

KANTOROVICH TYPE OPERATOR INEQUALITIES VIA THE SPECHT RATIO II

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ABSTRACT. Yamazaki [14] showed new order preserving operator inequalities on the usual order and the chaotic order by estimating the lower bound of the difference. Mond and Shisha [7, 8] gave an estimate of the difference of the arithmetic mean to the geometric one, as a converse of the arithmetic-geometric mean inequality. In this report, we shall present other order preserving operator inequalities on the usual order and the chaotic one via the Mond-Shisha difference. Among others, as an application of the Furuta inequality, we show that if A and B are positive operators on a Hilbert space H and $k \geq B \geq 1/k$ for some $k \geq 1$, then for a given $\delta \in [0, 1]$, $A^\delta \geq B^\delta$ implies

$$A^p + 2k^{p-2\delta} L(1, k^{2p-2\delta}) \log M_k(p - \delta) I \geq B^p \quad \text{holds for all } p \geq 2\delta,$$

where the case $\delta = 0$ means the chaotic order and the Specht ratio $M_k(r)$ is defined for each $r > 0$ as

$$M_k(r) = \frac{(k^r - 1)k^{r-1}}{r \log k} \quad (k > 0, k \neq 1) \quad \text{and} \quad M_1(r) = 1.$$

1. INTRODUCTION

We shall consider a bounded linear operator on a complex Hilbert space H . An operator A is said to be positive (in symbol: $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$. The Löwner-Heinz theorem asserts that $A \geq B \geq 0$ ensures $A^p \geq B^p$ for all $p \in [0, 1]$. However $A \geq B$ does not always ensure $A^p \geq B^p$ for $p > 1$ in general. Yamazaki [13] showed that t^2 is order preserving in the following sense:

$$(1) \quad A \geq B \geq 0 \quad \text{and} \quad M \geq B \geq m > 0 \quad \text{imply} \quad A^2 + \frac{(M - m)^2}{4} I \geq B^2.$$

Moreover, he showed the following order preserving operator inequality as an extension of (1):

Theorem A. *Let A and B be positive operators on a Hilbert space H satisfying $M \geq B \geq m > 0$. If $A \geq B > 0$, then*

$$A^p + M(M^{p-1} - m^{p-1})I \geq A^p + C(m, M, p)I \geq B^p \quad \text{for all } p \geq 1,$$

where

$$C(m, M, p) = \frac{mM^p - Mm^p}{M - m} + (p - 1) \left(\frac{M^p - m^p}{p(M - m)} \right)^{\frac{p}{p-1}}.$$

For positive invertible operators A and B on a Hilbert space H , the order defined by $\log A \geq \log B$ is called the chaotic order. Since $\log t$ is an operator monotone function, the chaotic order is weaker than the usual one $A \geq B$. J.I.Fujii and the author [1] showed

the following order preserving operator inequalities on the chaotic order which is parallel to Theorem A.

Theorem B . *Let A and B be positive invertible operators on a Hilbert space H satisfying $M \geq B \geq m > 0$. If $\log A \geq \log B$, then*

$$A^p + \frac{M}{m}(M^p - m^p)I \geq A^p + \frac{1}{m}C(m, M, p+1)I \geq B^p \quad \text{for all } p \geq 0,$$

In fact, $\log A \geq \log B$ does not always ensure $A \geq B$ in general. However, by Theorem B, it follows that

$$\log A \geq \log B \quad \text{and} \quad M \geq B \geq m > 0 \quad \text{imply} \quad A + \frac{(M-m)^2}{4m}I \geq B.$$

On the other hand, Specht [9] estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1, \dots, x_n \in [m, M]$ with $M \geq m > 0$,

$$M_h(1) \sqrt[h]{x_1 \cdots x_n} \geq \frac{x_1 + \cdots + x_n}{n} \geq \sqrt[h]{x_1 \cdots x_n},$$

where $h = \frac{M}{m} (\geq 1)$ is a generalized condition number in the sense of Turing [12] and the Specht ratio $M_h(1)$ is defined for $h \geq 1$ as

$$(2) \quad M_h(1) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \quad (h > 1) \quad \text{and} \quad M_1(1) = 1.$$

Yamazaki [14] showed a new characterization of chaotic order as follows:

