

On a class of indecomposable modules with trivial source

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Abstract The concept of p -radical groups was introduced by Motose-Ninomiya [MN]. Later Tsushima [Ts] investigated p -radical blocks, a block-wise version of p -radical groups. Here we consider more general blocks and introduce module-theoretical viewpoint.

Introduction

Let p be a prime. Let k be an algebraically closed field of characteristic p . Tsushima [Ts] has defined p -radical blocks. In this paper we consider a more general concept and give a module theoretical consideration. We need to introduce some terminology. Let G be a group and P a p -subgroup of G . An indecomposable (right) kG -module S is said to be weakly P -radical if it is P -projective and the number of indecomposable summands (counting multiplicity) of S_P equals $\frac{\dim S |vx(S)|}{|P|}$ (for notation, see below). A simple kG -module S is said to be P -radical if $(1_P)^G \simeq mS \oplus V$, where m is an integer and V is a kG -module not involving S , in which case m is positive, since $\text{Hom}_{kG}(S, (1_P)^G) \neq 0$. We call a p -block B of G weakly P -radical if any simple kG -module in B is weakly P -radical. We call B P -radical if any simple kG -module in B is P -radical. Clearly B is P -radical if and only if $(1_P)^G e_B$ is semi-simple, where e_B is a block idempotent of kG corresponding to B . So when P is a Sylow p -subgroup of G , a P -radical block is a p -radical block in the sense of Tsushima [Ts].

In Section 1 we show for any p -subgroup P of G , B is P -radical if and only if B is weakly P -radical. In Section 2 we consider relationship between weakly P -radical simple modules and subgroups of G for a Sylow p -subgroup P of G . We obtain an alternative proof of a theorem of Laradji [La]. In Section 3 we consider D -radical blocks B for a defect group D of B and strengthen a theorem of Hida-Koshitani [HK].

For a k -module X $\dim X$ denotes the k -dimension of X . For an indecomposable kG -module S , let $vx(S)$ be a vertex of S . For a group H and kH -modules X, Y , $\text{Hom}(X, Y)$ denotes $\text{Hom}_{kH}(X, Y)$ and let $P(X)$ be the projective cover of X . For subgroups H, K of G , $H \backslash G / K$ denotes a complete set of representatives of (H, K) -double cosets in G .

1. Weakly P -radical and P -radical modules

Let P be a p -subgroup of the group G . For an indecomposable kG -module S , let $n_{S,P}$ be the number of indecomposable summands of S_P (counting multiplicity), let $n'_{S,P}$ be the number of indecomposable summands of S_P (counting multiplicity) whose vertices are G -conjugate to $\text{vx}(S)$. Note that $n'_{S,P}$ is positive, if S is P -projective, cf. [Fe, III 4.6]. Let $m_{S,P}$ be the multiplicity of S in $(1_P)^G$ as direct summands. If S is simple, let $k_{S,P}$ be the multiplicity of S in $(1_P)^G$ as irreducible constituents.

Lemma 1. *Let S be a P -projective indecomposable kG -module. Then $n_{S,P} \leq \frac{\dim S|\text{vx}(S)|}{|P|}$ and the following are equivalent.*

$$(i) \ n_{S,P} = \frac{\dim S|\text{vx}(S)|}{|P|}.$$

(ii) S_P is a direct sum of modules of the form $(1_A)^P$ where A is a vertex of S contained in P .

(iii) $S_P \simeq \oplus_i (1_{Q_i})^P$, where Q_i are subgroups of P of the same order.

If these conditions hold, then S has a trivial source.

Proof. We have $S_P = \oplus_{i=1}^{n_{S,P}} W_i^P$, where W_i are indecomposable kQ_i -modules for $Q_i \leq P$ with $Q_i \leq_G \text{vx}(S)$. So $\dim S = \sum_i |P : Q_i| \dim W_i \geq (\sum_i \dim W_i) |P| / |\text{vx}(S)| \geq n_{S,P} |P| / |\text{vx}(S)|$. Thus $n_{S,P} \leq \frac{\dim S|\text{vx}(S)|}{|P|}$. The rest follows from [Fe, III 4.6]. $\square$

As stated in Introduction, for a p -subgroup P of G , we say an indecomposable kG -module S weakly P -radical if S is P -projective and $n_{S,P} = \frac{\dim S|\text{vx}(S)|}{|P|}$. For this definition we have the following, which is straightforward to see.

