Study of Equivariant maps and applications to Quantum Information Theory

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A thesis submitted to the

Department of Mathematics, Graduate school of Science
Kyoto University

In Partial Fulfillment of the requirements
For the Degree of Doctor of Philosophy
Kyoto University

Kyoto
December 2018

Abstract

In [**1**], Bhat characterizes a class of linear maps which respect unitary conjugation. We generalize this idea and introduce a new class of linear maps, which we call equivariant maps. These maps satisfy an algebraic property which makes it easier to detect the k-positivity for this class of linear maps. They are therefore particularly suitable for entanglement detection in quantum information theory. The problem of detecting entanglement is an NP-hard problem [**2**]. This thesis serves two purposes:

(i) It provides a criterion to detect the k-positivity of equivariant linear maps and we study some new examples of equivariant maps, which act as entanglement witness.

(ii) Subsequently, we study a subclass of equivariant maps, which we call (a, b)-unitarily equivariant maps. We give a characterization of this subclass and study its graphical representation. We then apply this class to study the problem of entanglement detection.

Declaration

I hereby declare that this thesis is an original report of my research work, has been written by me and has not been submitted for any previous degree. I have duly acknowledged all sources of information which have been used in this thesis.

Gunjan Sapra
(20 December 2018)

Acknowledgement

First, I would like to acknowledge my sincere gratitude to my supervisor Prof. Benoît Collins for his guidance through our weekly discussions for years, his patience, motivation, and immense knowledge. I would like to thank him for the numerous discussions and useful comments over skype and emails. My sincere thanks go to Prof. Hiroyuki Osaka for his guidance and support at all stages of my Ph.D. studies. I am very grateful for his time and all the beautiful mathematical ideas he provided me during the course of this work.
This work would not have been possible without their support. They carefully read the draft of my thesis and provided useful suggestions and corrections.
I would like to thank the committee members who accepted to evaluate my Thesis, for their time and attention. I received very constructive feedback from at least three committee members. It allowed to improved substantially the appearance and the mathematical content of my thesis and I am very grateful for that.
I owe special acknowledgment to Prof. Arno Suzuki at Kyoto University, who was always willing to discuss about my career. I thank her for her valuable advice.

I greatly acknowledge the support and guidance of my master supervisor Prof. G. Ramesh at IIT Hyderabad, who motivated me to go for doctoral studies. I would like to thank him for his scientific advice on my career as well as research.

I would like to thank my friend cum colleague, Yuki Ueda, for organizing seminars and for his useful discussions over various topics of mathematics. My sincere thanks go to Ivan Bardet for stimulating discussions.
My special thanks to the staff and in particular Ms. Yukari Shinozaki, at the Mathematics Department, Kyoto University for their constant support and who made my stay pleasant at Kyoto.
I sincerely acknowledge FRIENDSHIP project of Japan International Corporation Agency (JICA) for research fellowship (D-15-90284), for providing me an opportunity to pursue my doctoral studies at Kyoto University. I would like to thank all the JICA coordinators who have helped me during my stay in Japan. My special thanks to Ms. Yuko Yokota, Ms. Miki Matsuo, Ms. Mizuho Uotsu for their kind cooperation. I thank my fellow colleagues Sushma, Prashant, Reddy, Mathieu, and Felix.
Last but not the least, My deepest gratitude goes to my parents Shashi Sapra and Lalit Kumar Sapra, sister and my beloved family for their unflagging love and support.

Dedication

To my family.

Introduction

Background

The study of positive linear maps started in 1950's with Kadison generalized Schwartz inequality and characterization of isometries of C^*-algebra [**3, 4**]. It was confined to operator algebra until the 1990s when it became clear that they are extremely important to detect entanglement, which was shown in a series of papers where dual cone relation between the positive linear maps and quantum entanglement was established [**5, 6**]. Since then, positive linear maps have been extensively used to solve problems in quantum information theory. Yet the structure of positive linear maps is far from being completely understood, even on low dimensional matrix algebras. There is no simple technique available to detect the positivity of linear maps. Therefore, it is important to study various concepts of positivity.
On the other hand, there is another class of maps whose structure is well understood, the class of completely positive maps. In 1952, Stinespring [**7**] introduced completely positive maps and gave his famous Stinespring dilation theorem. In 1975, Choi [**8**] gave a characterization to this class of linear maps. Many efforts have been made to find examples of linear maps and study various concepts of positivity. Choi gave the first example of a linear map to distinguish n-positivity from $(n-1)$-positivity. He gave an example of a linear map on $M_3(\mathbb{C})$, which is 2-positive but not completely positive.