Theorem C . *Let A and B be positive invertible operators on a Hilbert space H satisfying $M \geq B \geq m > 0$. Then $\log A \geq \log B$ is equivalent to*

$$A^p + L(m^p, M^p) \log M_h(p)I \geq B^p \quad \text{holds for all } p > 0,$$

where $h = \frac{M}{m} > 1$, the logarithmic mean $L(m, M) = \frac{M-m}{\log M - \log m}$ and a generalized Specht ratio $M_h(p)$ is defined as

$$(3) \quad M_h(p) = \frac{(h^p - 1)h^{\frac{p}{h^p-1}}}{pe \log h} \quad (h > 0, h \neq 1) \quad \text{and} \quad M_1(p) = 1.$$

What is the meaning of the constant $L(m^p, M^p) \log M_h(p)$ in Theorem C ? Mond and Shisha [7, 8] made an estimate of the difference between the arithmetic mean and the geometric one: For $x_1, \dots, x_n \in [m, M]$ with $M \geq m > 0$,

$$\sqrt[h]{x_1 \cdots x_n} + D(m, M) \geq \frac{x_1 + \cdots + x_n}{n},$$

where $h = \frac{M}{m} (\geq 1)$ and

$$D(m, M) = \theta M + (1 - \theta)m - M^\theta m^{1-\theta} \quad \text{and} \quad \theta = \log \left(\frac{h-1}{\log h} \right) \frac{1}{\log h}$$

which we call the Mond-Shisha difference. As a matter of fact, J.I.Fujii and the author [1] showed that the Mond-Shisha difference exactly coincides with the constant in Theorem C via the Specht ratio: If $M > m > 0$, then

$$D(m^p, M^p) = L(m^p, M^p) \log M_h(p)$$

where $h = \frac{M}{m}$.

Comparing Theorem A with Theorem B, we observe the difference between p and $p-1$ in the power of the constant. Hence one might expect that the following result holds under the usual order as a parallel result to Theorem C via the Mond-Shisha difference: Let A and B be positive invertible operators satisfying $M \geq B \geq m > 0$. Then

$A \geq B$ implies $A^p + mL(m^{p-1}, M^{p-1}) \log M_h(p-1)I \geq B^p$ for all $p \geq 2$, where $h = \frac{M}{m} \geq 1$. However, we have a counterexample to this conjecture. Put

$$A = \begin{pmatrix} 3 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

then $A \geq B \geq 0$ and $2I \geq B \geq \frac{1}{2}I$. Then we have $mL(m^1, M^1) \log M_h(1) = 0.126638$. On the other hand, $A^2 + \alpha I \geq B^2$ holds if and only if $\alpha \geq \frac{-35 + \sqrt{1465}}{8} = 0.409415$. Therefore $A^2 + mL(m^1, M^1) \log M_h(1)I \not\geq B^2$.

We collect the difference between the usual order and the chaotic one in the following table.

TABLE 1. The difference between the usual order and the chaotic order

$A \geq B$	$\log A \geq \log B$
$A^p + M(M^{p-1} - m^{p-1})I \geq B^p$ for $p \geq 1$	$A^p + \frac{M}{m}(M^p - m^p)I \geq B^p$ for $p > 0$
$A^p + C(m, M, p)I \geq B^p$ for $p \geq 1$	$A^p + \frac{1}{m}C(m, M, p+1)I \geq B^p$ for $p > 0$
$A^p + \frac{1}{m^{r-1}}C(m^r, M^r, 1 + \frac{p-1}{r})I \geq B^p$ for $p, r \geq 1$	$A^p + \frac{1}{m^r}C(m^r, M^r, 1 + \frac{p}{r})I \geq B^p$ for $p, r > 0$
$A^p + \frac{(M^{p-1} - m^{p-1})^2}{4m^{p-2}}I \geq B^p$ for $p \geq 2$	$A^p + \frac{(M^p - m^p)^2}{4m^p}I \geq B^p$ for $p > 0$
???	$A^p + L(m^p, M^p) \log M_h(p)I \geq B^p$ for $p > 0$

In this report, we shall present order preserving operator inequalities on the usual order and the chaotic one in terms of the Mond-Shisha difference. As an application of the Furuta inequality, we show that if A and B are positive operators and $k \geq B \geq 1/k$ for some $k \geq 1$, then for a given $\delta \in [0, 1]$ $A^\delta \geq B^\delta$ implies

$$A^p + 2k^{p-2\delta}L(1, k^{2p-2\delta}) \log M_{k^2}(p-\delta)I \geq B^p \quad \text{holds for all } p \geq 2\delta.$$

2. MAIN RESULTS

First of all, we present other characterizations of the chaotic order via the Mond-Shisha difference.