Lemma 2. *Let P be a p -subgroup of G and let S be an indecomposable kG -module. Let x be any element of G . Then $n_{S,P} = n_{S,P^x}$, $n'_{S,P} = n'_{S,P^x}$, $m_{S,P} = m_{S,P^x}$ and if S is simple $k_{S,P} = k_{S,P^x}$. In particular, if S is weakly P -radical, then S is weakly P^x -radical.*

Recall that a weight U for G is a projective simple $k[N_G(Q)/Q]$ -module for a p -subgroup Q of G ([Al]). So as a $kN_G(Q)$ -module U is indecomposable with trivial source and has Q as a vertex. The Green correspondent of U with respect to $(G, Q, N_G(Q))$ is said to be an Alperin (kG -)module.

The following strengthens Lemma 1 of [Al].

Theorem 3. *Let P be a p -subgroup of G . Let S be a P -projective indecomposable kG -module with trivial source. Then $m_{S,P} \leq n'_{S,P}$ and the equality holds if and only if S is an Alperin module.*

Proof. We compute $n'_{S,P}$. Let Q be a vertex of S . Let U be the Green correspondent of S with respect to $(G, Q, N_G(Q))$. By Green's theorem, $U^G = S \oplus V$, where V is $\mathcal{X}$ -projective for $\mathcal{X} = \{Q \cap Q^x; x \notin N_G(Q)\}$. Since V_P has no

direct summands whose vertex is G -conjugate to Q , it suffices to consider $(U^G)_P$. By Mackey decomposition, $(U^G)_P \simeq \oplus_{x \in N_G(Q) \backslash G/P} ((U^x)_{N_G(Q)^x \cap P})^P$. Assume that $((U^x)_{N_G(Q)^x \cap P})^P$ has an indecomposable summand with vertex G -conjugate to Q . Then for a vertex R of some indecomposable summand of $(U^x)_{N_G(Q)^x \cap P}$, we have $R^u \geq Q^g$ for some $u \in P$ and $g \in G$. Since U^x has a vertex Q^x , which is normal in $N_G(Q)^x$, we have $R \leq Q^x$. Therefore $R = Q^x$. Hence $P \geq Q^x$. Conversely assume $P \geq Q^x$. Then $N_G(Q)^x \cap P \geq Q^x$. Therefore $(U^x)_{N_G(Q)^x \cap P} = \overline{U^x}_{N_G(Q)^x \cap P}$, where $N_G(Q)^x \cap P = N_G(Q)^x \cap P/Q^x$ and $\overline{U^x}$ is the $N_G(Q)^x/Q^x$ -module corresponding to U^x .

Since $\overline{U^x}$ is projective, we obtain $\overline{U^x}_{N_G(Q)^x \cap P} \simeq \frac{\dim U}{|N_G(Q)^x \cap P|} k[\overline{N_G(Q)^x \cap P}]$.

Since $k[\overline{N_G(Q)^x \cap P}] = (1_{Q^x})^{N_G(Q)^x \cap P}$, we have $((U^x)_{N_G(Q)^x \cap P})^P \simeq \frac{\dim U}{|N_G(Q)^x \cap P|} (1_{Q^x})^P$.

Therefore $n'_{S,P} = \sum_{x \in N_G(Q) \backslash G/P, P \geq Q^x} \frac{\dim U}{|N_G(Q)^x \cap P|}$.

Now we consider $m_{S,P}$. By the Burry-Carlson-Puig theorem, $m_{S,P}$ equals the multiplicity of U in $((1_P)^G)_{N_G(Q)}$ as direct summands. By Mackey decomposition, we have

$$((1_P)^G)_{N_G(Q)} \simeq \oplus_{x \in P \backslash G/N_G(Q)} (1_{P^x \cap N_G(Q)})^{N_G(Q)}.$$

Since U has vertex Q it suffices to consider those $x \in P \backslash G/N_G(Q)$ for which $Q \leq P^x$. Then, $(1_{P^x \cap N_G(Q)})^{N_G(Q)} = (1_{\overline{P^x \cap N_G(Q)}})^{\overline{N_G(Q)}}$, where $\overline{N_G(Q)} = N_G(Q)/Q$.

Put $(1_{\overline{P^x \cap N_G(Q)}})^{\overline{N_G(Q)}} \simeq n_x \bar{U} \oplus V_x$, where $\bar{U}$ is the $\overline{N_G(Q)}$ -module corresponding to U and

V_x has no summands isomorphic to $\bar{U}$. Then $m_{S,P} = \sum_{x \in P \backslash G/N_G(Q), Q \leq P^x} n_x$. Now

$$\dim \text{Hom}((1_{\overline{P^x \cap N_G(Q)}})^{\overline{N_G(Q)}}, \bar{U}) = n_x \dim \text{Hom}(\bar{U}, \bar{U}) + \dim \text{Hom}(V_x, \bar{U}) \geq n_x.$$

