mathcal{P})$.) Hence (i) holds.

Case 4. Δ' occurs in Δ but neither Case 2 nor Case 3 (i.e., Δ' overlaps with the pattern l of $\Delta \equiv l\theta$). Then, there exists a conditional critical pair $[p_1 = q_1, \dots, p_m = q_m] \vdash \langle q, q' \rangle$ between r_1 and r_2 , and we can write $\mathcal{P}: t \equiv C[q\theta] \xrightarrow{\Delta} p \equiv C[\Delta] \xrightarrow{\Delta'} s \equiv C[q'\theta]$ with subproofs $\mathcal{P}_i: p_i\theta \xrightarrow{*} q_i\theta$ ($i = 1, \dots, m$). Thus $w(\mathcal{P}) = \sum_{i=1}^m w(\mathcal{P}_i) + 2$. From the assumption about critical pairs the possible relations between q and q' are given in the following subcases.

Subcase 4.1. $q \sim_{E'} q'$ for some E' such that $E' \subseteq E \sqcup [\bullet]$. By Lemma 3 and $E' \subseteq E \sqcup [\bullet]$, we have a proof $\mathcal{Q}': q\theta \xrightarrow{*} q'\theta$ with $w(\mathcal{Q}') \leq \sum_{i=1}^m w(\mathcal{P}_i) + 1 < w(\mathcal{P})$. Hence it is obtained that $\mathcal{Q}: t \equiv C[q\theta] \xrightarrow{*} s \equiv C[q'\theta]$ with $w(\mathcal{Q}) < w(\mathcal{P})$. Thus, (i) holds.

Subcase 4.2. $q \sim_{E_1} \cdot \sim_{E_2} q'$ and $q \sim_{E'_1} \cdot \sim_{E'_2} q'$ for some E_1, E_2, E'_1 , and E'_2 such that $E_1 \sqcup E_2 \subseteq E \sqcup [\bullet]$ and $E'_1 \sqcup E'_2 \subseteq E \sqcup [\bullet]$. By Lemma 3 and $E_1 \sqcup E_2 \subseteq E \sqcup [\bullet]$, we have a proof $\mathcal{Q}': q\theta \rightarrow \cdot \xrightarrow{*} q'\theta$ with $w(\mathcal{Q}') \leq \sum_{i=1}^m w(\mathcal{P}_i) + 2 = w(\mathcal{P})$. Hence we can take $\mathcal{Q}_1: t \equiv C[q\theta] \rightarrow \cdot \xrightarrow{*} s \equiv C[q'\theta]$ with $w(\mathcal{Q}_1) \leq w(\mathcal{P})$. Similarly we have $\mathcal{Q}_2: t \equiv C[q\theta] \xrightarrow{*} \cdot \leftarrow s \equiv C[q'\theta]$ with $w(\mathcal{Q}_2) \leq w(\mathcal{P})$. Thus, (ii) follows.

Subcase 4.3. $q \sim_{E'} \triangleright q'$ (or $q \sim_{E'} \triangleleft q'$) and $E' \subseteq E \sqcup [\bullet]$. By Lemma 3 and $E' \subseteq E \sqcup [\bullet]$, we have a proof $\mathcal{Q}': q\theta \rightarrow q'\theta$ with $w(\mathcal{Q}') \leq \sum_{i=1}^m w(\mathcal{P}_i) + 2 = w(\mathcal{P})$. Hence we obtain $\mathcal{Q}: t \equiv C[q\theta] \rightarrow s \equiv C[q'\theta]$ with $w(\mathcal{Q}) \leq w(\mathcal{P})$. For the case of $q \sim_{E'} \triangleleft q'$ we can obtain $\mathcal{Q}: s \leftarrow t$ with $w(\mathcal{Q}) \leq w(\mathcal{P})$ similarly. Thus, (iii) holds. $\square$

Corollary 1 *Let R be a left-right separated conditional term rewriting system. Then R is weight decreasing joinable if R is non-overlapping.*

Example 4 *Let R be the left-right separated conditional term rewriting system with the following rewrite rules:*

$$R \quad \left\{ \begin{array}{l} f(x', x'') \rightarrow h(x, f(x, b)) \Leftarrow x' = x, x'' = x \\ f(g(y'), y'') \rightarrow h(y, f(g(y), a)) \Leftarrow y' = y, y'' = y \\ a \rightarrow b \end{array} \right.$$