Theorem 1. *Let A and B be positive invertible operators on a Hilbert space H satisfying $k \geq B \geq \frac{1}{k}$ for some $k \geq 1$. Then the following are mutually equivalent:*

- (i) $\log A \geq \log B$
- (ii) $(A^{\frac{1}{2}} B^p A^{\frac{1}{2}})^s + q k^r L(1, k^{\frac{(p+t)s+r}{q}}) \log M_k((p+t)s+r) I \geq B^{(p+t)s+r}$
holds for $p \geq 0, t \geq 0, s \geq 0, q \geq 1$ with $(t+r)q \geq (p+t)s+r$.
- (iii) $(A^{\frac{1}{2}} B^p A^{\frac{1}{2}})^s + 2k^{(p+t)s-2t} L(1, k^{2(p+t)s-2t}) \log M_k(2(p+t)s-2t) I \geq B^{(p+t)s}$
holds for $p \geq 0, t \geq 0, s \geq 0$ with $(p+t)s \geq 2t$.
- (iv) $A^p + 2k^p L(1, k^{2p}) \log M_k(2p) I \geq B^p$ holds for $p > 0$,
where $M_k(r)$ is defined as (3).

Let A and B be positive invertible operators on a Hilbert space H . We consider an order $A^\delta \geq B^\delta$ for $\delta \in (0, 1]$ which interpolates the usual order $A \geq B$ and the chaotic one $\log A \geq \log B$ continuously, where the case of $\delta = 0$ means the chaotic order. By virtue of the Furuta inequality, we show the following order preserving operator inequality associated with the Mond-Shisha difference.

Theorem 2. *Let A and B be positive invertible operators on a Hilbert space H satisfying $k \geq B \geq \frac{1}{k} > 0$. If $A^\delta \geq B^\delta$ for some $\delta \in [0, 1]$, then*

$$A^p + 2k^{p-2\delta} L(1, k^{2p-2\delta}) \log M_{k^2}(p-\delta) I \geq B^p \quad \text{holds for all } p \geq 2\delta,$$

where $M_k(r)$ is defined as (3).

If we put $\delta = 1$ in Theorem 2, then we have a usual order version via the Mond-Shisha difference.

Theorem 3. *Let A and B be positive invertible operators on a Hilbert space H satisfying $k \geq B \geq \frac{1}{k} > 0$. If $A \geq B$, then*

$$A^p + 2k^{p-2} L(1, k^{2p-2}) \log M_{k^2}(p-1) I \geq B^p \quad \text{holds for all } p \geq 2,$$

where $M_k(r)$ is defined as (3).

Remark 4. *Theorem 2 interpolates Theorems 1 (iv) and Theorem 3 by means of the Mond-Shisha difference. Let A and B be positive invertible operators on a Hilbert space H satisfying $k \geq B \geq \frac{1}{k} > 0$. Then the following assertions hold:*

- (i) $A \geq B$ implies $A^p + 2k^{p-2} L(1, k^{2p-2}) \log M_{k^2}(p-1) I \geq B^p$ for all $p \geq 2$.
- (ii) $A^\delta \geq B^\delta$ implies $A^p + 2k^{p-2\delta} L(1, k^{2p-2\delta}) \log M_{k^2}(p-\delta) I \geq B^p$ for all $p \geq 2\delta$.
- (iii) $\log A \geq \log B$ implies $A^p + 2k^p L(1, k^{2p}) \log M_{k^2}(p) I \geq B^p$ for all $p > 0$.

It follows that the Mond-Shisha difference of (ii) interpolates the scalars of (i) and (iii) continuously. In fact, if we put $\delta = 1$ in (ii), then we have (i), also if we put $\delta \rightarrow 0$ in (ii), then we have (iii).

