

Rarefied gas flows through a curved channel: Application of a diffusion-type equation

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Rarefied gas flows through a curved two-dimensional channel, caused by a pressure or a temperature gradient, are investigated numerically by using a macroscopic equation of convection-diffusion type. The equation, which was derived systematically from the Bhatnagar–Gross–Krook model of the Boltzmann equation and diffuse-reflection boundary condition in a previous paper [K. Aoki *et al.*, “A diffusion model for rarefied flows in curved channels,” *Multiscale Model. Simul.* **6**, 1281 (2008)], is valid irrespective of the degree of gas rarefaction when the channel width is much shorter than the scale of variations of physical quantities and curvature along the channel. Attention is also paid to a variant of the Knudsen compressor that can produce a pressure raise by the effect of the change of channel curvature and periodic temperature distributions without any help of moving parts. In the process of analysis, the macroscopic equation is (partially) extended to the case of the ellipsoidal-statistical model of the Boltzmann equation. © 2010 American Institute of Physics. [doi:10.1063/1.3496315]

I. INTRODUCTION

Rarefied gas flows in a channel driven by an imposed pressure gradient in the gas (Poiseuille flow) and by an imposed temperature gradient along the channel walls (thermal transpiration), which are fundamental problems in kinetic theory of gases, play important roles in applications related to micromechanical and vacuum systems. The case when the gas flows through a straight channel is a classical problem and has been the subject of many papers (see, e.g., Refs. 1–6 for the Poiseuille flow and Refs. 3–9 for the thermal transpiration between two parallel plates). In contrast, there are few studies for the flows through a curved channel (Refs. 10–12 are among the few examples). In practical applications in microfluid dynamics and vacuum technology, however, one encounters various complex channel shapes. In the present paper, we focus our attention on rarefied gas flows through a twisty channel.

In order to analyze rarefied gas flows for a wide range of gas rarefaction, i.e., for arbitrary Knudsen numbers (the ratio of the mean free path of gas molecules to the characteristic length of the system), we need numerical analysis of the Boltzmann equation. However, when the channel is long and twisty, it is practically impossible to analyze the flows by the prevailing numerical methods, such as the direct simulation Monte Carlo (DSMC) method^{13,14} and the finite-difference (or discrete-ordinate) method using model Boltzmann equations. In contrast, a macroscopic system, consisting of a partial differential equation of convection-diffusion type and its connection condition, that describes rarefied gas flows in a two-dimensional (2D) curved channel was proposed recently.¹⁵ With the help of this system, we can obtain the properties of gas flows through a twisty 2D channel at arbitrary Knudsen numbers with a small computational load. The system is derived from the Boltzmann equation and its boundary condition by a systematic asymptotic analysis un-

der the assumption that the channel width is small compared with the length scale of variation of the temperature, the density, and the channel curvature along the channel. We should mention that the Bhatnagar–Gross–Krook (BGK) model^{16,17} is used in Ref. 15 in place of the original Boltzmann equation, but the analysis for the latter equation is essentially the same. It should be emphasized that the macroscopic system is valid for any Knudsen number.

In the present study, we investigate the gas flows through curved channels of various shapes for a wide range of the Knudsen number by exploiting the macroscopic system derived in Ref. 15. At the same time, we extend the system, which was originally derived on the basis of the BGK model, to the case of the more sophisticated ellipsoidal-statistical (ES) model.^{18–20} This paper is organized as follows. The formulation of the problem is given in Sec. II, and the macroscopic system derived in Ref. 15 is summarized in Sec. III. The system is extended to the case of the ES model in Sec. IV, and its applications are shown in Sec. V. Finally, some concluding remarks are given in Sec. VI.

II. FORMULATION OF THE PROBLEM

A. Problem, assumptions, and notations

Let us consider a curved 2D channel with constant width D (Fig. 1). The temperatures of the channel walls do not change in time, but their distribution along the channel is arbitrary, except that they are the same at the normal cross section of the channel. The pressure (or density) of the gas in the channel may vary in time and along the channel (the change is determined depending on individual problems). We investigate the behavior of the gas in this situation.

We assume that the 2D channel is on the $\tilde{x}_1\tilde{x}_2$ plane and the physical quantities do not depend on $\tilde{x}_3$, where $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is a Cartesian coordinate system in space. Let $\tilde{\mathbf{x}}, \tilde{\mathbf{x}}_c(\tilde{s}), \boldsymbol{\tau}(\tilde{s})$,

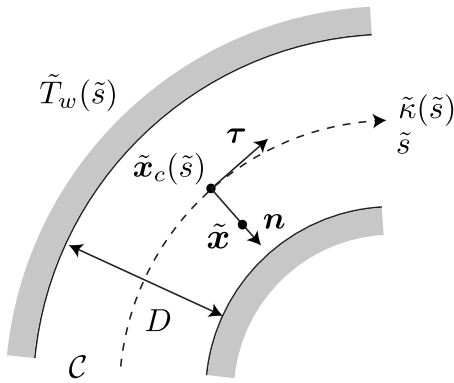


FIG. 1. Curved channel with some notations.

and $\mathbf{n}(\tilde{s})$ be the two-dimensional vectors on the $\tilde{x}_1\tilde{x}_2$ plane (i.e., the vectors with vanishing $\tilde{x}_3$ components), indicating the position $(\tilde{x}_1, \tilde{x}_2, 0)$, the median curve C of the channel, the unit tangential vector to C , and the unit normal vector to C , respectively. Here, $\tilde{s}$ is the length along C , $\boldsymbol{\tau}(\tilde{s})$ is directed in the direction of increasing $\tilde{s}$, i.e., $\boldsymbol{\tau} = d\tilde{\mathbf{x}}_c(\tilde{s})/d\tilde{s}$, and $\mathbf{n}(\tilde{s})$ is directed to the right when facing in the direction of increasing $\tilde{s}$. Then, we have the following relation (the Frenet formula²¹):

$$\frac{d\boldsymbol{\tau}}{d\tilde{s}} = \tilde{\kappa}\mathbf{n}, \quad \frac{d\mathbf{n}}{d\tilde{s}} = -\tilde{\kappa}\boldsymbol{\tau}, \quad (1)$$

where $\tilde{\kappa}(\tilde{s})$ is the curvature of the median curve C and is positive when the center of curvature lies in the side of $\mathbf{n}$. Then, we introduce the curvilinear coordinate system $(\tilde{s}, \tilde{r})$ on the $\tilde{x}_1\tilde{x}_2$ plane by expressing the point $\tilde{\mathbf{x}}$ in terms of $\tilde{s}$ and $\tilde{r}$ as

$$\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_c(\tilde{s}) + \tilde{r}\mathbf{n}, \quad (2)$$

where $-D/2 \leq \tilde{r} \leq D/2$, and we assume that the center of curvature of C does not lie inside the channel ($|\tilde{\kappa}| < 2/D$). Since the temperatures of the two walls are common at the normal cross section (the cross section along $\mathbf{n}$), we denote them by $\tilde{T}_w(\tilde{s})$. We analyze the behavior of the gas in the channel under the following assumptions:

- (i) The behavior of the gas is described by the BGK model of the Boltzmann equation (the ES model will also be employed later).
- (ii) The gas molecules undergo diffuse reflection on the walls.^{22,23} That is, the reflected molecules are distributed according to the half-range Maxwellian with flow velocity zero and temperature $\tilde{T}_w(\tilde{s})$, and there is no net mass flux across the walls.
- (iii) The curvature $\tilde{\kappa}(\tilde{s})$, the wall temperature $\tilde{T}_w(\tilde{s})$, and the pressure of the gas change slowly along the channel; that is, the length scale of variation L_s of these quantities along the channel is much longer than the channel width D ($L_s \gg D$).

Before presenting the basic equations, we summarize the additional notations used in this paper. We first introduce dimensional quantities: $\tilde{t}$ is the time variable, $\tilde{\boldsymbol{\zeta}}$ is the molecular velocity, $\tilde{f}(\tilde{s}, \tilde{r}, \tilde{\boldsymbol{\zeta}}, \tilde{t})$ is the velocity distribution func-

tion of the gas molecules, $\tilde{\rho}$ is the density of the gas, $\tilde{\mathbf{u}}$ is the flow velocity, $\tilde{T}$ is the temperature, and $\tilde{p} = R\tilde{\rho}\tilde{T}$ is the pressure, where R is the gas constant per unit mass, i.e., $R = k/m$ with k as the Boltzmann constant and m as the mass of a gas molecule. In addition, $\tilde{\zeta}_s = \tilde{\boldsymbol{\zeta}} \cdot \boldsymbol{\tau}$, $\tilde{\zeta}_r = \tilde{\boldsymbol{\zeta}} \cdot \mathbf{n}$, and $\tilde{\zeta}_3$ are the components of $\tilde{\boldsymbol{\zeta}}$ in the directions of $\boldsymbol{\tau}$, $\mathbf{n}$, and the $\tilde{x}_3$ axis, respectively; $\tilde{u}_s = \tilde{\mathbf{u}} \cdot \boldsymbol{\tau}$, $\tilde{u}_r = \tilde{\mathbf{u}} \cdot \mathbf{n}$, and $\tilde{u}_3 (=0)$ are the components of $\tilde{\mathbf{u}}$ in the corresponding directions.