In 1983, Takasaki and Tomiyama [**9**] obtained examples of linear maps which are $(k-1)$-positive but not k-positive. They considered the line segment between the Choi map and the transpose map on $M_n(\mathbb{C})$. In 1985, Tomiyama [**10**] generalized this construction and study linear maps on the extended segment. In the same note, he proved a necessary and sufficient condition for a linear map to be k-positive.

Given a linear map, we try to express it as a positive sum of other class of linear maps which are more tractable, namely the class of completely positive and completely copositive maps, such maps are called decomposable maps. Woronowicz [**11**] showed that every positive map in $\mathcal{B}(M_2(\mathbb{C}), M_n(\mathbb{C}))$ is decomposable if and only if $n \leq 3$ (See also [**12**]). Størmer [**13**] gave a criterion to detect the decomposability of positive linear maps. However, there are maps which are not decomposable. Choi [**14**] gave the first example of an indecomposable map. In 1988, Tanahashi and Tomiyama [**15**] considered the extension of Choi map $\tau_{n,k}$ and proved their indecomposability (See [**16, 17, 18, 19, 3**] for variants of Choi map). In 1991, Osaka [**20**] showed the positivity of $\tau_{5,2}$ and provided a stronger indecomposable map. Finally, S. Yamagami [**21**] proved that all $\tau_{n,k}$ are positive.
There is another class of maps, called atomic maps. Osaka [**22**] showed that all $\tau_{n,1}$ are atomic maps for $n \geq 4$. Tanahashi and Tomiyama [**15**] provided examples of atomic maps. Robertson [**23**] also provided examples of maps which are atomic.

In 1992, S.J Cho, S.-H Kye, and S.G Lee [**24**] gave generalized Choi maps on $M_3(\mathbb{C})$ and studied the properties of this class of linear maps. They proved that if this linear map is 2-positive or 2-co-positive then it is decomposable [**24**, Corollary 4.3]. In 2016, Yu Yang, Denny H. Leung, Wai-Shing Tang [**25**] generalized this and proved that this is even true for every 2-positive map in $\mathcal{B}(M_3(\mathbb{C}))$. It is natural to ask about the existence of 2-positive and indecomposable map in higher dimensions.

Entanglement detection is one of the fundamental problems in quantum information theory. This is the problem of determining whether a given quantum state is separable or not. Schmidt decomposition plays an important role, in the sense that the problem of separability of pure quantum states is completely solved as the Schmidt number of pure quantum states is rather easy to obtain, but for a mixed quantum state, this approach fails. In the past, many efforts have been made in this direction [**26**]. First operational characterization of entanglement was obtained by Peres [**27**] (See also [**28, 29, 30, 31**]), which is based on the partial transpose, applying to one of the quantum subsystems. It says all separable states have positive partial transpose. States which have positive partial transpose are called PPT states. It gives a clear distinction between separable and entangled states. It provides a property which separable state satisfy but entangled states do not, that is, it gives a necessary condition for separability. Therefore, this criterion is more useful to detect entanglement.
It is known that for 2×2 and 2×3, systems PPT and separability are same. but for higher dimensions, this is not the case, and states which do not fall in this class are called positive partial transpose entangled states or PPTES in short. The first example of a PPT entangled state was given by Choi [**32**] for the case of 3×3 system.

Michal, Pawel and Ryszard Horodecki [**33**] gave a criterion, which is a necessary and sufficient condition to detect separability of quantum states and is called Horodecki criterion. It says that a state ρ is separable iff for any positive map $\Lambda : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$, the operator $(I \otimes \Lambda)(\rho)$ is positive. Clearly, the map Λ is not completely positive. Therefore, maps which are positive and not completely positive lead to entanglement detectors. They used positive maps to formulate a much more generalized criterion for separability. PPT criterion then becomes a special case, which uses transpose map as this map is positive and not completely positive.
Detection of PPT entangled states is one of the central problems in quantum information theory. It has been proved that indecomposable maps are used to detect PPT entangled states. Therefore, detecting the decomposability of linear maps is of significant interest in quantum information theory.

Motivation

Bhat [**1**] gave a characterization of linear maps which respect unitary conjugation. We generalize this idea and define a new class of linear maps on finite-dimensional matrix algebras, called them equivariant maps. We give a criterion to detect k-positivity of equivariant linear maps. Its implementation to detect k-positivity is rather easy. With this method in hand, it is of interest to study examples of equivariant maps and detect their k-positivity. The study of equivariant maps began with an example of $(1, 1)$-unitarily equivariant maps. We prove that this family of maps is an entanglement witness. It provides enough evidence to dig further about this class of linear maps. We then generalize this construction and define a class of $V(U)$-equivariant maps and classify its subclasses namely, the class of unitarily equivariant maps and (a, b)-unitarily equivariant maps. In the following section, we discuss the main results of this thesis and some interesting questions in quantum information theory which may be solved with the theory developed in this thesis.

