

Kannan mapping theorems in partially ordered sets

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1 Introduction

Let X be a metric space and let T be a mapping from X into itself. T is contractive if there exists $k \in [0, 1)$ such that for any $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y).$$

Moreover T is Kannan if there exists $\alpha \in [0, \frac{1}{2})$ such that for any $x, y \in X$,

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty).$$

For these mappings, we can consider the existence and uniqueness of fixed points; see, for example, [3].

On the other hand, in [2], Nieto and López consider fixed point theorems for contractive mappings in partially ordered sets. They introduce a mapping T such that there exists $k \in [0, 1)$ such that for any $x, y \in X$,

$$x \geq y \text{ implies } d(Tx, Ty) \leq kd(x, y).$$

For this mapping, they show the existence and uniqueness of fixed points.

In this paper, motivated by [2], we consider fixed point theorems for Kannan mappings [1] in partially ordered sets.

2 Fixed point theorem for contractive mappings

The following theorem is proved in [2]. For the sake of completeness, we show the proof.

Let X be a partially ordered set with a metric d and let T be a mapping from X into itself. We say that T is monotone nondecreasing if for any $x, y \in X$, $x \leq y$ implies $Tx \leq Ty$.

Theorem 1 ([2]). *Let X be a partially ordered set with a metric d such that (X, d) is a complete metric space. If a nondecreasing sequence $\{x_n\}$ converges to x , then we have $x_n \leq x$ for any n . Let T be a monotone nonincreasing mapping from X into itself such that there exists $k \in [0, 1)$ such that for any $x, y \in X$,*

$$x \geq y \text{ implies } d(Tx, Ty) \leq kd(x, y)$$

Assume that there exists x_0 in X with $x_0 \leq Tx_0$. Then there exists a fixed point of T . Moreover, if for any $x, y \in X$, there exists $z \in X$ which is comparable to x and y , then the fixed point of T is unique.

Proof. Since $x_0 \leq Tx_0$ and T is monotone nondecreasing, we obtain that

$$x_0 \leq Tx_0 \leq T^2x_0 \leq \cdots \leq T^nx_0 \leq T^{n+1}x_0 \leq \cdots.$$

Since $x_0 \leq Tx_0$, we have

$$d(T^2x_0, Tx_0) \leq kd(Tx_0, x_0).$$

Moreover, since $Tx_0 \leq T^2x_0$, we have

$$d(T^3x_0, T^2x_0) \leq kd(T^2x_0, Tx_0) \leq k^2d(Tx_0, x_0).$$

Hence we have

$$d(T^{n+1}x_0, T^nx_0) \leq k^nd(Tx_0, x_0)$$

for any n . For $n < m$, we have

$$\begin{aligned} & d(T^mx_0, T^nx_0) \\ & \leq d(T^mx_0, T^{m-1}x_0) + d(T^{m-1}x_0, T^{m-2}x_0) + \cdots + d(T^{n+1}x_0, T^nx_0) \\ & \leq (k^{m-1} + k^{m-2} + \cdots + k^n)d(Tx_0, x_0) \\ & < (k^n + k^{n+1} + \cdots)d(Tx_0, x_0) \\ & = \frac{k^n}{1-k}d(Tx_0, x_0). \end{aligned}$$

Then $\{T^n x_0\}$ is a Cauchy sequence in X . Since X is complete, there exists $p \in X$ such that $\lim_{n \rightarrow \infty} T^n x_0 = p$. Since $x_0 \leq Tx_0 \leq T^2 x_0 \leq \dots \leq T^n x_0 \leq T^{n+1} x_0 \leq \dots$ and $T^n x_0 \rightarrow p$, we have $T^n x_0 \leq p$ for all n . Then we have

$$\begin{aligned} d(Tp, p) &\leq d(Tp, T^{n+1} x_0) + d(T^{n+1} x_0, p) \\ &\leq kd(p, T^n x_0) + d(T^{n+1} x_0, p). \end{aligned}$$

As $n \rightarrow \infty$, we have $d(Tp, p) = 0$. Hence we have $Tp = p$.