Here, R has the conditional critical pair

$[g(y') = x, y'' = x, y' = y, y'' = y] \vdash \langle h(x, f(x, b)), h(y, f(g(y), a)) \rangle$.
 Since $h(x, f(x, b)) \underset{[y''=x]}{\sim} h(y'', f(x, b)) \underset{[g(y')=x]}{\sim} h(y'', f(g(y'), b)) \underset{[y''=y, y'=y]}{\sim}$
 $h(y, f(g(y), b)) \underset{[\bullet]}{\sim} h(y, f(g(y), a))$, we have $h(x, f(x, b)) \underset{E'}{\sim} h(y, f(g(y), a))$ where $E' = [g(y') = x, y'' = x, y'' = y, y' = y, \bullet]$. Thus, from Theorem 1 it follows that R is weight decreasing joinable.

We say that $E = [p_1 = q_1, \dots, p_m = q_m]$ is satisfiable (in R) if there exist proofs $\mathcal{P}_i: p_i \theta^* \rightarrow q_i \theta$ ($i = 1, \dots, m$) for some θ ; otherwise E is unsatisfiable. Note that the satisfiability of E is generally undecidable. Theorem 1 requests that every conditional critical pair $E \vdash \langle q, q' \rangle$ satisfies (i), (ii) or (iii). However, it is clear that we can ignore conditional critical pairs having unsatisfiable E . Thus, we can strengthen Theorem 1 as follows.

Corollary 2 *Let R be a left-right separated conditional term rewriting system. Then R is weight decreasing joinable if any conditional critical pair $E \vdash \langle q, q' \rangle$ such that E is satisfiable in R satisfies (i), (ii) or (iii) in Theorem 1.*

6 Conditional Linearization

The original idea of the conditional linearization of non-left-linear term rewriting systems was introduced by De Vrijer (1990), Klop and De Vrijer (1989) for giving a simpler proof of Chew's theorem (Chew, 1981; Ogawa, 1992). In this section, we introduce a new conditional linearization based on left-right separated conditional term rewriting systems. The point of our linearization is that by replacing traditional conditional systems with left-right separated conditional systems we can easily relax the non-overlapping limitation.

Now we explain a new linearization of non-left-linear rules. For instance, let consider a non-duplicating non-left-linear rule $f(x, x, x, y, y, z) \rightarrow g(x, x, x, z)$. Then, by replacing all the variable occurrences x, x, x, y, y, z from left to right in the left handside with distinct fresh variable occurrences $x', x'', x''', y', y'', z'$ respectively and connecting every fresh variable to corresponding original one with equation, we can make a left-right separated conditional rule $f(x', x'', x''', y', y'', z') \rightarrow g(x, x, x, z) \Leftarrow x' = x, x'' = x, x''' = x, y' = y, y'' = y, z' = z$. More formally we have the following definition, the framework of which originates essentially from De Vrijer (1990), Klop and De Vrijer (1989).

Definition 10 (i) *If r is a non-duplicating rewrite rule $l \rightarrow r$ and $l \equiv C[y_1, \dots, y_m]$ (all the variable occurrences of l are displayed), then the (left-right separated) conditional linearization of r is a left-right separated conditional rewrite rule $r_L: l' \rightarrow r \Leftarrow x_1 = y_1, \dots, x_m = y_m$ where $l' \equiv C[x_1, \dots, x_m]$ and $x_1, \dots, x_m$ are distinct fresh variables. Note that $l'\theta \equiv l$ for the substitution $\theta = [x_1 := y_1, \dots, x_m := y_m]$.*

(ii) *If R is a non-duplicating term rewriting system, then R_L , the conditional linearization of R , is defined as the set of the rewrite rules $\{r_L | r \in R\}$.*

Note. The non-duplicating limitation of R in the above definition is necessary to guarantee that R_L is a left-right separated conditional term rewriting system. Otherwise R_L does not satisfy the condition (iv) of Definition 6 in general.

The above conditional linearization is different from the original one by Klop and De Vrijer (1989) and De Vrijer (1990) in which the left-linear version of a rewrite rule r is a traditional conditional rewrite rule without extra variables in the right handside and the conditional part. Hence, in the case r is already left-linear, Klop and De Vrijer (1989) and De Vrijer (1990) can take r itself as its conditional linearization. On the other hand, in our definition we cannot take r itself as its conditional linearization since r is not a left-right separated rewrite rule.