On the other hand,

$$\begin{aligned} \dim \text{Hom}((1_{\overline{P^x \cap N_G(Q)}})^{\overline{N_G(Q)}}, \bar{U}) &= \dim \text{Hom}((1_{\overline{P^x \cap N_G(Q)}}), \bar{U}_{\overline{P^x \cap N_G(Q)}}) \\ &= \frac{\dim U}{|\overline{P^x \cap N_G(Q)}|}, \end{aligned}$$

since $\bar{U}$ is projective. Therefore

$$\sum_{x \in P \backslash G/N_G(Q), Q \leq P^x} \dim \text{Hom}((1_{\overline{P^x \cap N_G(Q)}})^{\overline{N_G(Q)}}, \bar{U}) = \sum_{x \in P \backslash G/N_G(Q), Q \leq P^x} \frac{\dim U}{|\overline{P^x \cap N_G(Q)}|} = n'_{S,P}.$$

Here the last equality follows by considering the correspondence $x \mapsto x^{-1}$. Hence $n'_{S,P} \geq \sum_x n_x = m_{S,P}$. If the equality holds then $m_{S,P} = n'_{S,P} \neq 0$, since S is P -projective, cf. [Fe, III 4.6]. So $n_x \neq 0$ for some x . Thus $\dim \text{Hom}(\bar{U}, \bar{U}) = 1$. Since $\bar{U}$ is projective we see $\bar{U}$ is simple and S is Alperin. Conversely assume U is simple. Then equality holds throughout. Hence $m_{S,P} = n'_{S,P}$. The proof is complete. $\square$

Corollary 4. *Let S be a P -projective simple module with trivial source. Then*

$$m_{S,P} \leq n'_{S,P} \leq n_{S,P} \leq k_{S,P} = \frac{\dim P(S)}{|P|}.$$

Proof. The first inequality follows from Theorem 3. The second is trivial. To prove the third, put $S_P \simeq \bigoplus_{i=1}^{n_{S,P}} (1_{Q_i})^P$ for suitable Q_i . Then $n_{S,P} = \dim \text{Hom}(S_P, 1_P) = \dim \text{Hom}(S, (1_P)^G) \leq k_{S,P}$. Further $k_{S,P} = \dim \text{Hom}(P(S), (1_P)^G) = \dim \text{Hom}(P(S)_P, 1_P) = \frac{\dim P(S)}{|P|}$, since $P(S)_P$ is projective. $\square$

Proposition 5. *Let S be a simple module. If S is P -radical, then S is weakly P -radical.*

Proof. Since S is P -projective and has trivial source, by Corollary 4 we have $m_{S,P} \leq n'_{S,P} \leq n_{S,P} \leq k_{S,P}$. By assumption $m_{S,P} = k_{S,P}$, so that $n'_{S,P} = n_{S,P}$. Thus S is weakly P -radical. $\square$

When P is a Sylow p -subgroup of G , a weakly P -radical module is said to be just a weakly radical module. The same is true for other terminology. Also $n_{S,P}$ is denoted by n_S etc. Such convention is justified by Lemma 2. Note that then radical blocks are p -radical blocks as defined by Tsushima [Ts].

Corollary 6. ([Ok2, Lemma 1]) *A radical (simple) module is weakly radical.*

The following is fundamental.

Proposition 7. ([Ok1, Lemma 2.2]). *A simple kG -module with trivial source is an Alperin module.*

Lemma 8. *If S is an Alperin module, then $(\dim S)_p = |P : \text{vx}(S)|$, where P is a Sylow p -subgroup of G with $\text{vx}(S) \leq P$.*

Proof. See the proof of Lemma 2.2 of [Ok1]. $\square$

Proposition 9. *Let S be a simple kG -module. Let P be a Sylow p -subgroup of G .*

(i) *If S is weakly radical, $\dim P(S) \geq |\text{vx}(S)| \dim S \geq |P|(\dim S)_{p'}$.*

Furthermore the following conditions are equivalent.

(ii) *S is radical.*

(iii) *S is weakly radical and $\dim P(S) = |P|(\dim S)_{p'}$.*

(iv) *S is weakly radical and $\dim P(S) = |\text{vx}(S)| \dim S$.*

(v) *$n_S = \frac{\dim P(S)}{|P|}$.*

Proof. We may assume $P \geq \text{vx}(S)$.

(i) By Corollary 4 we have $\frac{\dim P(S)}{|P|} = k_S \geq n_S = \frac{\dim S}{|P:\text{vx}(S)|}$, from which the

first inequality follows. The second inequality follows from Green's theorem.