Let $\tilde{\rho}_0$, $\tilde{T}_0$, and $\tilde{p}_0 = R\tilde{\rho}_0\tilde{T}_0$ be the reference density, temperature, and pressure, respectively, and l_0 be the mean free path of the gas molecules at the reference equilibrium state at rest with density $\tilde{\rho}_0$ and temperature $\tilde{T}_0$. For the BGK model, $l_0 = (8R\tilde{T}_0/\pi)^{1/2}/A_{\text{BGK}}\tilde{\rho}_0$, where A_{BGK} is a constant such that $A_{\text{BGK}}\tilde{\rho}_0$ is the collision frequency of the gas molecules. Then, we introduce the following dimensionless quantities:

$$\begin{aligned} s &= \tilde{s}/L_s, & r &= \tilde{r}/D, \\ x_3 &= \tilde{x}_3/D, & t &= \tilde{t}/[L_s^2/D(2R\tilde{T}_0)^{1/2}], \\ \zeta &= \tilde{\boldsymbol{\zeta}}/(2R\tilde{T}_0)^{1/2}, & (\zeta_s, \zeta_r, \zeta_3) &= (\tilde{\zeta}_s, \tilde{\zeta}_r, \tilde{\zeta}_3)/(2R\tilde{T}_0)^{1/2}, \\ f &= \tilde{f}/[\tilde{\rho}_0(2R\tilde{T}_0)^{-3/2}], & \rho &= \tilde{\rho}/\tilde{\rho}_0, \\ \mathbf{u} &= \tilde{\mathbf{u}}/(2R\tilde{T}_0)^{1/2}, & (u_s, u_r, u_3) &= (\tilde{u}_s, \tilde{u}_r, \tilde{u}_3)/(2R\tilde{T}_0)^{1/2}, \\ T &= \tilde{T}/\tilde{T}_0, & p &= \tilde{p}/\tilde{p}_0, \\ \kappa(s) &= \tilde{\kappa}(\tilde{s})D, & T_w(s) &= \tilde{T}_w(\tilde{s})/\tilde{T}_0, \\ \text{Kn} &= l_0/D, \end{aligned} \quad (3)$$

where Kn is the Knudsen number. Because of assumption (iii), κ and T_w are assumed to be functions of s . In Eq. (3), we have also assumed that $f = f(s, r, \boldsymbol{\zeta}, t)$, so that ρ , u_s , u_r , T , and p are functions of (s, r, t) .

B. Basic equations

The BGK model in the dimensionless form in the (s, r, x_3) coordinate system reads as

$$\begin{aligned} \epsilon^2 \frac{\partial f}{\partial t} + \epsilon \frac{1}{1 - \kappa r} \zeta_s \frac{\partial f}{\partial s} + \zeta_r \frac{\partial f}{\partial r} + \frac{\kappa}{1 - \kappa r} \zeta_r \zeta_s \frac{\partial f}{\partial \zeta_s} \\ - \frac{\kappa}{1 - \kappa r} \zeta_s^2 \frac{\partial f}{\partial \zeta_r} = \frac{1}{K_0} J_{\text{BGK}}(f), \end{aligned} \quad (4a)$$

$$J_{\text{BGK}}(f) = \rho(\mathcal{M}[\rho, \mathbf{u}, T] - f), \quad (4b)$$

$$\mathcal{M}[\rho, \mathbf{u}, T] = \frac{\rho}{(\pi T)^{3/2}} \exp\left(-\frac{|\boldsymbol{\zeta} - \mathbf{u}|^2}{T}\right), \quad (4c)$$

$$\rho = \int f d\boldsymbol{\zeta}, \quad \mathbf{u} = \frac{1}{\rho} \int \boldsymbol{\zeta} f d\boldsymbol{\zeta}, \quad T = \frac{2}{3\rho} \int |\boldsymbol{\zeta} - \mathbf{u}|^2 f d\boldsymbol{\zeta}, \quad (4d)$$

where

$$\epsilon = D/L_s, \quad K_0 = (\sqrt{\pi}/2)\text{Kn} = (\sqrt{\pi}/2)(l_0/D). \quad (5)$$

The boundary condition (diffuse reflection) on the channel walls ($r = \pm 1/2$) can be written as

$$f = \frac{\rho_w}{(\pi T_w)^{3/2}} \exp\left(-\frac{|\xi|^2}{T_w}\right) \quad \text{for } \zeta_r \leq 0 \quad \text{at } r = \pm 1/2, \quad (6a)$$

$$\rho_w = \pm 2 \left(\frac{\pi}{T_w}\right)^{1/2} \int_{\zeta_r \geq 0} \zeta_r f d\zeta, \quad (6b)$$

where the upper and lower signs go together. The initial condition is

$$f = f^{\text{in}}(s, r, \zeta) \quad \text{at } t = 0, \quad (7)$$

with an appropriate function f^{in} .

We investigate the behavior of the gas described by Eqs. (4), (6), and (7) when $\epsilon \ll 1$ but K_0 is arbitrary.

III. SUMMARY OF MACROSCOPIC SYSTEM

In this section, we summarize the macroscopic system derived in Ref. 15. It is an extension to curved channels of the corresponding macroscopic systems for straight channels or pipes developed in Refs. 24–26.

A. Macroscopic equation

The solution to Eqs. (4), (6), and (7) is obtained in the form of power series in ϵ , i.e.,

$$f = f_{(0)} + f_{(1)}\epsilon + f_{(2)}\epsilon^2 + \cdots. \quad (8)$$

Correspondingly, the macroscopic quantities are expanded as

$$h = h_{(0)} + h_{(1)}\epsilon + h_{(2)}\epsilon^2 + \cdots, \quad (9)$$

where h stands for ρ , u , T , and p . The coefficients $f_{(m)}$ are given as

$$f_{(0)} = \frac{\rho_{(0)}(s, t)}{[\pi T_w(s)]^{3/2}} \exp\left(-\frac{|\xi|^2}{T_w(s)}\right), \quad (10a)$$

$$\begin{aligned} f_{(1)} = & f_{(0)} \left(\phi_C(s, t) + \phi_P\left(r, \frac{\xi}{\sqrt{T_w}}; K_{(0)}(s, t), \kappa(s)\right) \right. \\ & \times \frac{\partial}{\partial s} \ln \rho_{(0)}(s, t) + \left\{ \phi_P\left(r, \frac{\xi}{\sqrt{T_w}}; K_{(0)}(s, t), \kappa(s)\right) \right. \\ & \left. \left. + \phi_T\left(r, \frac{\xi}{\sqrt{T_w}}; K_{(0)}(s, t), \kappa(s)\right) \right\} \frac{d}{ds} \ln T_w(s) \right), \quad (10b) \end{aligned}$$

and so on, where $K_{(0)}(s, t)$ is a kind of local Knudsen number defined by

$$K_{(0)}(s, t) = T_w(s)^{1/2} K_0 / \rho_{(0)}(s, t), \quad (11)$$

$\rho_{(0)}(s, t)$ and $\phi_C(s, t)$ are functions of s and t (see below), and ϕ_P and ϕ_T are the solutions of the specific boundary-value problems of the linearized BGK model, which are described in Sec. II B. It turns out that ϕ_P and ϕ_T are odd in ζ_s and even in ζ_z (see Sec. II B). Thus, the macroscopic quantities obtained from the above $f_{(0)}$ and $f_{(1)}$ are as follows:

$$\rho_{(0)} = \rho_{(0)}(s, t), \quad u_{(0)} = 0, \quad T_{(0)} = T_w(s), \quad (12a)$$

$$\rho_{(1)} = \phi_C(s, t) \rho_{(0)}(s, t), \quad (12b)$$

$$\begin{aligned} u_{s(1)} = & u_P(r, s, t) \frac{\partial}{\partial s} \ln \rho_{(0)}(s, t) \\ & + [u_P(r, s, t) + u_T(r, s, t)] \frac{d}{ds} \ln T_w(s), \quad (12c) \end{aligned}$$

$$u_{r(1)} = 0, \quad T_{(1)} = 0, \quad (12d)$$

where

$$\begin{aligned} u_J = & \frac{1}{\rho_{(0)}} \int \zeta_s \phi_J\left(r, \frac{\xi}{\sqrt{T_w}}; K_{(0)}(s, t), \kappa(s)\right) f_{(0)} d\zeta \\ = & \sqrt{T_w} \int \eta_s \phi_J(r, \boldsymbol{\eta}; K_{(0)}(s, t), \kappa(s)) E(|\boldsymbol{\eta}|) d\boldsymbol{\eta} \quad (J = P, T), \quad (13) \end{aligned}$$

$$\boldsymbol{\eta} = \boldsymbol{\zeta} / \sqrt{T_w}, \quad E(|\boldsymbol{\eta}|) = \pi^{-3/2} \exp(-|\boldsymbol{\eta}|^2). \quad (14)$$