TABLE 2. Kantorovich constant

$A \geq B$ and $M \geq B \geq m$	$\log A \geq \log B$ and $M \geq B \geq m$
$A^2 + \frac{(M-m)^2}{4}I \geq B^2$	$A + \frac{(M-m)^2}{4m}I \geq B$

TABLE 3. Mond-Shisha difference

$A \geq B$ and $k \geq B \geq 1/k$	$\log A \geq \log B$ and $k \geq B \geq 1/k$
$A^2 + L(1, k^2) \log M_{k^2}(1)^2 I \geq B^2$	$A + kL(1, k^2) \log M_{k^2}(1)^2 I \geq B$

3. PROOF OF RESULTS

To prove our results, we collect several properties of the Specht ratio, see [11, 15]:

- Lemma 5.** (i) $M_k(r) = M_{kr}(1)$ for $k > 0$ and $r > 0$.
(ii) $k \rightarrow M_k(1)$ is increasing for $k > 1$ and decreasing for $1 > k > 0$.
(iii) $M_k(1) = M_{k^{-1}}(1)$ for $k > 0$.
(iv) For $k > 1$, $M_k(p)^{1/p} \rightarrow 1$ as $p \rightarrow 0$.

Lemma 6. Let A and B be positive operators on a Hilbert space H satisfying $k \geq B \geq \frac{1}{k}$ for some $k \geq 1$. If $A^p \geq B^p$ for some $p \in (0, 1]$, then

$$A + \frac{1}{p}L(1, k) \log M_k(1)I \geq B,$$

where $M_k(1)$ is defined as (2).

Proof. The following reverse inequality of Young's one is shown in [11]: For a positive operator A satisfying $k \geq A \geq \frac{1}{k}$ for some $k \geq 1$,

$$(4) \quad A^p + L(1, k) \log M_k(1)I \geq pA + (1-p)I$$

holds for all $1 > p > 0$. Then we have

$$\begin{aligned}
& L(1, k) \log M_k(1)I + pA + (1-p)I \\
& \geq L(1, k) \log M_k(1)I + A^p \quad \text{by the Young inequality and } 1 > p > 0 \\
& \geq L(1, k) \log M_k(1)I + B^p \quad \text{by } A^p \geq B^p \\
& \geq pB + (1-p)I \quad \text{by (4) and } k \geq B \geq 1/k > 0.
\end{aligned}$$

□

The following order preserving operator inequality is our key lemma in this report.

Lemma 7. Let A and B be positive operators on a Hilbert space H satisfying $k \geq B \geq \frac{1}{k} > 0$ for some $k \geq 1$. If $A \geq B$, then

$$A^p + pL(1, k^p) \log M_k(p)I \geq B^p \quad \text{for all } p \geq 1,$$

where $M_k(1)$ is defined as (2).

Proof. Since $(A^p)^{1/p} \geq (B^p)^{1/p}$ for $0 < \frac{1}{p} \leq 1$ and $k^p \geq B^p \geq k^{-p}$, it follows from Lemma 6 that

$$A^p + pL(1, k^p) \log M_k(p)I \geq B^p \quad \text{for all } p \geq 1.$$

□

To prove Theorem 1, we need the following result [3, Proposition 7]:

Theorem D . Let A and B be positive invertible operators on a Hilbert space H . If $\log A \geq \log B$, then

$$\{B^{\frac{r}{2}} (B^{\frac{t}{2}} A^p B^{\frac{t}{2}})^s B^{\frac{r}{2}}\}^{\frac{1}{q}} \geq B^{\frac{(p+t)s+r}{q}}$$

holds for $p, t, s, r \geq 0$ and $q \geq 1$ with $(t+r)q \geq (p+t)s+r$.

Proof of Theorem 1.

(i) $\implies$ (ii): By Theorem D, (i) ensures

$$(5) \quad \{B^{\frac{r}{2}} (B^{\frac{t}{2}} A^p B^{\frac{t}{2}})^s B^{\frac{r}{2}}\}^{\frac{1}{q}} \geq B^{\frac{(p+t)s+r}{q}}$$

holds for $p, t, s, r \geq 0$ and $q \geq 1$ with

$$(6) \quad (t+r)q \geq (p+t)s+r.$$

Put $A_1 = A^{\frac{(p+t)s+r}{q}}$ and $B_1 = (A^{\frac{r}{2}} (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s A^{\frac{r}{2}})^{1/q}$, then $A_1 \geq B_1$ by (5) and $k \geq A \geq 1/k > 0$ assures $k^{\frac{(p+t)s+r}{q}} \geq A^{\frac{(p+t)s+r}{q}} \geq k^{-\frac{(p+t)s+r}{q}}$. By applying Lemma 7 to A_1 and B_1 , we have

$$A_1^q + qL(1, k^{(p+t)s+r}) \log M_{k^{(p+t)s+r}}(1)I \geq B_1^q.$$

Multiplying $B^{-\frac{r}{2}}$ on both sides, we have (ii).