(ii) $\Rightarrow$ (iii): By Corollary 6, S is weakly radical. Also by Corollary 4 $n_S = k_S$. We have $n_S = \frac{\dim S}{|P:\text{vx}(S)|} = (\dim S)_{p'}$ by Proposition 7 and Lemma 8. And $k_S = \frac{\dim P(S)}{|P|}$. Thus the equality holds.

(iii) $\Rightarrow$ (iv): This follows from (i).

(iv) $\Rightarrow$ (v): Since S is weakly radical, $n_S = \frac{\dim S}{|P:\text{vx}(S)|}$. The result follows.

(v) $\Rightarrow$ (ii): Write $S_P = \oplus_{i=1}^{n_S} (W_i)^P$, where each W_i is an indecomposable kQ_i -module for some $Q_i \leq P$. Then $\dim \text{Hom}(1_P, S_P) = \sum_i \dim \text{Hom}(1_P, (W_i)^P) = \sum_i \dim \text{Hom}(1_{Q_i}, W_i) \geq n_S$. So we have

$$n_S \leq \dim \text{Hom}(1_P, S_P) = \dim \text{Hom}((1_P)^G, S) \leq k_S = \frac{\dim P(S)}{|P|}.$$

Hence equality holds throughout. Likewise we have $\dim \text{Hom}(S, (1_P)^G) = k_S$. Hence there exist submodules U and V of $(1_P)^G$ with the following properties: $U \simeq k_S S$, $(1_P)^G/V \simeq k_S S$ and V does not involve S . Then $U \cap V = 0$ and hence $(1_P)^G = U \oplus V$. Thus S is radical. The proof is complete. $\square$

Corollary 10. *Let S be a simple kG -module for a p -solvable group G . Then S is radical if and only if S is weakly radical.*

Proof. "only if" part: This follows from Corollary 6.

"if" part: Let P be a Sylow p -subgroup of G . Since G is p -solvable $\dim P(S) = |P|(\dim S)_{p'}$ by Fong's theorem [Na, Corollary 10.14]. Thus Proposition 9 yields the result. $\square$

Remark. There does exist a simple kG -module which is weakly radical but not radical. Indeed, clearly 1_G is always weakly radical. Let G be the alternating group of degree 5 and $p = 3$. Then $\dim P(1_G) = 6$ ([HB, p.222]). So by Proposition 9, 1_G is not radical.

Corollary 11. *If B is radical, then $(1_P)^G e_B \simeq \oplus_S (\dim S)_{p'} S$, where S runs through simple modules in B up to isomorphism.*

Theorem 12. *Let P be a p -subgroup of G . Then B is P -radical if and only if B is weakly P -radical.*

Proof. "if" part: Let $(1_P)^G e_B \simeq \oplus_S m_{S,P} S \oplus X$, where S runs through simple modules in B up to isomorphism. Assume $X \neq 0$ and let T be a simple submodule of X . Then $\dim \text{Hom}(T, (1_P)^G) > m_{T,P}$. But $\dim \text{Hom}(T, (1_P)^G) = \dim \text{Hom}(T_P, 1_P) = n_{T,P} = n'_{T,P} = m_{T,P}$ by Proposition 7 and Theorem 3, a contradiction. Hence $X = 0$ and B is P -radical.

"only if" part: This follows from Proposition 5. $\square$

The group G is said to be p -radical, if $(1_P)^G$ is semi-simple for a Sylow p -subgroup P of G ([Ts, p.80]),

Corollary 13. G is p -radical if and only if any simple kG -module is weakly radical.

Lemma 14. *If an Alperin module S is weakly radical, then S is simple.*

Proof. By Theorem 3 $m_S = n_S$. From $(1_P)^G = m_S S \oplus V$, we have

$$n_S = \dim \text{Hom}(S_P, 1_P) = \dim \text{Hom}(S, (1_P)^G) = m_S \dim \text{Hom}(S, S) + \dim \text{Hom}(S, V)$$
Thus $\text{Hom}(S, S) = k$ and $\text{Hom}(S, V) = 0$. Let T be a simple module in the head of S . Since $\text{Hom}(T, (1_P)^G) = \text{Hom}(T_P, 1_P) \neq 0$, T is a submodule of V or S . The former is impossible, since $\text{Hom}(S, V) = 0$. Thus the latter holds. Then there is a non-zero homomorphism $\varphi : S \rightarrow \text{Soc}(S)$. Of course $\varphi(J(S)) = 0$. Since $\text{Hom}(S, S) = k$, φ must be a monomorphism. Therefore $J(S) = 0$. Thus S is simple.

Proposition 15. *Let B be a block of G . Assume that Alperin's weight conjecture [Al] is true for B . Then the following are equivalent.*

- (i) B is radical.
- (ii) $(1_P)^G e_B$ is a direct sum of weakly radical indecomposable modules.
- (iii) All Alperin modules in B are weakly radical.

