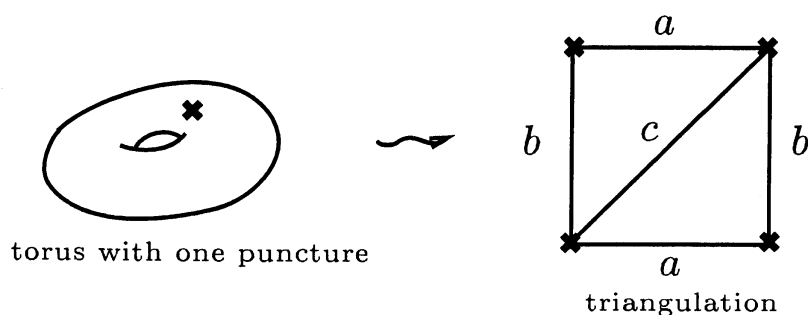


On $PGL_3(\mathbb{C})$ -torsions

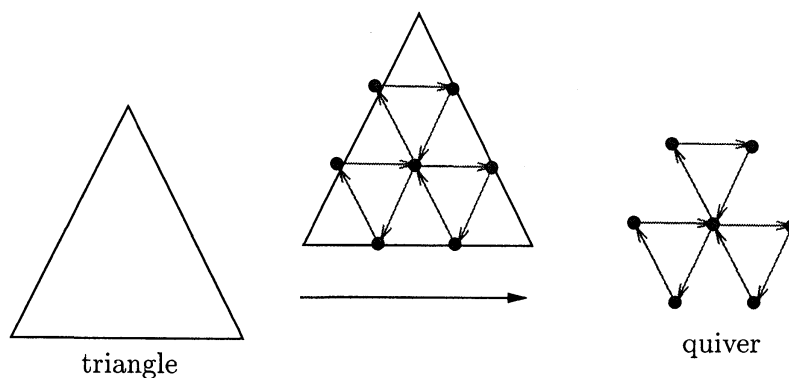
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In this note, we explain how to obtain a $PGL_3(\mathbb{C})$ -torsion for a mapping torus of a surface with punctures, using a concrete description of the action of a mapping class on Fock-Goncharov parameters.

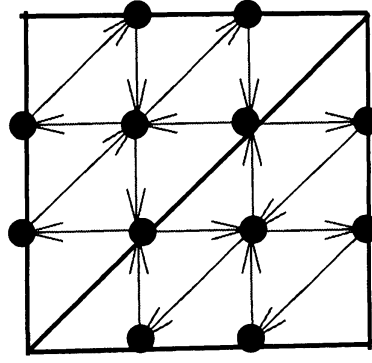
Step 1. First, we fix an ideal triangulation of a surface S with punctures. In the case of a torus with one puncture, we choose the following triangulation:



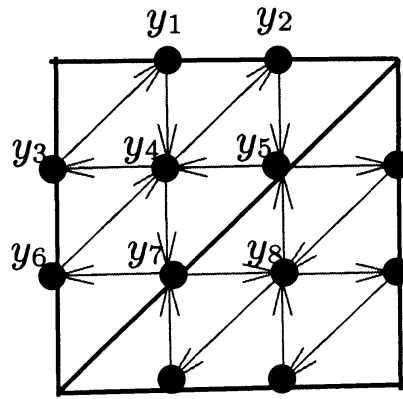
Step 2. For each triangle in the ideal triangulation, we put the following quiver:



In the case of the triangulation of a torus with one puncture in Step 1, we have the following quiver:



Step 3. For each vertex v in the quiver, we put a variable y_v . These variable $\{y_v\}$ are called Fock-Goncharov parameters [FG]. In the case of the quiver for a torus with one puncture in Step 2, we have variables $y_1, y_2, \dots, y_8$:



Step 4. For a Dehn twist φ on a surface S , we write down the action $\varphi^\#$ on Fock-Goncharov parameters. This step can be done concretely according to the appendix in [TY] (see also [NTY]). In the case of the Dehn twists L, R on a torus with one puncture, we have the following

action:

$$\begin{aligned}
L^\#(y_1) &= \frac{y_1 y_3^2 (1 + y_6) y_8}{(1 + y_3)(1 + y_3 + y_3 y_8 + y_3 y_6 y_8)} \\
L^\#(y_2) &= \frac{y_2 (1 + y_3) y_4 y_6^2}{(1 + y_6)(1 + y_6 + y_4 y_6 + y_3 y_4 y_6)} \\
L^\#(y_3) &= \frac{(1 + y_3) y_5 (1 + y_6 + y_4 y_6 + y_3 y_4 y_6)}{1 + y_6} \\
L^\#(y_4) &= \frac{(1 + y_6 + y_4 y_6 + y_3 y_4 y_6) y_8}{1 + y_3 + y_3 y_8 + y_3 y_6 y_8} \\
L^\#(y_5) &= \frac{1 + y_6}{(1 + y_3) y_4 y_6} \\
L^\#(y_6) &= \frac{(1 + y_6) y_7 (1 + y_3 + y_3 y_8 + y_3 y_6 y_8)}{1 + y_3} \\
L^\#(y_7) &= \frac{1 + y_3}{y_3 (1 + y_6) y_8} \\
L^\#(y_8) &= \frac{y_4 (1 + y_3 + y_3 y_8 + y_3 y_6 y_8)}{1 + y_6 + y_4 y_6 + y_3 y_4 y_6} \\
\\
R^\#(y_1) &= \frac{y_1^2 (1 + y_2) y_4 y_7}{(1 + y_1)(1 + y_1 + y_1 y_4 + y_1 y_2 y_4)} \\
R^\#(y_2) &= \frac{(1 + y_1) y_2^2 y_5 y_8}{(1 + y_2)(1 + y_2 + y_2 y_8 + y_1 y_2 y_8)} \\
R^\#(y_3) &= \frac{(1 + y_1) y_3 (1 + y_2 + y_2 y_8 + y_1 y_2 y_8)}{1 + y_2} \\
R^\#(y_4) &= \frac{y_4 (1 + y_2 + y_2 y_8 + y_1 y_2 y_8)}{1 + y_1 + y_1 y_4 + y_1 y_2 y_4} \\
R^\#(y_5) &= \frac{1 + y_2}{(1 + y_1) y_2 y_8} \\
R^\#(y_6) &= \frac{(1 + y_2)(1 + y_1 + y_1 y_4 + y_1 y_2 y_4) y_6}{1 + y_1} \\
R^\#(y_7) &= \frac{1 + y_1}{y_1 (1 + y_2) y_4} \\
R^\#(y_8) &= \frac{(1 + y_1 + y_1 y_4 + y_1 y_2 y_4) y_8}{1 + y_2 + y_2 y_8 + y_1 y_2 y_8}
\end{aligned}$$

Step 5. For a mapping class φ of a surface S , we consider the equation of fixed points:

$$y_i = \varphi^\#(y_i)$$

This equation should be the equation of a "geometric part" of the moduli space of $PGL(3; \mathbb{C})$ -representations of the fundamental group of the mapping torus M_φ with φ . In the case of the

mapping class LR of a torus with one puncture, we have the following equation:

$$\begin{aligned}
y_1 &= (y_1^2 y_3^3 (1 + 2y_6 + y_4 y_6 + y_3 y_4 y_6 + y_6^2 + y_4 y_6^2 + y_2 y_4 y_6^2 + y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2) y_8^2) / \\
&\quad ((1 + 2y_3 + y_3^2 + y_3 y_8 + y_3^2 y_8 + y_1 y_3^2 y_8 + y_3 y_6 y_8 + y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8) \\
&\quad (1 + 2y_3 + y_3^2 + 2y_3 y_8 + 2y_3^2 y_8 + y_1 y_3^2 y_8 + 2y_3 y_6 y_8 + 2y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8 + \\
&\quad y_3^2 y_8^2 + y_1 y_3^2 y_8^2 + 2y_3^2 y_6 y_8^2 + 2y_1 y_3^2 y_6 y_8^2 + y_1 y_3^2 y_4 y_6 y_8^2 + \\
&\quad y_3^2 y_6 y_8^2 + y_1 y_3^2 y_6 y_8^2 + y_1 y_3^2 y_4 y_6 y_8^2 + y_1 y_2 y_3^2 y_4 y_6 y_8^2)) \\
y_2 &= (y_2^2 y_4^2 y_6^3 (1 + 2y_3 + y_3^2 + y_3 y_8 + y_3^2 y_8 + y_1 y_3^2 y_8 + y_3 y_6 y_8 + y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8)) / \\
&\quad ((1 + 2y_6 + y_4 y_6 + y_3 y_4 y_6 + y_6^2 + y_4 y_6^2 + y_2 y_4 y_6^2 + y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2) \\
&\quad (1 + 2y_6 + 2y_4 y_6 + 2y_3 y_4 y_6 + y_6^2 + 2y_4 y_6^2 + y_2 y_4 y_6^2 + 2y_3 y_4 y_6^2 + \\
&\quad y_2 y_3 y_4 y_6^2 + y_4^2 y_6^2 + y_2 y_4^2 y_6^2 + 2y_3 y_4^2 y_6^2 + 2y_2 y_3 y_4^2 y_6^2 + y_3^2 y_4^2 y_6^2 + \\
&\quad y_2 y_3^2 y_4^2 y_6^2 + y_2 y_3 y_4^2 y_6^2 y_8 + y_2 y_3^2 y_4^2 y_6^2 y_8 + y_1 y_2 y_3^2 y_4^2 y_6^2 y_8)) \\
y_3 &= (y_5 (1 + 2y_3 + y_3^2 + y_3 y_8 + y_3^2 y_8 + y_1 y_3^2 y_8 + y_3 y_6 y_8 + y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8) \\
&\quad (1 + 2y_6 + 2y_4 y_6 + 2y_3 y_4 y_6 + y_6^2 + 2y_4 y_6^2 + y_2 y_4 y_6^2 + 2y_3 y_4 y_6^2 + \\
&\quad y_2 y_3 y_4 y_6^2 + y_4^2 y_6^2 + y_2 y_4^2 y_6^2 + 2y_3 y_4^2 y_6^2 + 2y_2 y_3 y_4^2 y_6^2 + y_3^2 y_4^2 y_6^2 + \\
&\quad y_2 y_3^2 y_4^2 y_6^2 + y_2 y_3 y_4^2 y_6^2 y_8 + y_2 y_3^2 y_4^2 y_6^2 y_8 + y_1 y_2 y_3^2 y_4^2 y_6^2 y_8)) / \\
&\quad ((1 + 2y_6 + y_4 y_6 + y_3 y_4 y_6 + y_6^2 + y_4 y_6^2 + y_2 y_4 y_6^2 + y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2) \\
&\quad (1 + y_3 + y_3 y_8 + y_3 y_6 y_8)) \\
y_4 &= (y_8 (1 + y_3 + y_3 y_8 + y_3 y_6 y_8) (1 + 2y_6 + 2y_4 y_6 + 2y_3 y_4 y_6 + y_6^2 + 2y_4 y_6^2 + y_2 y_4 y_6^2 + \\
&\quad 2y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2 + y_4^2 y_6^2 + y_2 y_4^2 y_6^2 + 2y_3 y_4^2 y_6^2 + 2y_2 y_3 y_4^2 y_6^2 + \\
&\quad y_3^2 y_4^2 y_6^2 + y_2 y_3^2 y_4^2 y_6^2 + y_2 y_3 y_4^2 y_6^2 y_8 + y_2 y_3^2 y_4^2 y_6^2 y_8 + y_1 y_2 y_3^2 y_4^2 y_6^2 y_8)) / \\
&\quad ((1 + y_6 + y_4 y_6 + y_3 y_4 y_6) (1 + 2y_3 + y_3^2 + 2y_3 y_8 + 2y_3^2 y_8 + y_1 y_3^2 y_8 + 2y_3 y_6 y_8 + \\
&\quad 2y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8 + y_3^2 y_8^2 + y_1 y_3^2 y_8^2 + 2y_3^2 y_6 y_8^2 + 2y_1 y_3^2 y_6 y_8^2 + \\
&\quad y_1 y_3^2 y_4 y_6 y_8^2 + y_3^2 y_6 y_8^2 + y_1 y_3^2 y_6 y_8^2 + y_1 y_3^2 y_4 y_6 y_8^2 + y_1 y_2 y_3^2 y_4 y_6 y_8^2)) \\
y_5 &= ((1 + y_6 + y_4 y_6 + y_3 y_4 y_6) (1 + 2y_6 + y_4 y_6 + y_3 y_4 y_6 + y_6^2 + y_4 y_6^2 + \\
&\quad y_2 y_4 y_6^2 + y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2)) / \\
&\quad (y_2 y_4^2 y_6^2 (1 + 2y_3 + y_3^2 + y_3 y_8 + y_3^2 y_8 + y_1 y_3^2 y_8 + y_3 y_6 y_8 + y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8)) \\
y_6 &= ((1 + 2y_6 + y_4 y_6 + y_3 y_4 y_6 + y_6^2 + y_4 y_6^2 + y_2 y_4 y_6^2 + y_3 y_4 y_6^2 + y_2 y_3 y_4 y_6^2) y_7 \\
&\quad (1 + 2y_3 + y_3^2 + 2y_3 y_8 + 2y_3^2 y_8 + y_1 y_3^2 y_8 + 2y_3 y_6 y_8 + 2y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8 + \\
&\quad y_3^2 y_8^2 + y_1 y_3^2 y_8^2 + 2y_3^2 y_6 y_8^2 + 2y_1 y_3^2 y_6 y_8^2 + y_1 y_3^2 y_4 y_6 y_8^2 + y_3^2 y_6 y_8^2 + \\
&\quad y_1 y_3^2 y_6 y_8^2 + y_1 y_3^2 y_4 y_6 y_8^2 + y_1 y_2 y_3^2 y_4 y_6 y_8^2)) / ((1 + y_6 + y_4 y_6 + y_3 y_4 y_6) \\
&\quad (1 + 2y_3 + y_3^2 + y_3 y_8 + y_3^2 y_8 + y_1 y_3^2 y_8 + y_3 y_6 y_8 + y_3^2 y_6 y_8 + y_1 y_3^2 y_6 y_8))
\end{aligned}$$