Theorem 2 *If a conditional linearization R_L of a non-duplicating term rewriting system R is Church-Rosser, then R has unique normal forms.*

Proof. By Proposition 1, similar to Klop and De Vrijer (1989). $\square$

Example 5 *Let R be the non-duplicating term rewriting system with the following rewrite rules:*

$$R \quad \left\{ \begin{array}{l} f(x, x) \rightarrow h(x, f(x, b)) \\ f(g(y), y) \rightarrow h(y, f(g(y), a)) \\ a \rightarrow b \end{array} \right.$$

Note that R is non-left-linear and non-terminating. Then we have the following R_L as the linearization of R :

$$R_L \quad \left\{ \begin{array}{l} f(x', x'') \rightarrow h(x, f(x, b)) \Leftarrow x' = x, x'' = x \\ f(g(y'), y'') \rightarrow h(y, f(g(y), a)) \Leftarrow y' = y, y'' = y \\ a \rightarrow b \end{array} \right.$$

In Example 4 the Church-Rosser property of R_L has already been shown. Thus, from Theorem 2 it follows that R has unique normal forms.

7 Church-Rosser Property of Non-Duplicating Systems

In the previous section we have shown a general method based on the conditional linearization technique to prove the unique normal form property of non-left-linear overlapping non-duplicating term rewriting systems. In this section we show that the same conditional linearization technique can be used as a general method for proving the Church-Rosser property of some class of non-duplicating term rewriting systems.

Theorem 3 *Let R be a term rewriting system in which every rewrite rule $l \rightarrow r$ is right-linear (i.e., r is linear) and no non-linear variables in l occur in r . If the conditional linearization R_L of R is weight decreasing joinable then R is Church-Rosser.*

Proof. Let R and R_L have reduction relations $\rightarrow$ and $\xrightarrow{L}$ respectively. Since $\xrightarrow{L}$ extends $\rightarrow$ and R_L is weight decreasing joinable, the theorem clearly holds if we show the claim: for any t, s and $\mathcal{P}: t \xrightarrow{L}^* s$ there exist proofs $\mathcal{Q}: t \xrightarrow{L}^* r \xleftarrow{L}^* s$ with $w(\mathcal{P}) \geq w(\mathcal{Q})$ and $t \xrightarrow{L}^* r \xleftarrow{L}^* s$ for some term r . We will prove this claim by induction on $w(\mathcal{P})$. *Base Step* ($w(\mathcal{P}) = 0$) is trivial. *Induction Step:* Let $w(\mathcal{P}) = \rho > 0$. Form the weight decreasing joinability of R_L , we have a proof $\mathcal{P}': t \xrightarrow{L}^* \cdot \xleftarrow{L}^* s$ with $\rho \geq w(\mathcal{P}')$. Let $\mathcal{P}'$ have the form $t \xrightarrow{L} \hat{s} \xrightarrow{L}^* \cdot \xleftarrow{L}^* s$. Without loss of generality we may assume that $C_L[x_1, \dots, x_m, y_1, \dots, y_n, \dots, z_1, \dots, z_p, v_1, \dots, w_1] \rightarrow C_R[v, \dots, w] \Leftarrow x_1 = x, \dots, x_m = x, y_1 = y, \dots, y_n = y, \dots, z_1 = z, \dots, z_p = z, v_1 = v, \dots, w_1 = w$ (all the variable occurrences are displayed) is a linearization of a right-linear rewrite rule $C_L[x, \dots, x, y, \dots, y, \dots, z, \dots, z, v, \dots, w] \rightarrow C_R[v, \dots, w]$ and $t \equiv C[C_L[t_1^x, \dots, t_m^x, t_1^y, \dots, t_n^y, \dots, t_1^z, \dots, t_p^z, t_1^v, \dots, t_1^w]] \xrightarrow{L} \hat{s} \equiv C[C_R[t^v, \dots, t^w]]$ with subproofs $\mathcal{P}_i^x: t_i^x \xrightarrow{L}^* t^x$ ($i = 1, \dots, m$), $\mathcal{P}_j^y: t_j^y \xrightarrow{L}^* t^y$ ($j = 1, \dots, n$), $\dots$, $\mathcal{P}_k^z: t_k^z \xrightarrow{L}^* t^z$ ($k = 1, \dots, p$) for some $t^x, t^y, \dots, t^z$, and $\mathcal{P}^v: t_1^v \xrightarrow{L}^* t^v, \dots, \mathcal{P}^w: t_1^w \xrightarrow{L}^* t^w$. Then, we can take $t \equiv C[C_L[t_1^x, \dots, t_m^x, t_1^y, \dots, t_n^y, \dots, t_1^z, \dots, t_p^z, t_1^v, \dots, t_1^w]] \xrightarrow{L} s' \equiv C[C_R[t_1^v, \dots, t_1^w]] \xleftarrow{L} \hat{s} \equiv C[C_R[t^v, \dots, t^w]] \xrightarrow{L}^* \cdot \xleftarrow{L}^* s$ with the weight $w(\mathcal{P}')$. Let $\mathcal{P}'': t \equiv C[C_L[t_1^x, \dots, t_m^x, t_1^y, \dots, t_n^y, \dots, t_1^z, \dots, t_p^z, t_1^v, \dots, t_1^w]] \xrightarrow{L}^* \cdot \xleftarrow{L}^* s$ with the weight $w(\mathcal{P}')$. Let $\mathcal{P}'': t \equiv C[C_L[t_1^x, \dots, t_m^x, t_1^y, \dots, t_n^y, \dots, t_1^z, \dots, t_p^z, t_1^v, \dots, t_1^w]] \xrightarrow{L}^* \cdot \xleftarrow{L}^* s$ with the weight $w(\mathcal{P}')$.