Proof. (i) $\Rightarrow$ (ii): Any simple module S in B is radical. Hence S is weakly radical by Corollary 6.

(ii) $\Rightarrow$ (iii): Let S be an Alperin module in B . Then $m_S = n'_S > 0$ by Theorem 3 and [Fe, III 4.6]. Hence S is weakly radical.

(iii) $\Rightarrow$ (i): Let S be an Alperin module in B . Then S is weakly radical. Hence S is simple by Lemma 14. Thus, by Alperin's weight conjecture, any simple module T in B is an Alperin module. Hence T is weakly radical. So B is weakly radical and radical by Theorem 12.

Proposition 16. *Let S be an indecomposable kG -module. If $\dim S$ is prime to p , then S is weakly radical if and only if $G/\text{Ker } S$ is a p' -group.*

Proof. (i) "only if" part: Let P be a Sylow p -subgroup of G . By Lemma 1, we have $S_P \simeq \oplus_i (1_{Q_i})^P$, where Q_i are vertices of S . Thus $Q_i = P$ for all i and $P \leq \text{Ker } S$.

"if" part: Since $P \leq \text{Ker } S$, the result follows by Lemma 1.

2. Weakly radical simple modules and subgroups

In this section we consider relationship between weakly radical simple modules and subgroups.

Proposition 17. *Let S be a simple kG -module with trivial source. Let H be a subgroup of G and let U be a simple kH -module such that $S \simeq U^G$.*

- (i) If S is weakly radical, then U is weakly radical.
- (ii) Let P be a Sylow p -subgroup of G . The following are equivalent.
 - (iia) S is radical and $P(S) \simeq P(U)^G$.
 - (iib) $\dim \text{Inv}_{P^x \cap H}(U) = \frac{\dim P(U)}{|P^x \cap H|}$ for any $x \in G$.
 - (iic) U is radical and S is weakly radical.

Proof. (i) Choose a Sylow p -subgroup P of G such that $Q = P \cap H$ is a Sylow p -subgroup of H . We have $(U_Q)^P | S_P$ by Mackey decomposition. Since S has a trivial source, so does U . So we can put $U_Q \simeq \oplus_i (1_{R_i})^Q$ for some subgroups R_i of Q . Then $(U_Q)^P \simeq \oplus_i (1_{R_i})^P$. Since S is weakly radical, all R_i have the same order. Thus U is weakly radical by Lemma 1.

(iia) $\Rightarrow$ (iib): We have $n_S = \dim \text{Hom}(1_P, S_P) = \dim \text{Hom}((1_P)^G, S) = \dim \text{Hom}(((1_P)^G)_H, U) = \sum_{x \in P \backslash G/H} \dim \text{Hom}((1_{P^x \cap H})^H, U)$. Here

$$\dim \text{Hom}((1_{P^x \cap H})^H, U) = \dim \text{Hom}(1_{P^x \cap H}, U_{P^x \cap H}) = \dim \text{Inv}_{P^x \cap H}(U).$$

And

$$\begin{aligned} \dim \text{Hom}((1_{P^x \cap H})^H, U) &\leq \dim \text{Hom}(P(U), (1_{P^x \cap H})^H) \\ &= \dim \text{Hom}(P(U)_{P^x \cap H}, 1_{P^x \cap H}) = \frac{\dim P(U)}{|P^x \cap H|}. \end{aligned}$$

Further, $\sum_x \frac{|H|_p}{|P^x \cap H|} = |G : H|_{p'}$. Therefore $n_S \leq \frac{\dim P(U) |G : H|_{p'}}{|H|_p} = \frac{\dim P(S)}{|G|_p} = k_S$. Since S is radical, equality holds throughout by Proposition 9, and the result follows.

(iib) $\Rightarrow$ (iia): From the above proof we obtain $n_S = \frac{\dim P(U) |G : H|}{|G|_p}$. Since $P(S) | P(U)^G$, $\frac{\dim P(U) |G : H|}{|G|_p} \geq \frac{\dim P(S)}{|G|_p} = k_S$. Therefore $n_S = k_S$ by Corollary 4, and S is radical by Proposition 9. Further, $P(S) \simeq P(U)^G$.

(iia) $\Rightarrow$ (iic): Since S is weakly radical by Corollary 6, U is weakly radical by (i). So by Proposition 9 it suffices to show $\dim P(U) = |vx(U)| \dim U$. We have $\dim P(S) = |G : H| \dim P(U)$. Since S is radical, by Proposition 9 $\dim P(S) = |vx(S)| \dim S = |vx(S)| |G : H| \dim U$. Since $vx(S) =_G vx(U)$, the result follows.