The dimensionless density of the gas per unit length of s (linear density) is defined by

$$\bar{\rho}(s, t) = \int_{-1/2}^{1/2} \rho(r, s, t) (1 - \kappa r) dr. \quad (15)$$

With Eqs. (9) and (12), we have

$$\bar{\rho}(s, t) = \rho_{(0)}(s, t) [1 + \phi_C(s, t)\epsilon] + O(\epsilon^2). \quad (16)$$

If we admit the error of $O(\epsilon^2)$ in $\bar{\rho}(s, t)$, it is described by the following equation:

$$\frac{\partial}{\partial t} \bar{\rho}(s, t) + \frac{\partial}{\partial s} M(s, t) = 0, \quad (17a)$$

$$\begin{aligned} M(s, t) = & \sqrt{T_w(s)} M_P(K(s, t), |\kappa(s)|) \frac{\partial}{\partial s} \bar{\rho}(s, t) \\ & + \frac{\bar{\rho}(s, t)}{\sqrt{T_w(s)}} \{M_P(K(s, t), |\kappa(s)|) \\ & + M_T(K(s, t), |\kappa(s)|)\} \frac{d}{ds} T_w(s), \quad (17b) \end{aligned}$$

where

$$K(s, t) = T_w(s)^{1/2} K_0 / \bar{\rho}(s, t) \quad [K_0 = (\sqrt{\pi}/2)\text{Kn}], \quad (18)$$

which corresponds to the local Knudsen number, and $M_P(K, |\kappa|)$ and $M_T(K, |\kappa|)$ are the functions of two variables obtained numerically in Ref. 15. Equation (17) is solvable numerically with the database¹⁵ of M_P and M_T . Since the unknown $\bar{\rho}(s, t)$ is contained in $K(s, t)$, Eq. (17) is a nonlinear equation of diffusion type. Note that $\bar{\rho} = \rho$ within the present approximation.

We note here that M_P and M_T are, respectively, expressed in terms of ϕ_P and ϕ_T as

$$M_J(K, |\kappa|) = \int_{-1/2}^{1/2} \int \eta_s \phi_J(r, \boldsymbol{\eta}; K, \kappa) E(|\boldsymbol{\eta}|) d\boldsymbol{\eta} dr \quad (J = P, T). \quad (19)$$

Equation (17) is the consequence of the solvability conditions for $f_{(2)}$ and $f_{(3)}$ incorporated suitably. Therefore, in order to derive Eq. (17), we need to proceed to $f_{(3)}$. However, the essential information is contained in the first-order solution $f_{(1)}$.

Equation (17) describes the mass conservation, and M indicates the mass-flow rate of the gas. More precisely, if we denote by $\tilde{M}(\tilde{s}, \tilde{t})$ the (dimensional) mass flow of the gas (per unit time and per unit thickness in the $\tilde{x}_3$ direction) in the direction of $\boldsymbol{\tau}$ through the normal cross section at position $\tilde{s}$ and at time $\tilde{t}$, then it is related to M as

$$\tilde{M}(\tilde{s}, \tilde{t}) = \rho_0 (2RT_0)^{1/2} D [M(s, t) + O(\epsilon^2)] \epsilon. \quad (20)$$

As seen from Eq. (12), T is equal to T_w except for the error of $O(\epsilon^2)$.

It should be mentioned that macroscopic models like Eq. (17), based on the assumption that the width of a channel is much shorter than its length, have a long history; see, e.g., Refs. 27–29 in addition to Refs. 24–26 for applications and Refs. 30–33 for rigorous mathematical study.

B. Connection condition

When the curvature $\kappa(s)$ of the median curve of the channel changes discontinuously, Eq. (17) is, in general, not applicable to the vicinity of the point. However, if we use suitable connection conditions there, Eq. (17) can be extended to the whole channel in such a way that it describes the correct behavior of the gas except in the neighborhood of the point where the curvature changes discontinuously.

Let $s=s_0$ be the point at which the curvature $\kappa(s)$ changes discontinuously. We assume that the wall temperature $T_w(s)$ is continuous there, although its derivative dT_w/ds may be discontinuous. Then, the following connection condition is imposed at $s=s_0$:

$$(\bar{\rho})^+ = (\bar{\rho})^-(1 + d\epsilon), \quad M^+ = M^-, \quad (21)$$

where the superscripts $+$ and $-$ indicate the limiting values at $s=s_0+0$ and $s=s_0-0$, respectively. It should be noted that there arises a jump of $O(\epsilon)$ in $\bar{\rho}$. The coefficient for the jump d depends on the limiting values of $K(s, t)$, $\kappa(s)$, $T_w(s)$, the unknown $\rho(s, t)$, and so on. Condition (21), including the jump coefficient d , is obtained by analyzing the original BGK system in the vicinity of $s=s_0$ (see Ref. 15 for details).

In the present paper, when we apply the connection condition (21), we use $(\bar{\rho})^+ = (\bar{\rho})^-$, neglecting the terms of $O(\epsilon)$. Therefore, the result contains the error of the order of ϵ .

Equations (17) and (21) can be solved numerically with appropriate boundary conditions and initial condition.

IV. EXTENSION TO THE ES MODEL

The macroscopic system summarized in Sec. III A is based on the BGK model. In this section, we try to extend it (partially) for the ES model.

A. ES model

The ES model in the dimensionless form is given by Eq. (4a) with $J_{\text{BGK}}(f)$ replaced by the following $J_{\text{ES}}(f)$:^{18–20}

$$J_{\text{ES}}(f) = \rho(\mathcal{G}[\rho, \underline{u}, \underline{T}] - f), \quad (22a)$$

$$\mathcal{G}[\rho, \underline{u}, \underline{T}] = \frac{\rho}{\pi^{3/2} |\underline{T}|^{1/2}} \exp(-(\underline{\zeta} - \underline{u})^t \underline{T}^{-1} (\underline{\zeta} - \underline{u})), \quad (22b)$$

$$\rho = \int f d\underline{\zeta}, \quad \underline{u} = \frac{1}{\rho} \int \underline{\zeta} f d\underline{\zeta}, \quad T = \frac{2}{3\rho} \int |\underline{\zeta} - \underline{u}|^2 f d\underline{\zeta}, \quad (22c)$$

$$\underline{T} = (1 - \nu) \underline{T} I + \nu (\underline{p}/\rho), \quad (22d)$$

$$\underline{p} = 2 \int (\underline{\zeta} - \underline{u})(\underline{\zeta} - \underline{u})^t f d\underline{\zeta}, \quad (22e)$$

where $\underline{T}$ and $\underline{p}$ are the 3×3 symmetric matrices, I is the 3×3 identity matrix, $|\underline{T}|$ and $\underline{T}^{-1}$ are the determinant and the inverse matrix of $\underline{T}$, respectively, the superscript t indicates the transpose operation, and $\nu \in [-1/2, 1)$ is the parameter to adjust the Prandtl number. $\underline{p}$ is the dimensionless stress tensor, whose dimensional counterpart is $\tilde{\underline{p}} = \tilde{\rho}_0 \underline{p}$. The mean free path l_0 in Eq. (5) is defined, as in the case of the BGK model, by $l_0 = (8R\tilde{T}_0/\pi)^{1/2}/A_{\text{ES}}\tilde{\rho}_0$, where A_{ES} is a constant such that $A_{\text{ES}}\tilde{\rho}$ is the collision frequency of the gas molecules. This model leads to the following expressions of the viscosity μ , the thermal conductivity λ , and the Prandtl number Pr :

$$\mu = \frac{1}{1 - \nu} \frac{\tilde{\rho}}{A_{\text{ES}}\tilde{\rho}}, \quad \lambda = \frac{5R}{2} \frac{\tilde{\rho}}{A_{\text{ES}}\tilde{\rho}}, \quad \text{Pr} = \frac{5R\mu}{2\lambda} = \frac{1}{1 - \nu}. \quad (23)$$

For the BGK model μ , λ , and Pr are given by Eq. (23) with $\nu=0$ and A_{ES} replaced by A_{BGK} .