Finally, we show the uniqueness of fixed points of T . Let q is another fixed point of T . If q is comparable to p , then $T^n q = q$ is comparable to $T^n p = p$ for any n . Then we have

$$d(p, q) = d(T^n p, T^n q) \leq k^n d(p, q),$$

which implies $d(p, q) = 0$.

If q is not comparable to p , then there exists $z \in X$ comparable to p and q . Then $T^n z$ is comparable to $T^n p = p$ and $T^n q = q$ for all n . Then we have

$$\begin{aligned} d(p, q) &\leq d(T^n p, T^n z) + d(T^n z, T^n q) \\ &\leq k^n d(p, z) + k^n d(z, q). \end{aligned}$$

As $n \rightarrow \infty$, we have $d(p, q) = 0$. □

3 Fixed point theorem for Kannan mappings

In this section, we consider Kannan mappings in partially ordered sets.

Let X be a partially ordered set with a metric d and let T be a mapping from X into itself. To prove the uniqueness of fixed point of Kannan mappings in partially ordered sets, we assume that X satisfies the following;

for any x, y there exists z with $z \leq Tz$, which is comparable to x, y . (1)

Theorem 2. *Let X be a partially ordered set with a metric d such that (X, d) is a complete metric space. If a nondecreasing sequence $\{x_n\}$ converges to x , then we have $x_n \leq x$ for any n . Let T be a monotone nonincreasing mapping from X into itself such that there exists $\alpha \in [0, \frac{1}{2})$ such that for any $x, y \in X$,*

$$x \geq y \text{ implies } d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty).$$

Assume that there exists x_0 in X with $x_0 \leq Tx_0$. Then there exists a fixed point of T . Moreover, if X satisfies (1), the fixed point of T is unique.

Proof. Since $x_0 \leq Tx_0$ and T is monotone nondecreasing, we obtain that

$$x_0 \leq Tx_0 \leq T^2x_0 \leq \cdots \leq T^n x_0 \leq T^{n+1}x_0 \leq \cdots .$$

Then we have

$$d(T^n x_0, T^{n-1}x_0) \leq \alpha d(T^{n-1}x_0, T^n x_0) + \alpha d(T^{n-2}x_0, T^{n-1}x_0).$$

Thus $(1 - \alpha)d(T^{n-1}x_0, T^n x_0) \leq \alpha d(T^{n-2}x_0, T^{n-1}x_0)$ holds. Therefore we have

$$d(T^{n-1}x_0, T^n x_0) \leq \frac{\alpha}{1 - \alpha} d(T^{n-2}x_0, T^{n-1}x_0)$$

for any n . Then we have

$$\begin{aligned} d(T^n x_0, T^{n+1}x_0) &\leq \frac{\alpha}{1 - \alpha} d(T^{n-1}x_0, T^n x_0) \\ &\leq \left(\frac{\alpha}{1 - \alpha} \right)^2 d(T^{n-2}x_0, T^{n-1}x_0) \\ &\leq \cdots \\ &\leq \left(\frac{\alpha}{1 - \alpha} \right)^n d(x_0, Tx_0). \end{aligned}$$

Therefore we obtain that for any n ,

$$d(T^n x_0, T^{n+1}x_0) \leq \left(\frac{\alpha}{1 - \alpha} \right)^n d(x_0, Tx_0).$$

Then $\{T^n x_0\}$ is a Cauchy sequence in X . In fact, for $n \leq m$, we have

$$\begin{aligned} d(T^m x_0, T^n x_0) &\leq d(T^m x_0, T^{m-1}x_0) + \cdots + d(T^{n+1}x_0, T^n x_0) \\ &\leq \left(\frac{\alpha}{1 - \alpha} \right)^{m-1} d(x_0, Tx_0) + \cdots + \left(\frac{\alpha}{1 - \alpha} \right)^n d(x_0, Tx_0) \\ &< \frac{1 - \alpha}{1 - 2\alpha} \left(\frac{\alpha}{1 - \alpha} \right)^n d(x_0, Tx_0). \end{aligned}$$