(iic) $\Rightarrow$ (iia): Since U is radical, $\dim P(U)^G = |G : H| |vx(U)| \dim U$. Since S is weakly radical, by Proposition 9 $\dim P(S) \geq |vx(S)| \dim(S) = |vx(S)| |G : H| \dim U$. Hence $\dim P(S) \geq \dim P(U)^G$. But $P(S) | P(U)^G$. So the equality holds throughout. Therefore $P(S) \simeq P(U)^G$ and S is radical by Proposition 9. $\square$

Theorem 18([La, Theorem]) *Let P be a Sylow p -subgroup of G . The following are equivalent.*

- (i) G is p -radical.
- (ii) For any simple kG -module S , there are a subgroup H of G and a simple kH -module U with the following properties: $S = U^G$, $vx(U) \leq \text{Ker } U$, $P^x \cap H$ is a Sylow p -subgroup of H for any $x \in G$.
- (iii) For any simple kG -module S , there are a subgroup H of G and a simple kH -module U with the following properties: $S = U^G$, $vx(S) \leq \text{Ker } U$, $P^x \cap H$ is a Sylow p -subgroup of H for any $x \in G$.

Proof. (i) $\Rightarrow$ (ii) G is p -solvable by [Ok2]. So there are H and U as above such that $S = U^G$ and that $\dim U$ is a p' -number by [Na, Theorem 10.11]. Since G is p -solvable, $P(S) \simeq P(U)^G$ by Fong's theorem [Na, Corollary 10.14]. Hence U is radical by Proposition 17. Therefore $\text{vx}(U) \leq \text{Ker } U$ by Corollary 6 and Proposition 16. Further, for any $x \in G$, $\dim U = \dim \text{Inv}_{P^x \cap H}(U) = \frac{\dim P(U)}{|P^x \cap H|} = \frac{|H|_p \dim U}{|P^x \cap H|}$ by Proposition 16, Proposition 17 (iib) and Fong's theorem [Na, Corollary 10.14]. So $P^x \cap H$ is a Sylow p -subgroup of H for any $x \in G$.

(ii) $\Rightarrow$ (i) By Corollary 13, it suffices to show S is weakly radical. From the condition that $\text{vx}(U) \leq \text{Ker}(U)$, we see $U|(1_{\text{Ker}(U)})^H$. This implies U is weakly radical. We have $S_P \simeq \sum_{x \in H \setminus G/P} (U_{H^x \cap P}^x)^P$. Since U^x is a weakly radical kH^x -module and $H^x \cap P$ is a Sylow p -subgroup of H^x , we have $U_{H^x \cap P}^x \simeq \oplus_i (1_{Q_{x,i}})^{H^x \cap P}$ and $|Q_{x,i}| = |\text{vx} U|$. Therefore $S_P \simeq \oplus_{x,i} (1_{Q_{x,i}})^P$. So S is weakly radical by Lemma 1.

(ii) $\Rightarrow$ (iii). Since $\text{vx}(U)$ is a vertex of S , the result follows.

(iii) $\Rightarrow$ (ii). Since $\text{vx}(S) \leq \text{Ker } U$, $\text{vx}(S) \leq \text{vx}(U)$ for a vertex of U ([NT, Theorem 4.7.8 (i)]). But $\text{vx}(S) =_G \text{vx}(U)$. So $\text{vx}(U) = \text{vx}(S) \leq \text{Ker } U$. The proof is complete. $\square$

In case of normal subgroups we have the following

Proposition 19. *Let N be a normal subgroup of G . Let S (resp. X) be a simple kG - (resp. kN -) module.*

(i) *If $S|X^G$ and X is weakly radical, then S is weakly radical.*

(ii) *If $X|S_N$ and S is weakly radical, then X is weakly radical.*

Proof. Let P be a Sylow p -subgroup of G .

(i) We have $S_P|(X^G)_P$. By Mackey decomposition,

$$(X^G)_P \simeq \oplus_{x_i \in N \setminus G/P} ((X^{x_i})_{P \cap N})^P.$$

It is straightforward to check that for each x_i , X^{x_i} is also weakly radical. So by Lemma 1, for each i , $(X^{x_i})_{P \cap N} \simeq \oplus_j (1_{Q_{ij}})^{P \cap N}$, where Q_{ij} are subgroups of $P \cap N$ such that $|Q_{ij}| = |\text{vx}(X)|$. Hence S is weakly radical by Lemma 1.