B. Macroscopic system

By repeating the same asymptotic analysis as in Ref. 15 on the basis of the ES model, we can derive the macroscopic system corresponding to Eqs. (17) and (21). Here, we have carried out the analysis up to one order less than that in Ref. 15. According to the result, $f_{(0)}$ is the same as in Eq. (10a), and $f_{(1)}$ is of the same form as in Eq. (10b), with ϕ_P and ϕ_T being the solutions of the specific boundary-value problems of the linearized ES model (see Sec. II B). In addition, if we admit the error of $O(\epsilon)$ in the linear density $\bar{\rho}(s, t)$ [Eq. (15)] as

$$\bar{\rho}(s, t) = \rho_{(0)}(s, t) + O(\epsilon), \quad (24)$$

instead of Eq. (16), then the equation for $\bar{\rho}(s, t)$ is of the same form as Eq. (17). The difference between the BGK model and the ES model arises only in the functions $M_P(K, |\kappa|)$ and $M_T(K, |\kappa|)$. Corresponding to Eq. (24), we need to replace Eq. (20) by

$$\tilde{M}(\tilde{s}, \tilde{t}) = \rho_0 (2RT_0)^{1/2} D [M(s, t) + O(\epsilon)] \epsilon, \quad (25)$$

and the connection condition (21) by

$$(\bar{\rho})^+ = (\bar{\rho})^-, \quad M^+ = M^-. \quad (26)$$

C. Basic problems and M_P and M_T

The functions $\phi_P(r, \boldsymbol{\eta}; K, \kappa)$ and $\phi_T(r, \boldsymbol{\eta}; K, \kappa)$ occurring in Eqs. (10b), (13), and (19) are given by the solutions of the auxiliary boundary-value problems of the linearized BGK or ES model (see Sec. 2.2.2 of Ref. 15). To be more specific, the problem for ϕ_P corresponds to a circulating flow between two coaxial circular cylinders driven by a constant (small) pressure gradient in the circumferential direction, and that for ϕ_T corresponds to a circulating flow in the same geometry driven by a constant (small) gradient of the surface temperature of each cylinder in the circumferential direction. As stated in Ref. 15, such flows are physically unrealistic because the pressure of the gas or the temperature of the

cylinders becomes multivalued. However, the problems for ϕ_P and ϕ_T themselves make sense mathematically, and M_P and M_T in Eq. (19) are interpreted as the mass-flow rates of the artificial Poiseuille flow and thermal transpiration induced in the circumferential direction between two coaxial circular cylinders.

The solutions ϕ_P and ϕ_T can be obtained consistently by assuming that they are odd in η_s and even in η_z . With this assumption, the equation becomes

$$\mathcal{D}\phi_J = (1/K)\mathcal{L}[\phi_J] - I_J \quad (J = P, T), \quad (27a)$$

$$\mathcal{D} = \eta_r \frac{\partial}{\partial r} + \frac{\kappa}{1 - \kappa r} \eta_r \eta_s \frac{\partial}{\partial \eta_s} - \frac{\kappa}{1 - \kappa r} \eta_s^2 \frac{\partial}{\partial \eta_r}, \quad (27b)$$

$$I_P = (1 - \kappa r)^{-1} \eta_s, \quad I_T = (1 - \kappa r)^{-1} \eta_s (|\boldsymbol{\eta}|^2 - \frac{5}{2}), \quad (27c)$$

$$\mathcal{L}[\phi_J] = \begin{cases} \mathcal{L}_{\text{BGK}}[\phi_J] = 2U_s[\phi_J]\eta_s - \phi_J & \text{(for BGK model)} \\ \mathcal{L}_{\text{ES}}[\phi_J] = 2U_s[\phi_J]\eta_s + 2\nu\eta_s\eta_r P_{sr}[\phi_J] - \phi_J & \text{(for ES model),} \end{cases} \quad (27d)$$

$$U_s[\phi_J] = \int \eta_s \phi_J E(|\boldsymbol{\eta}|) d\boldsymbol{\eta}, \quad (27e)$$

$$P_{sr}[\phi_J] = 2 \int \eta_s \eta_r \phi_J E(|\boldsymbol{\eta}|) d\boldsymbol{\eta}, \quad (27f)$$

and the boundary condition reduces to

$$\phi_J = 0 \quad \text{for } \eta_r \leq 0 \quad \text{at } r = \pm 1/2. \quad (28)$$

Now we derive the relation between the solution of Eqs. (27) and (28) for the BGK model and that for the ES model. Let us denote the former solution by ϕ_J^{BGK} and the latter by ϕ_J^{ES} . Suppose that we have a function $C_J(r)$ that satisfies

$$\mathcal{D}(\eta_s C_J) = (2\nu/K) \eta_r \eta_s P_{sr}[\phi_J^{\text{BGK}}], \quad (29)$$

and vanishes at $r = \pm 1/2$. If we let

$$\psi_J = \phi_J^{\text{BGK}} + \eta_s C_J, \quad (30)$$

then we have

$$U_s[\psi_J] = U_s[\phi_J^{\text{BGK}}] + (1/2)C_J, \quad P_{sr}[\psi_J] = P_{sr}[\phi_J^{\text{BGK}}]. \quad (31)$$

Therefore, it follows from Eq. (27) that

$$\begin{aligned} \mathcal{D}\psi_J &= \mathcal{D}\phi_J^{\text{BGK}} + \mathcal{D}(\eta_s C_J) \\ &= (1/K)(2U_s[\phi_J^{\text{BGK}}]\eta_s - \phi_J^{\text{BGK}}) \\ &\quad + (2\nu/K)\eta_r \eta_s P_{sr}[\phi_J^{\text{BGK}}] - I_J \\ &= (1/K)(2U_s[\psi_J]\eta_s - \psi_J) + (2\nu/K)\eta_r \eta_s P_{sr}[\psi_J] - I_J \\ &= (1/K)\mathcal{L}_{\text{ES}}[\psi_J] - I_J, \end{aligned} \quad (32)$$

and from Eq. (28) that

$$\psi_J = 0 \quad \text{for } \eta_r \leq 0 \quad \text{at } r = \pm 1/2. \quad (33)$$

Equations (32) and (33) mean that ψ_J is the solution of Eqs. (27) and (28) for the ES model, i.e., $\psi_J = \phi_J^{\text{ES}}$.

Then, the problem is to find $C_J(r)$. Equation (29) gives

$$\frac{dC_J(r)}{dr} + \frac{\kappa}{1 - \kappa r} C_J(r) = \frac{2\nu}{K} P_{sr}[\phi_J^{\text{BGK}}]. \quad (34)$$

On the other hand, by integrating Eq. (27) (for the BGK model) multiplied by $\eta_s E$ over the whole molecular velocity space, we have

$$\frac{d}{dr} \{ (1 - \kappa r)^2 P_{sr}[\phi_J^{\text{BGK}}] \} = - (1 - \kappa r) \delta_J, \quad (35)$$

where $\delta_P = 1$ and $\delta_T = 0$. If we eliminate $P_{sr}[\phi_J^{\text{BGK}}]$ from Eqs. (34) and (35), we obtain the second-order ordinary differential equation for $C_J(r)$, which can be solved analytically under the condition $C_J(\pm 1/2) = 0$. To summarize, ϕ_J^{ES} is expressed in terms of ϕ_J^{BGK} as follows:

$$\phi_J^{\text{ES}} = \phi_J^{\text{BGK}} + \eta_s C_J(r) \quad (J = P, T), \quad (36)$$

where

$$\begin{aligned} C_P(r) &= -(\nu/\kappa^2 K) [(1 - \kappa r) \ln(1 - \kappa r) + c_0(1 - \kappa r)^{-1} \\ &\quad + c_1(1 - \kappa r)], \end{aligned} \quad (37a)$$

$$c_0 = \frac{1}{2\kappa} (1 - \kappa^2/4)^2 [\ln(1 + \kappa/2) - \ln(1 - \kappa/2)], \quad (37b)$$

$$c_1 = -\frac{1}{2\kappa}[(1 + \kappa/2)^2 \ln(1 + \kappa/2) - (1 - \kappa/2)^2 \ln(1 - \kappa/2)], \quad (37c)$$

and

$$C_T(r) = 0. \quad (38)$$

Note that concerning ϕ_T , the ES model gives the same solution as the BGK model.

From Eqs. (13), (19), (36), (37a)–(37c), and (38), we obtain

$$u_P^{\text{ES}} = u_P^{\text{BGK}} + (\sqrt{T_w}/2)C_P(r), \quad (39a)$$

$$\begin{aligned} M_P^{\text{ES}} &= M_P^{\text{BGK}} + \frac{1}{2} \int_{-1/2}^{1/2} C_P(r) dr \\ &= M_P^{\text{BGK}} + \frac{\nu}{4\kappa^2 K} \\ &\quad \times \left\{ 1 - \frac{1}{\kappa^2} \left[(1 - \kappa^2/4) \ln \left(\frac{1 + \kappa/2}{1 - \kappa/2} \right) \right]^2 \right\}, \end{aligned} \quad (39b)$$

and

$$u_T^{\text{ES}} = u_T^{\text{BGK}}, \quad M_T^{\text{ES}} = M_T^{\text{BGK}}, \quad (40)$$

where u_J^{ES} and M_J^{ES} indicate u_J and M_J obtained with $\phi_J = \phi_J^{\text{ES}}$, and u_J^{BGK} and M_J^{BGK} those obtained with $\phi_J = \phi_J^{\text{BGK}}$. Thus, we can obtain the numerical data of $M_P(K, |\kappa|)$ and $M_T(K, |\kappa|)$ for the ES model immediately from those for the BGK model.¹⁵

In summary, if one admits the error of $O(\epsilon)$, the macroscopic equation (17) and the connection condition (21) [or Eq. (26)] have been extended to the ES model.