$\dots, t_p^z, t_1^v, \dots, t_1^w] \xrightarrow{L} s' \equiv C[C_R[t_1^v, \dots, t_1^w]]$. Then, from Lemma 2 and the induction hypothesis we have proofs $t_i^x \xrightarrow{*} \tilde{t}_x$ ($i = 1, \dots, m$), $t_j^y \xrightarrow{*} \tilde{t}_y$ ($j = 1, \dots, n$), $\dots, t_k^z \xrightarrow{*} \tilde{t}_z$ ($k = 1, \dots, p$). Hence we can take the reduction $t \equiv C[C_L[t_1^x, \dots, t_m^x, t_1^y, \dots, t_n^y, \dots, t_1^z, \dots, t_p^z, t_1^v, \dots, t_1^w]] \xrightarrow{*} C[C_L[\tilde{t}_x, \dots, \tilde{t}_x, \tilde{t}_y, \dots, \tilde{t}_y, \dots, \tilde{t}_z, \dots, \tilde{t}_z, t_1^v, \dots, t_1^w]] \rightarrow s' \equiv C[C_R[t_1^v, \dots, t_1^w]]$. Let $\hat{\mathcal{P}}: s' \xrightarrow{L} \hat{s} \xrightarrow{L} \cdot \xrightarrow{L} s$. From $\rho > w(\hat{\mathcal{P}})$ and induction hypothesis, we have $\hat{\mathcal{Q}}: s' \xrightarrow{L} r \xrightarrow{L} s$ with $w(\hat{\mathcal{P}}) \geq w(\hat{\mathcal{Q}})$ and $s' \xrightarrow{*} r \xrightarrow{*} s$ for some r . Thus, the theorem follows. $\square$

Corollary 3 *Let R be a term rewriting system in which every rewrite rule $l \rightarrow r$ is right-linear and no non-linear variables in l occur in r . If the conditional linearization R_L of R is non-overlapping then R is Church-Rosser.*

The following corollary was originally proven by Oyamaguchi and Ohta (1993).

Corollary 4 [Oyamaguchi] *Let R be a right-ground term rewriting system having a non-overlapping conditional linearization R_L . Then R is Church-Rosser.*

Example 6 *Let R be the term rewriting system with the following rewrite rules:*

$$R \quad \left\{ \begin{array}{l} f(x, x, y) \rightarrow h(y, c) \\ g(x) \rightarrow f(x, c, g(c)) \\ c \rightarrow h(c, c) \end{array} \right.$$

Note that R is non-left-linear and non-terminating. Then we have the following R_L as the linearization of R :

$$R_L \quad \left\{ \begin{array}{l} f(x', x'', y') \rightarrow h(y, c) \Leftarrow x' = x, x'' = x, y' = y \\ g(x') \rightarrow f(x, c, g(c)) \Leftarrow x' = x \\ c \rightarrow h(c, c) \end{array} \right.$$

From Corollary 1, R_L is Church-Rosser. Thus, from Corollary 3 it follows that R is Church-Rosser.

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