Therefore we have $d(T^m x_0, T^n x_0) \rightarrow 0$. Since X is complete, there exists $p \in X$ such that $\lim_{n \rightarrow \infty} T^n x_0 = p$. Since $T^n x_0 \rightarrow p$ and $\{T^n x_0\}$ is nondecreasing, we obtain that $T^n x_0 \leq p$ for any n . Then we have

$$d(Tp, T^{n+1}x_0) \leq \alpha d(p, Tp) + \alpha d(T^n x_0, T^{n+1}x_0).$$

As $n \rightarrow \infty$, we have

$$d(Tp, p) \leq \alpha d(p, Tp).$$

So we have $(1 - \alpha)d(p, Tp) \leq 0$. Thus $d(p, Tp) \leq 0$ holds. Hence we have $Tp = p$.

Next we show the uniqueness of fixed points of T . We assume that X satisfies (1) and $q \in X$ is another fixed point of T .

If p is comparable to q , then $p \geq q$ implies $T^n p \geq T^n q$. Thus $T^n p = p$ is comparable to $T^n q = q$ for any n . Then we have

$$\begin{aligned} d(p, q) &= d(T^n p, T^n q) \\ &\leq \alpha d(T^{n-1} p, T^{n-1} p) + \alpha d(T^{n-1} q, T^n q) \\ &\leq \alpha \left(\frac{\alpha}{1 - \alpha} \right)^{n-1} d(p, Tp) + \alpha \left(\frac{r}{1 - r} \right)^{n-1} d(q, Tq) \end{aligned}$$

for any n . As $n \rightarrow \infty$, we have $p = q$.

If p is not comparable to q . By (1), for p and q , there exists $z \in X$ such that $z \leq Tz$ and z is comparable to p, q . Since $Tz \leq z$ and T is monotone nondecreasing, we obtain that

$$z \leq Tz \leq T^2 z \leq \dots \leq T^n z \leq T^{n+1} z \leq \dots$$

Then we have

$$d(T^{n-1} z, T^n z) \leq \frac{\alpha}{1 - \alpha} d(T^{n-2} z, T^{n-1} z)$$

for any n . Then we have

$$\begin{aligned} d(p, q) &= d(T^n p, T^n q) \\ &\leq d(T^n p, T^n z) + d(T^n z, T^n q) \\ &\leq \alpha(d(T^{n-1} p, T^n p) + d(T^{n-1} z, T^n z)) \\ &\quad + \alpha(d(T^{n-1} z, T^n z) + d(T^{n-1} q, T^n q)) \\ &= 2\alpha d(T^{n-1} z, T^n z) \\ &\leq 2\alpha \cdot \frac{\alpha}{1 - \alpha} d(T^{n-2} z, T^{n-1} z) \\ &\leq \dots \\ &\leq 2\alpha \left(\frac{\alpha}{1 - \alpha} \right)^{n-1} d(z, Tz). \end{aligned}$$

As $n \rightarrow \infty$, we have $d(p, q) = 0$. Hence we have $p = q$. □

The following mappings satisfy conditions of Theorem 2.

Example 3. Let $X = \{0, 1, 2\}$ and the distance function p is the ordinary Euclidean distance on the line. Let T be a mapping of X into itself defined by $Tx = 1$ for $x \in X$. Then T is a monotone nondecreasing mapping satisfying (1). Moreover if we take $\alpha = \frac{1}{2}$, then $x \geq y$ implies $d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty)$.

Example 4. Let $X = [0, 1]$ and the distance function p is the ordinary Euclidean distance on the line. Let T be a mapping of X into itself defined by

$$Tx = \begin{cases} \frac{1}{5}x & 0 \leq x < \frac{1}{2}, \\ \frac{1}{4}x & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Here T is a monotone nondecreasing mapping satisfying (1). Moreover if we take $\alpha = \frac{1}{3}$, then $x \geq y$ implies $d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty)$.

Remark. In [4], we apply Theorem 1 to boundary value problems for fourth order differential equations. We want to apply Theorem 2 to some problems for differential equations. This is a further topic.

References

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