V. APPLICATIONS OF MACROSCOPIC SYSTEM

In this section, we show some examples of the application of the macroscopic system summarized in Sec. III. Under appropriate initial and boundary conditions, the system consisting of Eqs. (17) and (21) describes the time evolution of the density distribution and mass-flow rate along a curved channel. Here, we consider two types of boundary condition at the end of the channel.

- (i) Open end: the gas flows through the end.
- (ii) Closed end: the mass flow vanishes at the end.

The results that will be shown in Sec. V A are based on the BGK model, and the comparison between the BGK model and the ES model will be given in Sec. V B.

A. Flow caused by pressure difference or temperature gradient

We first consider a twisty channel with length L_s and width D . Both ends are open, and the pressure at one end ($\tilde{s}=0$) is kept at $\tilde{p}_0$, whereas that at the other end ($\tilde{s}=L_s$) is kept at $\tilde{p}_1$. The temperature of the channel walls changes from $\tilde{T}_0$ to $\tilde{T}_1$ linearly in $\tilde{s}$, so that $\tilde{T}_w(\tilde{s}) = \tilde{T}_0 + [(\tilde{T}_1 - \tilde{T}_0)/L_s]\tilde{s}$. Therefore, the density at $\tilde{s}=0$ and that at

$\tilde{s}=L_s$ are $\tilde{p}_0/R\tilde{T}_0$ and $\tilde{p}_1/R\tilde{T}_1$, respectively. We investigate a steady flow through the channel in this situation with interest in the effect of curvature of the channel. The flow caused by the pressure difference is the Poiseuille-type flow, and that caused by the temperature gradient is the thermal transpiration.

Since $\partial \bar{p} / \partial t = 0$ for steady flows, Eq. (17) reduces to an ordinary differential equation, which is described, together with the end conditions, as follows:

$$dM/ds = 0 \quad (0 \leq s \leq 1), \quad (41a)$$

$$\begin{aligned} M &= \sqrt{T_w(s)} M_P(K(s), |\kappa(s)|) \frac{d\bar{p}(s)}{ds} \\ &\quad + \frac{\bar{p}(s)}{\sqrt{T_w(s)}} \{ M_P(K(s), |\kappa(s)|) + M_T(K(s), |\kappa(s)|) \} \frac{d}{ds} T_w(s), \end{aligned} \quad (41b)$$

$$K(s) = K_0 \sqrt{T_w(s)} / \bar{p}(s) \quad [K_0 = (\sqrt{\pi}/2) \text{Kn}], \quad (41c)$$

$$T_w(s) = 1 + (\tilde{T}_1/\tilde{T}_0 - 1)s, \quad (41d)$$

and

$$\bar{p} = 1 \quad \text{at } s = 0, \quad (42a)$$

$$\bar{p} = (\tilde{p}_1/\tilde{p}_0)(\tilde{T}_0/\tilde{T}_1) \quad \text{at } s = 1. \quad (42b)$$

Here, Kn is the Knudsen number at temperature $\tilde{T}_0$ and pressure $\tilde{p}_0$.

1. Archimedes' spiral

As the first example, let us consider a channel, whose median curve is a segment of Archimedes' spiral, which is expressed by

$$\tilde{r} = a\theta \quad (0 \leq \theta \leq \theta_{\max}), \quad (43)$$

where $(\tilde{r}, \theta)$ is a polar coordinate system in the $\tilde{x}_1\tilde{x}_2$ plane, and a and $\theta_{\max}$ are constants. Then, the curve length $\tilde{s}$ and the curvature $\tilde{\kappa}$ are expressed in terms of θ as follows:

$$\tilde{s}(\theta) = \frac{a}{2} [\theta \sqrt{\theta^2 + 1} + \log(\theta + \sqrt{\theta^2 + 1})], \quad (44a)$$

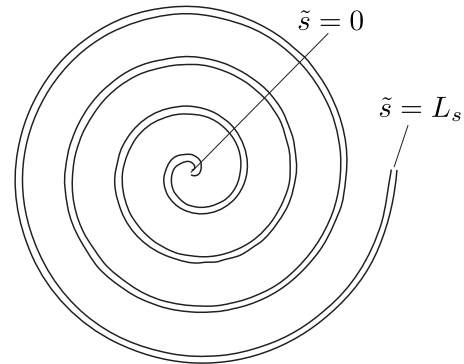


FIG. 2. Channel of Archimedes' spiral shape ($\theta_{\max} = 8\pi$).

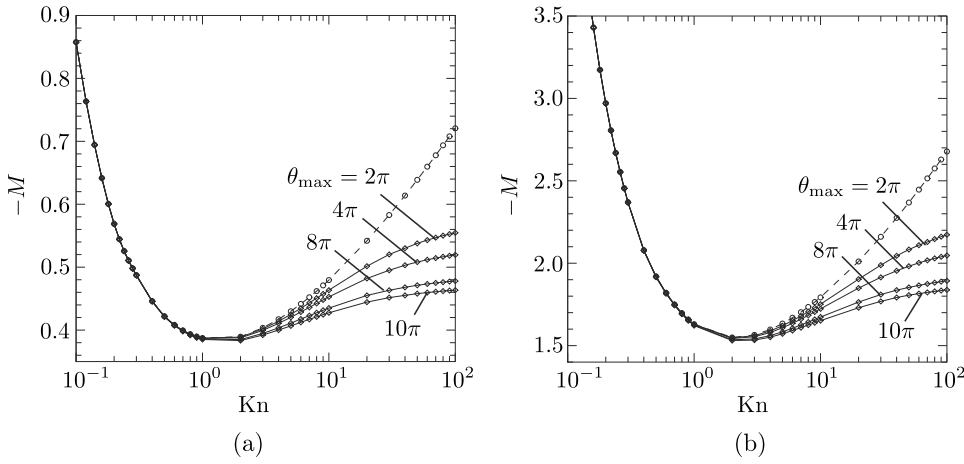


FIG. 3. Mass-flow rate of the flow through a channel of Archimedes' spiral shape ($\theta_{\max}=2\pi, 4\pi, 8\pi$, and 10π) induced by the pressure difference ($\tilde{T}_1/\tilde{T}_0=1$): (a) $\tilde{p}_1/\tilde{p}_0=1.5$ and (b) $\tilde{p}_1/\tilde{p}_0=3$. The dimensionless mass-flow rate M [cf. Eq. (20)] is plotted vs Kn . The dashed line indicates the result for the straight channel.

$$\tilde{\kappa}(\theta) = -\frac{\theta^2 + 2}{a(\theta^2 + 1)^{3/2}}. \quad (44b)$$

We consider four different channels given by $\theta_{\max}=2\pi, 4\pi, 8\pi$, and 10π and suppose that they are of the same length L_s . The channel with $\theta_{\max}=8\pi$ is shown in Fig. 2. For all these channels, we adjust the parameter a in such a way that the length to width ratio L_s/D is the same and is 636.08 (i.e., $\epsilon=D/L_s=0.001\,572\,1$). The value of a/D for each channel is as follows: $a/D=14.96$ ($\theta_{\max}=2\pi$), 3.935 ($\theta_{\max}=4\pi$), 1.000 ($\theta_{\max}=8\pi$), and 0.6415 ($\theta_{\max}=10\pi$).

Figure 3 shows the steady mass-flow rate when only the pressure difference is applied ($\tilde{T}_1/\tilde{T}_0=1$), whereas Fig. 4 shows that when only the temperature gradient is applied ($\tilde{p}_1/\tilde{p}_0=1$). Figures 3(a) and 3(b) show the results for $\tilde{p}_1/\tilde{p}_0=1.5$ and 3, respectively; Figs. 4(a) and 4(b) those for $\tilde{T}_1/\tilde{T}_0=1.5$ and 3, respectively. In each figure, the dimensionless mass-flow rate M [Eq. (20)] is shown as a function of the Knudsen number Kn . The dashed line indicates the result for the straight channel. The values of M for the free-molecular flow ($Kn=\infty$) corresponding to Figs. 3 and 4 are shown in Tables I and II, respectively. In the Poiseuille-type flow (Fig. 3), the effect of curvature manifests itself for Kn greater than around 2–3. The reduction of the mass-flow rate is more eminent for larger $\theta_{\max}$ because the channel has more turns. One observes the minimum of the magnitude of the

mass-flow rate at an intermediate Knudsen number (the so-called Knudsen minimum) also for spiral channels. In Table III, we show the dimensionless mass-flow rate of the Poiseuille-type flow ($\tilde{T}_1/\tilde{T}_0=1$) through the channel with $\theta_{\max}=8\pi$, together with that through the straight channel, for $Kn=2, 5$, and 10 and for some different values of $\tilde{p}_1/\tilde{p}_0$. In the thermal transpiration (Fig. 4), the effect of curvature is well visible only for Kn greater than around 6–8. It should be noted, however, that the mass-flow rate for the curved channels is slightly larger than that for the straight channel for a range of Kn , $2 \leq Kn \leq 10$, and the smaller the $\theta_{\max}$, the wider the range. In both the Poiseuille-type flow and the thermal transpiration, the magnitude of the dimensionless mass-flow rate becomes infinite as Kn tends to infinity (the free-molecular limit) for the 2D straight channel, whereas it remains finite for the spiral channels (see Tables I and II). Because of this difference at the free-molecular limit, the effect of the curvature is more significant for larger Kn .