(ii) We have $X_{P \cap N}|S_{P \cap N}$. Put $S_P \simeq \oplus_i (1_{Q_i})^P$ for suitable $Q_i \leq P$. Then for each i ,

$$((1_{Q_i})^P)_{P \cap N} \simeq \oplus_{u \in Q_i \setminus P/P \cap N} (1_{N \cap (Q_i)^u})^{P \cap N},$$

Since Q_i are G -conjugate, $|N \cap (Q_i)^u|$ are the same for all i and u . Thus X is weakly radical by Lemma 1. The proof is complete. $\square$

3. D -radical blocks

Let B be a block of G with defect group D . D -radical blocks have been investigated in [Hida-Koshitani]

Lemma 20. *Let P and Q be p -subgroups of G .*

- (i) If S is a weakly P -radical module and $P \leq Q$, then S is weakly Q -radical.
- (ii) If S is a P -radical module and $P \leq Q$, then S is Q -radical. In particular, if B is D -radical, then B is radical.
- (iii) If B is P -radical, P contains a defect group of B .

Proof (i) Let X be an indecomposable summand of S_Q . Then, since S is weakly P -radical, $(1_{Q_i})^P | X_P$ for some $Q_i \leq P$ with $Q_i = {}_G \text{vx}(S)$. Then there is a vertex $\text{vx}(X)$ of X with $\text{vx}(X) \geq Q_i$. But $\text{vx}(X) \leq {}_G \text{vx}(S)$, so $\text{vx}(X) = Q_i$. Since X has trivial source, we obtain $X = (1_{Q_i})^Q$. Thus S is weakly Q -radical.

(ii) Since there is an epi $(1_P)^Q \rightarrow 1_Q$, there is an epi $\varphi : (1_P)^G \rightarrow (1_Q)^G$. We have $(1_P)^G = U \oplus V$, where $U \simeq mS$ for some integer m and V does not involve S . Then $(1_Q)^G = \varphi(U) + \varphi(V)$. Here $\varphi(U) \simeq m'S$ for some integer m' and $\varphi(V)$ does not involve S . Hence $(1_Q)^G = \varphi(U) \oplus \varphi(V)$, and S is Q -radical.

(iii) Let S be a simple module in B with vertex D . Then S is P -radical, and S is weakly P -radical. Thus P contains a vertex of S , and the result follows. The proof is complete. $\square$

Lemma 21. *Let S be an indecomposable kG -module. Let $\text{vx}(S) = Q \leq P$ for a p -subgroup P of G . The following are equivalent.*

- (i) S is weakly P -radical and Q is strongly closed in P with respect to G .
- (ii) S is weakly P -radical and Q is weakly closed in P with respect to G .
- (iii) $S_P \simeq n(1_Q)^P$ for some integer n and $Q \triangleleft P$.
- (iv) $Q \leq \text{Ker} S$.

Proof. (i) $\Rightarrow$ (ii): This is trivial.

(ii) $\Rightarrow$ (iii): We have $S_P \simeq \oplus_i (1_{Q_i})^P$, where $Q_i = {}_G \text{vx}(S)$ for each i . Since $Q, Q_i \leq P$, we obtain $Q_i = Q$. Therefore $S_P \simeq n(1_Q)^P$ for some integer n . Clearly $Q \triangleleft P$.

(iii) $\Rightarrow$ (iv): Clearly $S_Q \simeq m1_Q$ for some integer m .

(iv) $\Rightarrow$ (i): We have $S_Q \simeq m1_Q$ for some integer m , so that S is weakly Q -radical. Thus S is weakly P -radical by Lemma 20. Put $N = \text{Ker} S$. Then $S_N \simeq m1_N$ and S is N -projective. Hence S and 1_N have a common vertex. Thus Q is a Sylow p -subgroup of N . Since $Q \leq P \cap N \leq N$, we obtain $Q = N \cap P$. Then for any $g \in G$, $Q^g \cap P \leq N \cap P = Q$. Thus Q is strongly closed in P with respect to G . The proof is complete. $\square$

Let $B_0(G)$ be the principal block of G .

Theorem 22 (Okuyama). *If $B_0(G)$ is radical, G is p -solvable.*

Proof. See the proof of Theorem 1 of [Ok2]. $\square$

Let $R_p(G)$ be the maximal normal p -solvable subgroup of G .

The following strengthens Theorem 1.1 of [HK].