Thesis structure

This thesis can be divided into two parts. In the first part (Chapter 1), we study the background needed to understand the construction and the characterization of a subclass of equivariant maps studied in part 2. The second part is devoted to the study of equivariant maps. This can be further divided into two sub-parts, Chapters 2 and 3. A part of the results in this thesis is obtained in a collaboration with my Supervisor Prof. Benoît Collins, Prof. Hiroyuki Osaka, and Ivan Bardet. Some results have already given rise to a joint published paper, a joint preprint, and I expect to convert the remaining results into a paper in the near future.

(i) In the first part (Chapter 2), we give a criterion to detect the k-positivity of equivariant linear maps on finite-dimensional matrix algebras. The structure of positive linear maps is difficult to understand. Therefore, it becomes important to have some simple techniques to detect k-positivity. Here, we give a criterion, which says k-positivity of linear maps depend on the positivity of a k-block sub-matrix of its Choi matrix, provided the map is equivariant.

Theorem 0.0.1. *Let $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a $V(U)$-equivariant linear map. Then, for $k \leq min\{n, m\}$, Φ is k-positive if and only if the block matrix $[\Phi(e_{ij})]_{i,j=1}^{k}$ is positive, where (e_{ij}) are the matrix units in $M_n(\mathbb{C})$.*

Subsequently, we study a parametric family of $(1, 1)$-unitarily equivariant maps and analyze the properties of positivity, 2-positivity (using aforementioned k-positivity criterion), and completely positivity. We prove that this family of parametric linear maps can act as an entanglement witness. This family of parametric equivariant maps is defined in Definition 2.2.1.

(ii) In the second part (Chapter 3), we study the characterization of certain subclasses of equivariant maps, which we call unitarily equivariant maps. We further study a subclass of unitarily equivariant maps, which we call (a, b)-unitarily equivariant maps. We give a characterization of this subclass of linear maps and also study their graphical representations. We give this characterization in the following theorem:

Theorem 0.0.2. *Let $a, b \in \mathbb{N}$ and $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})^{\otimes a} \otimes M_n(\mathbb{C})^{\otimes b}$. Then the following assertions are equivalent.*

(a) *Φ is (a, b)-unitarily equivariant.*

(b) *$[C_\Phi, \overline{U}^{\otimes a+1} \otimes U^{\otimes b}] = 0$ for all $U \in \mathcal{U}_n$, where C_Φ is the Choi matrix of Φ.*

(c) *There exists* $f : S_{k+1} \to \mathbb{C}$ *such that*

$$C_\Phi = \sum_{\pi \in S_{k+1}} f(\pi) \left(\theta_n^{\otimes a+1} \otimes i_n^{\otimes b}\right) [\sigma_{k+1}(\pi)] \ ,$$

where $k = a+b$, S_n *denotes the symmetric group on the set* $\{1, 2, ..., n\}$ *and* θ_n *is the transpose map on* $M_n(\mathbb{C})$.

We study the characterization of unitarily equivariant maps and deduce that if the map $U \mapsto V(U)$ becomes a unitary representation, then every unitarily equivariant map is "a corner of" (a, b)-unitarily equivariant map for some $a, b \in \mathbb{N}$. We summarize these results as follows:

Theorem 0.0.3. *Let* $\Phi : M_n(\mathbb{C}) \to M_N(\mathbb{C})$ *be a unitarily equivariant linear map. Let* $\mathcal{A}[\Im(\Phi)] = M_N(\mathbb{C})$ *and the map* $U \mapsto V(U)$ *in Equation* (3.1) *is continuous. Then there exists a* unitary representation $U \mapsto \tilde{V}(U)$ *of the special unitary group* $\mathcal{SU}_n$ *on* $\mathbb{C}^N$ *such that for all* $X \in M_n(\mathbb{C})$ *and* $U \in \mathcal{SU}_n$,

$$\Phi(UXU^*) = \tilde{V}(U)\Phi(X)\,\tilde{V}(U)^*$$

$\Im(\Phi)$ *denotes the image of* Φ.

The next corollary gives a relation between the two sub-classes of equivariant linear maps, which further characterize the sub-class of unitarily equivariant maps.