2. Channels of complex shape

Next, giving a formula of the curvature, we try to reconstruct channels of complex shape. Let us consider the curvature $\tilde{\kappa}(\tilde{s})$ defined by the following equation:

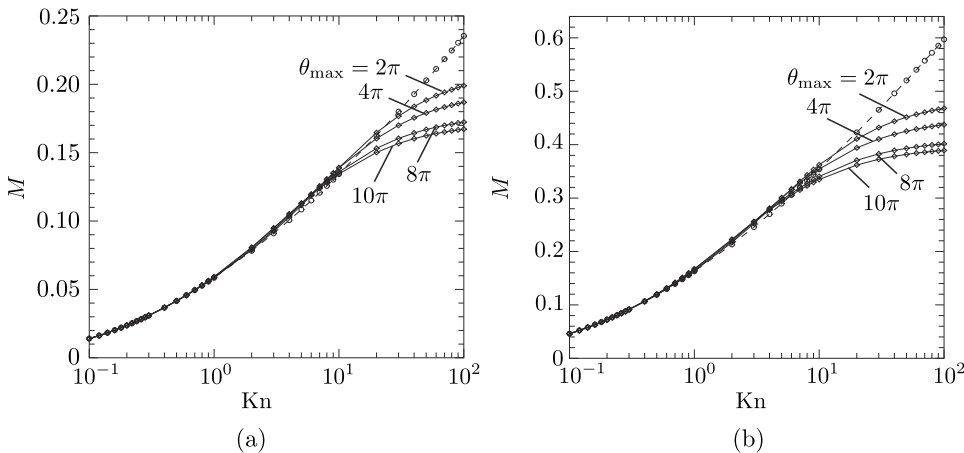


FIG. 4. Mass-flow rate of the flow through a channel of Archimedes' spiral shape ($\theta_{\max}=2\pi, 4\pi, 8\pi$, and 10π) induced by the temperature gradient ($\tilde{p}_1/\tilde{p}_0=1$): (a) $\tilde{T}_1/\tilde{T}_0=1.5$ and (b) $\tilde{T}_1/\tilde{T}_0=3$. The dimensionless mass-flow rate M [cf. Eq. (20)] is plotted vs Kn . The dashed line indicates the result for the straight channel.

TABLE I. Dimensionless mass-flow rate M [cf. Eq. (20)] through a channel of Archimedes' spiral shape at $\text{Kn}=\infty$ (free-molecular flow): flow induced by the pressure difference ($\tilde{T}_1/\tilde{T}_0=1$).

$\tilde{p}_1/\tilde{p}_0$	$\theta_{\max}=2\pi$	$\theta_{\max}=4\pi$	$\theta_{\max}=8\pi$	$\theta_{\max}=10\pi$	Straight
1.5	-0.575	-0.533	-0.485	-0.469	$-\infty$
3	-2.301	-2.133	-1.942	-1.878	$-\infty$

$$\tilde{\kappa}(\tilde{s}) = -(b/2)[\sin(10\pi\tilde{s}/L_s) + 1] \quad (0 \leq \tilde{s} \leq L_s). \quad (45)$$

If we let $L_s/D=200$ (i.e., $\epsilon=0.005$) and take $bD=0.5, 1.0$, and 1.5 , then we have the channels shown in Figs. 5(a)–5(c). These channels intersect themselves, but we assume that the intersections are fictitious two-level crossings, that is, the flow is unidirectional at each intersection and forbidden to go into the crossing channel. This setting is not realistic for 2D channels. Nevertheless, we consider such geometries as an example that demonstrates the powerfulness and usefulness of the macroscopic equation (17) for channels of complex shape. These geometries become realistic as soon as the macroscopic system is extended to a curved pipe in the three-dimensional space.

The steady mass-flow rate through the channels in Fig. 5 is shown in Fig. 6. More specifically, the mass-flow rate when only the pressure difference is imposed ($\tilde{p}_1/\tilde{p}_0=1.5$, $\tilde{T}_1/\tilde{T}_0=1$) is shown in Fig. 6(a), whereas that when only the temperature gradient is imposed ($\tilde{p}_1/\tilde{p}_0=1$, $\tilde{T}_1/\tilde{T}_0=1.5$) is shown in Fig. 6(b). In each figure, the dimensionless mass-flow rate M [Eq. (20)] is shown as a function of the Knudsen number Kn , and the dashed line indicates the result for the straight channel.

As in the case of the channels of Archimedes' spiral shape, the effect of curvature of the channel is more significant for large Knudsen numbers. However, the effect is visible for smaller Knudsen numbers in the present case, and it is more significant on the whole. In particular, in the Poiseuille-type flow (Fig. 6), the effect is quite visible even for small Knudsen numbers. In contrast to Fig. 3 for Archimedes' spiral, the minimum of the magnitude of the mass-flow rate at an intermediate Knudsen number is not clear in Fig. 6.

The Poiseuille-type flow is subject to a resistance exerted by the channel wall. One expects a larger resistance for curved channels intuitively, and it is true as seen from Figs. 3 and 6. However, in the case of thermal transpiration, the effect of the bend of the channel walls is not clear because the flow is induced by the walls themselves. In fact, the reduction of the magnitude of the mass-flow rate caused by

TABLE II. Dimensionless mass-flow rate M [cf. Eq. (20)] through a channel of Archimedes' spiral shape at $\text{Kn}=\infty$ (free-molecular flow): flow induced by the temperature gradient ($\tilde{p}_1/\tilde{p}_0=1$).

$\tilde{T}_1/\tilde{T}_0$	$\theta_{\max}=2\pi$	$\theta_{\max}=4\pi$	$\theta_{\max}=8\pi$	$\theta_{\max}=10\pi$	Straight
1.5	0.211	0.196	0.178	0.172	∞
3	0.486	0.451	0.414	0.397	∞

TABLE III. Dimensionless mass-flow rate M [cf. Eq. (20)] through a channel of Archimedes' spiral shape for the Poiseuille-type flow at $\text{Kn}=2, 5$, and 10 .

Kn	$\tilde{p}_1/\tilde{p}_0$	Archimedes ($\theta_{\max}=8\pi$)	Straight
2	1.1	-7.7701×10^{-2}	-7.8795×10^{-2}
2	1.5	-3.8487×10^{-1}	-3.8956×10^{-1}
2	2.0	-7.6620×10^{-1}	-7.7446×10^{-1}
2	3.0	-1.5352	-1.5493
2	5.0	-3.1485	-3.1725
5	1.1	-8.3576×10^{-2}	-8.8084×10^{-2}
5	1.5	-4.1108×10^{-1}	-4.2991×10^{-1}
5	2.0	-8.0951×10^{-1}	-8.4122×10^{-1}
5	3.0	-1.5837	-1.6337
5	5.0	-3.1018	-3.1773
10	1.1	-8.8235×10^{-2}	-9.8669×10^{-2}
10	1.5	-4.3500×10^{-1}	-4.7928×10^{-1}
10	2.0	-8.5699×10^{-1}	-9.3280×10^{-1}
10	3.0	-1.6734	-1.7926
10	5.0	-3.2411	-3.4152

the bend appears at higher Knudsen numbers than in the case of the Poiseuille-type flow, as seen from Figs. 3, 4, 6, and 7. In general, by using the thermal transpiration, one can transport the gas from a lower-pressure reservoir to a higher-pressure one against a pressure gradient. The present results indicate that when the Knudsen number is not very small, the transportation can be more efficient if a curved or twisty channel is used instead of a straight channel.

B. Knudsen pump using the effect of curvature

If two reservoirs containing a rarefied gas are connected by a channel with a temperature gradient, the gas is transported by the thermal transpiration. As the final steady state, one can sustain the pressure difference between the two reservoirs by the channel. In other words, the channel acts as a pump or a compressor without any moving parts. However, if we want to pump the gas against a large pressure difference or to sustain a large pressure difference, we need to impose a very large temperature difference between both ends of the channel, which is not practical. This difficulty can be overcome by using a periodic temperature distribution and a periodic change of the channel width. Such a device has long been known and is called the Knudsen pump or compressor.^{34,35} In recent years, it has been revived as a non-

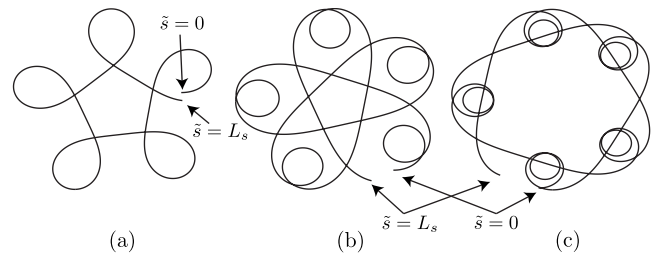


FIG. 5. Channels given by Eq. (45): (a) $bD=0.5$, (b) $bD=1.0$, and (c) $bD=1.5$.