(ii) $\implies$ (iii): Put $r = (p+t)s - 2t \geq 0$ and $q = 2$ in (ii). Then the condition (6) is satisfied and $(p+t)s \geq 2t$, so we have (iii).

(iii) $\implies$ (iv): If we put $t = 0$ and $s = 1$ in (iii), then we have (iv).

(iv) $\implies$ (i): If we put $p \rightarrow 0$ in (iv), then we have (i) by (iv) of Lemma 5. □

Related to the extension of the Löwner-Heinz theorem, Furuta [4] established the following ingenious order preserving inequality which is now called the Furuta inequality.

Theorem F (Furuta inequality)

If $A \geq B \geq 0$, then for each $r \geq 0$

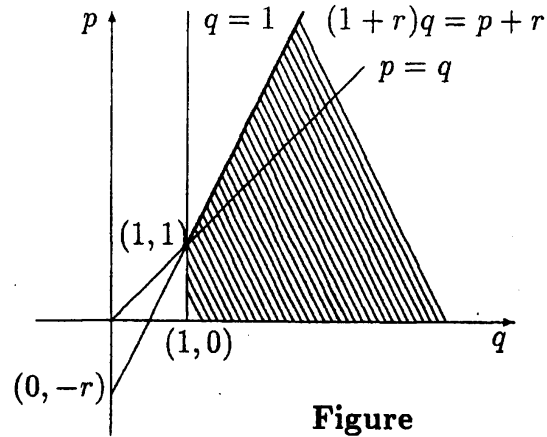
$$(i) \left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}} \right)^{\frac{1}{q}} \geq \left(B^{\frac{r}{2}} B^p B^{\frac{r}{2}} \right)^{\frac{1}{q}}$$

and

$$(ii) \left(A^{\frac{r}{2}} A^p A^{\frac{r}{2}} \right)^{\frac{1}{q}} \geq \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with

$$(1+r)q \geq p+r.$$



Figure

Alternative proofs of Theorem F have been given in [2], [6], and one-page proof in [5]. The domain drawn for p, q and r in Figure is the best possible one [10] for Theorem F.

To prove Theorem 2, we need the following Furuta inequality:

Theorem F'. Let A and B be positive invertible operators on a Hilbert space H and $\delta \in [0, 1]$. Then the following properties are mutually equivalent:

- (i) $A^\delta \geq B^\delta$
- (ii) $\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}} \right)^{\frac{\delta+r}{p+r}} \geq B^{\delta+r}$ for $p \geq \delta$ and $r \geq 0$.

Proof of Theorem 2.

Suppose that $A^\delta \geq B^\delta$ for some $\delta \in [0, 1]$. By the Furuta inequality, we have

$$\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}} \right)^{\frac{\delta+r}{p+r}} \geq B^{\delta+r} \quad \text{for } p \geq \delta \text{ and } r \geq 0.$$

and $k^{\delta+r} \geq B^{\delta+r} \geq k^{-\delta-r}$.

By Lemma 7 and $\frac{p+r}{\delta+r} \geq 1$, it follows that

$$B^{\frac{r}{2}} A^p B^{\frac{r}{2}} + \frac{p+r}{\delta+r} L(1, k^{p+r}) \log M_k(p+r) I \geq B^{p+r}.$$

Hence we have

$$A^p + \frac{p+r}{\delta+r} k^r L(1, k^{p+r}) \log M_k(p+r) I \geq B^p$$

for $p \geq \delta$ and $r \geq 0$. Put $r = p - 2\delta (\geq 0)$, then

$$A^p + 2k^{p-2\delta} L(1, k^{2p-2\delta}) \log M_k(2p-2\delta) I \geq B^p$$

for all $p \geq 2\delta$. □

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