Theorem 23. *Let P be a Sylow p -subgroup of G with $P \geq D$. The following are equivalent.*

- (i) B is D -radical.
- (ii) B is weakly D -radical.
- (iii) *There is a p -solvable normal subgroup N of G such that: B covers $B_0(N)$, D is a Sylow p -subgroup of N , and $B_0(N)$ is radical.*
- (iv) *For a block b of $R_p(G)$ covered by B , it holds that: D is a defect group of b , b is D -radical, and $G = N_G(D)R_p(G)$.*
- (v) B is radical and D is strongly closed in P with respect to G .
- (vi) B is radical and D is weakly closed in P with respect to G .
- (vii) B is radical and there is a simple kG -module S in B with $\text{Ker} S \geq D$.
- (viii) B is radical and there is a normal subgroup N of G such that D is a Sylow p -subgroup of N .

Proof. (i) $\Leftrightarrow$ (ii) This follows from Theorem 12.

(ii) $\Rightarrow$ (iii): Let S_1 be a simple kG -module in B with vertex D . Put $N = \text{Ker} S_1$. Since S_1 is weakly D -radical, $(S_1)_D \simeq n1_D$ for some integer n . So $D \leq N$. Since B covers $B_0(N)$, D is a defect group of $B_0(N)$. Thus D is a Sylow p -subgroup of N . For any simple kN -module X in $B_0(N)$, choose a simple kG -module S in B lying over X . Then, since S is weakly D -radical, we see X is weakly radical by Proposition 19 and Lemma 20. So $B_0(N)$ is radical by Theorem 12 and N is p -solvable by Theorem 22.

(iii) $\Rightarrow$ (iv): Let b be a block of $R_p(G)$ covered by B . Since $N \leq R_p(G)$ and b covers $B_0(N)$, we may assume D is a defect group of b . By the Frattini argument $G = N_G(D)N = N_G(D)R_p(G)$. Let S be a simple module in b . For any irreducible constituent X of S_N , X lies in $B_0(N)$ and X is weakly D -radical. Thus S is weakly D -radical. So b is weakly D -radical and hence D -radical by Theorem 12.

(iv) $\Rightarrow$ (ii): For any simple kG -module S in B , let X be an irreducible constituent in b of $S_{R_p(G)}$. Then, since b is D -radical and hence weakly D -radical, $X_D \simeq \oplus_i (1_{Q_i})^D$, where $Q_i =_{R_p(G)} \text{vx}(X)$. S_D is a direct sum of the modules of the form $(X^g)_D, g \in G$. Now there is $n \in N_G(D)$ such that $X^g \simeq X^n$. Then

$$(X^g)_D \simeq (X^n)_D \simeq (X_D)^n \simeq \oplus_i (1_{Q_i^n})^D.$$

Since $|Q_i^n| = |\text{vx}(X)|$, S is weakly D -radical. Hence (ii) follows.

(v) $\Rightarrow$ (vi): This is trivial.

(vi) $\Rightarrow$ (v): Let S be a simple module in B with vertex D . Since S is weakly radical and D is weakly closed in P with respect to G , D is strongly closed in P with respect to G by Lemma 21.

(v) $\Rightarrow$ (ii): Let S be a simple module in B . We have $S_P \simeq \oplus_i (1_{Q_i})^P$, where Q is a vertex of S and $Q_i = Q^{x_i}, x_i \in G$. We may assume $Q \leq D$. $((1_{Q_i})^P)_D \simeq \oplus_{u \in Q_i \setminus P/D} (1_{Q_i^u \cap D})^D$. We see $Q_i^u = Q^{x_i u} \leq D^{x_i u} \cap P \leq D$ by (v). Therefore $((1_{Q_i})^P)_D \simeq \oplus_u (1_{Q_i^u})^D$. Hence S is weakly D -radical.

(i) and (iii) $\Rightarrow$ (vii): By Lemma 20, B is radical. Let S be a simple module in B lying over 1_N . Then $D \leq N \leq \text{Ker} S$.

(vii) $\Rightarrow$ (viii): Let $N = \text{Ker}S$. Then B covers $B_0(N)$. Therefore $D = D \cap N$ is a defect group of $B_0(N)$.

(viii) $\Rightarrow$ (v): This follows from the fact that $D = P \cap N$. The proof is complete. $\square$

Remark. The implication (i) $\Rightarrow$ (ii) has been proved in Lemma 7 of [Ko] in a different way.

Corollary 24 ([HK], Corollary 1.3). *If $\text{vx}(S) \leq \text{Ker}S$ for any simple module S in B , then B is D -radical.*

Proof. Let S be a simple module in B . By Lemma 21 S is weakly D -radical. Hence B is weakly D -radical, and B is D -radical by Theorem 23. $\square$

The following extends Theorem 22.

Corollary 25. *Let B be a radical block of G with defect group D . If D is a Sylow p -subgroup of G , then G is p -solvable.*

Proof. We see B is D -radical. If N is as in (iii) of Theorem 23, then N is p -solvable and G/N is a p' -group. Hence G is p -solvable. $\square$

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