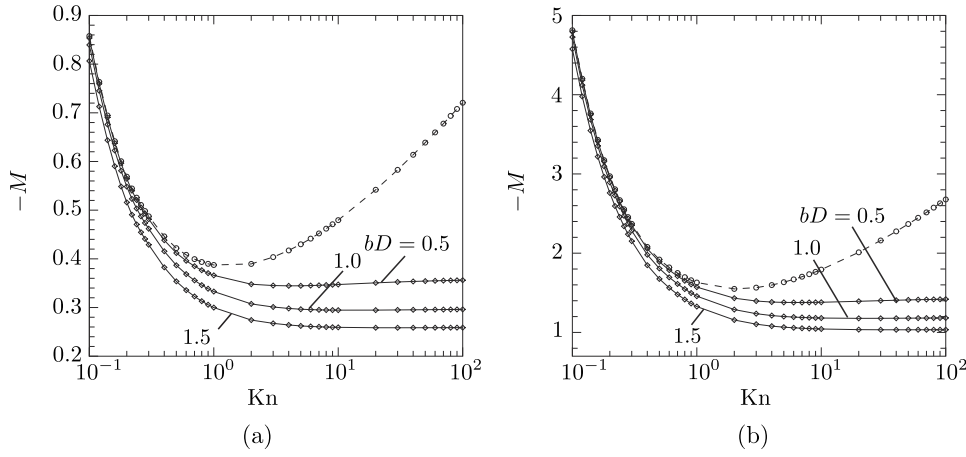


FIG. 6. Mass-flow rate of the flow through a channel of Fig. 5 [Eq. (45)] induced by the pressure difference ($\tilde{T}_1/\tilde{T}_0=1$): (a) $\tilde{p}_1/\tilde{p}_0=1.5$ and (b) $\tilde{p}_1/\tilde{p}_0=3$. The dimensionless mass-flow rate M [cf. Eq. (20)] is plotted vs Kn . The dashed line indicates the result for the straight channel.

mechanical pump in the fields of microfluidics and vacuum technology (see Ref. 23). We consider this problem in this subsection.

We start with the case of a single unit, that is, an S-shaped channel with one end closed and the other end open, where the pressure is kept at $\tilde{p}_0$ [Fig. 8(a)]. The channel consists of two semicircular channels, the median curves of which have the curvatures $\tilde{\kappa}_1=1/R_1$ and $\tilde{\kappa}_2=-1/R_2$ (thus, the radii R_1 and R_2), respectively, so that the channel length is $L_s=\pi(R_1+R_2)$. The temperature of the channel walls is $\tilde{T}_0$ at the open end ($\tilde{s}=0$) as well as at the closed end ($\tilde{s}=L_s$) but $\tilde{T}_1$ at the junction ($\tilde{s}=\pi R_1$), and it changes linearly in $\tilde{s}$ [Fig. 8(b)]. We assume that the channel width is much shorter than the length of each channel ($D/\pi R_1 \ll 1$, $D/\pi R_2 \ll 1$). We investigate the steady pressure distribution along the channel with the help of the macroscopic equation (17) and the connection condition (26) [we consider $\bar{p}$ up to $O(1)$, neglecting the terms of $O(\epsilon)$ in this subsection]. More specifically, the equation is given by Eqs. (41a)–(41c) in the ranges of $0 < s < s_*$ and $s_* < s < 1$ with

$$T_w(s) = \begin{cases} 1 + (\tilde{T}_1/\tilde{T}_0 - 1)(s/s_*) & (0 \leq s \leq s_*) \\ \tilde{T}_1/\tilde{T}_0 - (\tilde{T}_1/\tilde{T}_0 - 1)(s - s_*)/(1 - s_*) & (s_* < s < 1), \end{cases} \quad (46a)$$

$$\kappa(s) = \begin{cases} D/R_1 & (0 \leq s \leq s_*) \\ -D/R_2 & (s_* < s < 1), \end{cases} \quad (46b)$$

where $s_*= (R_2/R_1 + 1)^{-1}$, and $s=s_*$ indicates the junction; the boundary and connection conditions are as follows:

$$\bar{p} = 1 \quad \text{at } s=0, \quad (47a)$$

$$(\bar{p})^+ = (\bar{p})^-, \quad M^+ = M^- \quad \text{at } s=s_*, \quad (47b)$$

$$M = 0 \quad \text{at } s=1, \quad (47c)$$

where the superscripts $\pm$ indicate the values at $s=s_* \pm 0$. From the condition at $s=1$, the mass flow M vanishes identically.

The pressure distribution along the channel obtained by solving the system summarized in the preceding paragraph is shown in Fig. 9, i.e., the dimensionless pressure $\bar{p}/\tilde{p}_0 (= \bar{p}T_w)$ is plotted versus the dimensionless channel length $\tilde{s}/L_s$. Figure 9(a) shows the results for $R_1/D=1, 2$, and 5 in the case of $Kn=1$, $\tilde{T}_1/\tilde{T}_0=1.5$, and $R_2/R_1=2$; Fig. 9(b) shows those for $R_2/R_1=1, 2$, and 5 in the case of $Kn=1$, $\tilde{T}_1/\tilde{T}_0=1.5$, and $R_1/D=2$; Fig. 9(c) shows those for $\tilde{T}_1/\tilde{T}_0=1.5, 2, 3$, and 5 in the case of $Kn=1$, $R_2/R_1=2$, and $R_1/D=2$; and Fig. 9(d) shows those for $Kn=0.1, 0.5, 1$, and

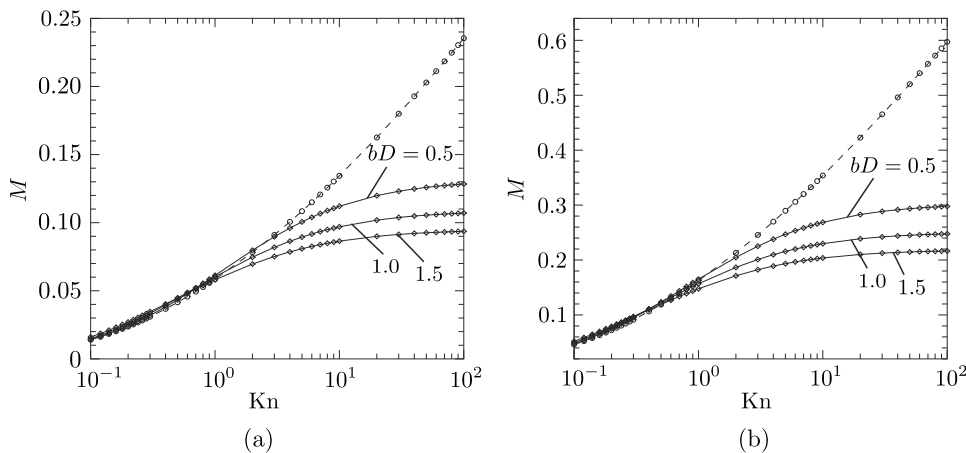


FIG. 7. Mass-flow rate of the flow through a channel of Fig. 5 [Eq. (45)] induced by the temperature gradient ($\tilde{p}_1/\tilde{p}_0=1$): (a) $\tilde{T}_1/\tilde{T}_0=1.5$ and (b) $\tilde{T}_1/\tilde{T}_0=3$. The dimensionless mass-flow rate M [cf. Eq. (20)] is plotted vs Kn . The dashed line indicates the result for the straight channel.

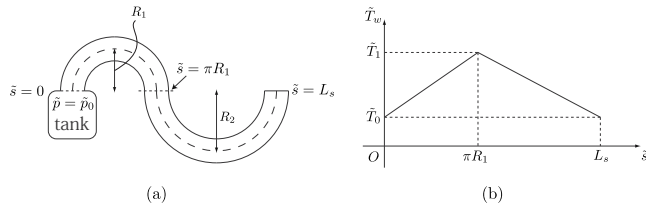


FIG. 8. An S-shaped channel: (a) configuration and (b) distribution of the wall temperature.

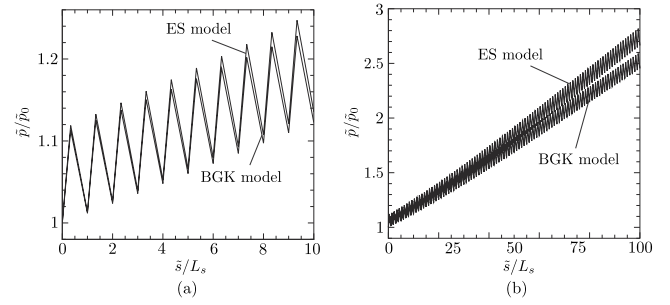


FIG. 11. Pressure distribution along the channel (multiple units) for $\tilde{T}_1/\tilde{T}_0=1.5$, $\text{Kn}=1$, $R_2/R_1=2$, and $R_1/D=2$: (a) 10 units and (b) 100 units.