$$\begin{aligned}
y_7 = & ((1 + y_3 + y_3y_8 + y_3y_6y_8)(1 + 2y_3 + y_3^2 + y_3y_8 + y_3^2y_8 + y_1y_3^2y_8 + \\
& y_3y_6y_8 + y_3^2y_6y_8 + y_1y_3^2y_6y_8))/ \\
& (y_1y_3^2(1 + 2y_6 + y_4y_6 + y_3y_4y_6 + y_6^2 + y_4y_6^2 + y_2y_4y_6^2 + y_3y_4y_6^2 + y_2y_3y_4y_6^2)y_8^2) \\
y_8 = & (y_4(1 + y_6 + y_4y_6 + y_3y_4y_6)(1 + 2y_3 + y_3^2 + 2y_3y_8 + 2y_3^2y_8 + y_1y_3^2y_8 + 2y_3y_6y_8 + \\
& 2y_3^2y_6y_8 + y_1y_3^2y_6y_8 + y_3^2y_8^2 + y_1y_3^2y_8^2 + 2y_3^2y_6y_8^2 + 2y_1y_3^2y_6y_8^2 + \\
& y_1y_3^2y_4y_6y_8^2 + y_3^2y_6^2y_8^2 + y_1y_3^2y_6^2y_8^2 + y_1y_3^2y_4y_6^2y_8^2 + y_1y_2y_3^2y_4y_6^2y_8^2))/ \\
& ((1 + y_3 + y_3y_8 + y_3y_6y_8)(1 + 2y_6 + 2y_4y_6 + 2y_3y_4y_6 + y_6^2 + 2y_4y_6^2 + y_2y_4y_6^2 + \\
& 2y_3y_4y_6^2 + y_2y_3y_4y_6^2 + y_4^2y_6^2 + y_2y_4^2y_6^2 + 2y_3y_4^2y_6^2 + 2y_2y_3y_4^2y_6^2 + \\
& y_3^2y_4^2y_6^2 + y_2y_3^2y_4^2y_6^2 + y_2y_3y_4^2y_6^2y_8 + y_2y_3^2y_4^2y_6^2y_8 + y_1y_2y_3^2y_4^2y_6^2y_8))
\end{aligned}$$

Remark that the mapping torus in this example is the complement of the figure-eight knot.

Step 6. For the equation of fixed points, we choose a solution. We can show that, for a pseudo-Anosov mapping class φ , we have always a solution of the equation of fixed points [KT].

In the case of LR , we have a solution:

$$\begin{aligned}
y_1 &= 1 \\
y_2 &= 1 \\
y_3 &= \frac{1}{2}(-1 + \sqrt{-3}) \\
y_4 &= 1 \\
y_5 &= \frac{1}{2}(-1 - \sqrt{-3}) \\
y_6 &= \frac{1}{2}(-1 + \sqrt{-3}) \\
y_7 &= \frac{1}{2}(-1 - \sqrt{-3}) \\
y_8 &= 1
\end{aligned}$$

Step 7. For a mapping class φ of a surface S , we evaluate at a solution $y = y^{\text{sol}}$ the characteristic polynomial $\Delta(t)$ of the Jacobi matrix of $\varphi^\#$:

$$\Delta(t) = \det \left(t - \left(\frac{\partial \varphi^\#(y_i)}{\partial y_j} \right) \right) \Big|_{y=y^{\text{sol}}}$$

In the case of the mapping class LR and the solution $y = y^{\text{sol}}$ in step 6, we have the following polynomial.

$$\Delta(t) = (-1 + t)^2(1 - 5t + t^2)(1 - 9t + 44t^2 - 9t^3 + t^4)$$

Step 8. Last, we consider the following limit:

$$\tau := \lim_{t \rightarrow 1} \frac{\Delta(t)}{(t-1)^{2p}},$$

where p is the number of punctures on S . One of our results is that this number is identified with $PGL_3(\mathbb{C})$ -torsion with the adjoint representation of the representation which corresponds the solution, except sign ambiguity [KT]. A $PGL_3(\mathbb{C})$ -torsion is a generalization of a Porti's torsion for a $PSL_2(\mathbb{C})$ -representation [P]. In the case of LR , we have:

$$\begin{aligned} \tau &= \lim_{t \rightarrow 1} \frac{\Delta(t)}{(t-1)^2} \\ &= -3 \times 28 \\ &= -84 \end{aligned}$$

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