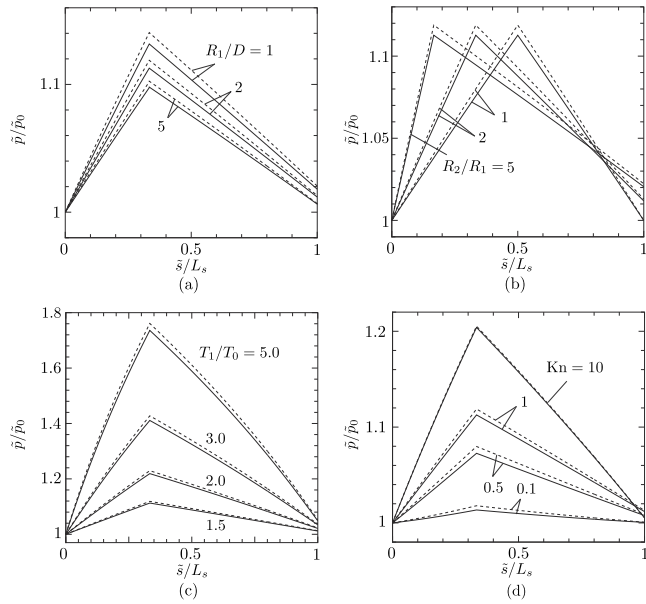


FIG. 9. Pressure distribution along the channel (single unit). (a) Different channel lengths: $R_1/D=1$, 2, and 5 ($\text{Kn}=1$, $\tilde{T}_1/\tilde{T}_0=1.5$, and $R_2/R_1=2$). (b) Different ratios of channel radii: $R_2/R_1=1$, 2, and 5 ($\text{Kn}=1$, $\tilde{T}_1/\tilde{T}_0=1.5$, and $R_1/D=2$). (c) Different temperature ratios: $\tilde{T}_1/\tilde{T}_0=1.5$, 2, 3, and 5 ($\text{Kn}=1$, $R_2/R_1=2$, and $R_1/D=2$). (d) Different Knudsen numbers: $\text{Kn}=0.1$, 0.5, 1, and 10 ($\tilde{T}_1/\tilde{T}_0=1.5$, $R_2/R_1=2$, and $R_1/D=2$). The solid line indicates the results based on the BGK model, and the dashed line those based on the ES model.

10 in the case of $\tilde{T}_1/\tilde{T}_0=1.5$, $R_2/R_1=2$, and $R_1/D=2$. The solid line indicates the results based on the BGK model, and the dashed line those based on the ES model with the Prandtl number $\text{Pr}=2/3$ [see Eq. (23)]. Figure 9 shows that when the radius of the second semicircular channel is larger than that of the first one, there arises a pressure rise at the closed end, i.e., the pressure there is higher than that at the open end. The pressure rise is larger for larger radius of the second channel [Fig. 9(b)]. For a fixed ratio of two radii (say $R_2/R_1=2$), the pressure rise becomes smaller for a longer channel [i.e., larger R_1/D ; Fig. 9(a)]. Here, we should admit that the cases of $R_1/D=1$ and 2 are not well compatible with the assumption $D/\pi R_1 \ll 1$. However, we may expect reasonable results even for these cases (see the last paragraph in Sec. V B).

We next consider the case of multiple units. Let us consider the S-shaped channel with both ends open and call it a unit [Fig. 10(a)]. We connect many units as in Fig. 10(b) to form a single and long twisty channel. If we join one end to the reservoir with pressure $\tilde{p}_0$ and close the other end of the long channel, we can obtain a larger pressure rise at the closed end. As an example, we show the steady pressure distribution along the channel. That is, $\tilde{p}/\tilde{p}_0$ versus $\tilde{s}/L_s$, where L_s is the length of the unit, is shown in Fig. 11 in the case of $\tilde{T}_1/\tilde{T}_0=1.5$, $\text{Kn}=1$, $R_2/R_1=2$, and $R_1/D=2$. Figure 11(a) is the result for 10 units, and Fig. 11(b) that for 100 units. In the figure, the result is shown for both the BGK model and the ES model with $\text{Pr}=2/3$. For the BGK model,

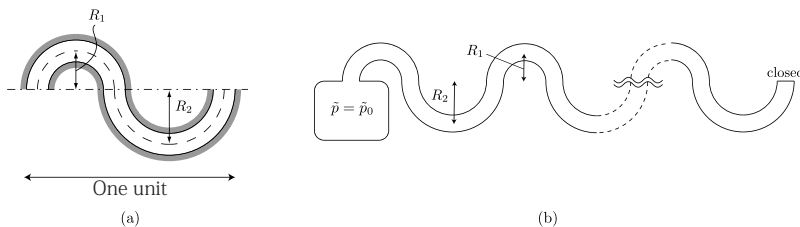


FIG. 10. The channel composed of S-shaped units: (a) unit and (b) channel composed of S-shaped units.

the pressure ratio $\bar{p}/\bar{p}_0$ at the closed end becomes 1.123 for 10 units and 2.454 for 100 units; for the ES model, that becomes 1.135 for 10 units and 2.664 for 100 units. That is, the ES model shows a compression ratio 8.6% higher than that of the BGK model for 100 units. The present channel is a variant of the Knudsen compressor composed of semicircular and straight channels proposed in Refs. 10–12 and 15.

Finally, we give some miscellaneous comments. In the comparison between the BGK and the ES models in Figs. 9 and 11, the mean free path for both models are assumed to be the same, i.e., $A_{\text{BGK}}=A_{\text{ES}}$. This means that the thermal conductivity for the ES model is the same as that for the BGK model [see Eq. (23) and the following sentence]. As mentioned at the end of the third sentence in Sec. V B, the cases of $R_1/D=1$ and $R_1/D=2$ do not meet the condition $D/\pi R_1 \ll 1$. In Ref. 15, however, some comparisons are made between the results based on the present macroscopic system and those based on the direct numerical solution of the BGK model, in the case of the Knudsen compressor composed of semicircular and straight channels with both ends closed (Figs. 9 and 10 in Ref. 15). These comparisons indicate that the macroscopic system is expected to give reasonably good results even for the cases of $R_1/D=1$ and $R_1/D=2$. Another important issue is to assess the validity of the BGK and ES models in comparison with the Boltzmann equation. However, the open-end condition with a specified pressure employed in the examples in Sec. V is an idealized condition, with the end effect being neglected. The open-end condition should be a good approximation when an open end of a long channel is connected to a very large reservoir. In this situation, however, it is hard to carry out a DSMC computation with acceptable accuracy even in the single-unit case of the present example shown in Fig. 9(a). In this connection, it should be mentioned that for the Knudsen compressor composed of semicircular and straight channels with both ends closed,¹² the direct numerical solution of the BGK model shows a good agreement with the DSMC result when the number of unit (consisting of a semicircular and a straight channel) is 2, 4, and 8 (see Fig. 15 in Ref. 12).

VI. CONCLUDING REMARKS

In the present study, we have investigated rarefied gas flows caused by a pressure gradient and/or by a wall-temperature gradient through a two-dimensional channel of various curved shapes numerically on the basis of the macroscopic system derived in Ref. 15. The principal aim is to investigate the effect of the curvature of the channel on these flows, which has not been studied much in spite of its practical importance. In the process of the analysis, we have extended the macroscopic system, which was originally derived using the BGK model, to the case of the ES model. With this extension, the macroscopic equation does not change its shape, whereas the two coefficients contained in it need to be modified. In order to obtain the modified coefficients, we have to solve the two auxiliary problems, the circulating flow in a two-dimensional ring caused by a pressure gradient and that by a wall-temperature gradient, using the ES model. However, we have shown that one of the coeffi-

cients is the same as the corresponding coefficient for the BGK model, and that the other could be obtained from the numerical data of the corresponding coefficient for the BGK model by a simple conversion formula. Thus, we have just exploited the database for the BGK model in Ref. 15 without the reconstruction of the database for the ES model. Then, the macroscopic system was applied to the Poiseuille-type flow, the flow caused by the pressure difference between both ends, and thermal transpiration, the flow caused by the temperature gradient along the channel wall, through curved channels of various shapes (Sec. V A). We have also demonstrated the possibility of making a variant of the Knudsen compressor by imposing a periodic temperature distribution on the channel of a periodic structure consisting of alternately arranged large- and small-curvature channels (Sec. V B).

It should be emphasized that it is very hard (or practically impossible) to analyze the examples shown in the present paper by the usual direct numerical simulation using the Boltzmann equation and its model equations. We have confirmed that the macroscopic system proposed in Ref. 15 is a very powerful tool to analyze rarefied gas flows through a curved channel or gas flows through a curved micro-channel.

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