Corollary 0.0.4. *Let* $\Phi \in \mathcal{B}(M_n(\mathbb{C}), M_N(\mathbb{C}))$ *be a unitarily equivariant map. Assume that the map* $U \mapsto V(U)$ *is an* irreducible representation *of the unitary group. Then there exist* a, b *natural integers, a partial isometric map* $W : \mathbb{C}^N \to (\mathbb{C}^n)^{\otimes a} \otimes (\mathbb{C}^n)^{\otimes b}$ *and an* (a, b)*-unitarily equivariant map* Ψ *on* $M_n(\mathbb{C})$ *such that for all* $X \in M_n(\mathbb{C})$,

$$W\Phi(X)W^* = \Psi(X)\,.$$

In chapter 4, we study the applications of equivariant maps to detect entanglement. More precisely, we prove that for every entangled state, there exists a positive unitarily equivariant map with the image on the infinite dimensional Hilbert space, which detects it, as it is shown in the following theorem.

Theorem 0.0.5. *Let* $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *be a positive - but not completely positive - map. Then there exists a unitarily equivariant map* $\Phi : M_n(\mathbb{C}) \to \mathcal{B}(\mathcal{H})$ *for some Hilbert space* $\mathcal{H}$ *such that:*

(1) $$\mathcal{S}_m(\Phi) \subset \mathcal{S}_m(\phi)\,.$$

where

$$\mathcal{S}_m(\psi) = \{\rho \in \mathcal{S}_{m\times n}\,;\, (i_m \otimes \psi)(\rho) \geq 0\}\,.$$

It means that all entangled density matrices that are detected by ϕ *are also detected by the unitarily equivariant map* Φ.

We conjecture that this requirement is not necessary and that this Hilbert space can be taken to be finite-dimensional. It would be interesting to study (a, b)-unitarily equivariant map for different values of a and b. In this spirit, we study one variant of the values of $a = 2, b = 0$, that is, we present an example of $(2, 0)$-unitarily equivariant map and study its various properties (See Section 4.4).

Chapter 5 is devoted to the study of decomposability of equivariant maps. We give a necessary condition for the decomposability of equivariant linear maps. We prove that for a decomposable (a, b)-unitarily equivariant map, its two constituents (two CP maps) in its decomposition can be chosen to be (a, b)-unitarily equivariant. We also give a sufficient condition of decomposability of the parametric family of $(1, 1)$-unitarily equivariant maps $\Phi_{\alpha,\beta,3}$ studied in Chapter 2 and show graphically the region of decomposability.

We believe that this is the only region of decomposability. An answer to this question may help to solve a long-standing problem of 2-positivity and decomposability. This problem was raised in [**34**] by Terhal. In [**24**], Cho, Kye, and Lee studied the maps which are 2-positive and decomposable. In 2016, Yu Yang, Denny H. Leung, and Wai-Shing Tang [**25**] proved that this is even true for every 2-positive map in $\mathcal{B}(M_3(\mathbb{C}))$.

We expect that there exist values of parameters α and β for which the map $\Phi_{\alpha,\beta,3}$ is 2-positive and not decomposable, which may help to obtain a partial understanding of the structure of positive linear maps.

The study and the characterization of equivariant maps open many new questions, which are of great interest in the theory of operator algebra and may help to solve problems in quantum information theory.

List of Notations

$\mathcal{B}(\mathcal{H})$	space of all bounded linear operators on $\mathcal{H}$
$\mathcal{B}(\mathcal{H},\mathcal{K})$	space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$
$\mathbb{1}_n$	identity element in $M_n(\mathbb{C})$
i_n	identity map in $\mathcal{B}(M_n(\mathbb{C}))$
θ_n	transpose map $A \mapsto A^t$ in $\mathcal{B}(M_n(\mathbb{C}))$
$\mathcal{A}^+$	space of all positive elements of an algebra $\mathcal{A}$
$P_k[n,m]$	space of all k-positive linear maps in $\mathcal{B}(M_n(\mathbb{C}), M_m(\mathbb{C}))$
$P^k[n,m]$	space of all k-co-positive linear maps in $\mathcal{B}(M_n(\mathbb{C}), M_m(\mathbb{C}))$
CP	completely positive maps
CoCP	completely copositive maps
$D_{n,m}$	space of all decomposable maps in $\mathcal{B}(M_n(\mathbb{C}), M_m(\mathbb{C}))$
$\mathcal{S}_{\mathcal{H}}$	space of all density matrices on $\mathcal{H}$
$\mathcal{S}_{n\times m}$	space of all density matrices on $(\mathbb{C}^n \otimes \mathbb{C}^m)$
$\mathcal{A}^{'}$	commutant of an algebra $\mathcal{A}$

Contents

CHAPTER 1

Preliminaries

1.1. Operator algebra basics. This chapter serves as an introduction, where we present to the reader the necessary background and all the notions needed to understand our new results. We start with the definition of a C^*-algebra and its properties. We next study about positive elements of a C^*-algebra and their properties. Our aim is to study the problems of k-positive and k-copositive linear maps on a finite-dimensional C^*-algebra, In particular on $M_n(\mathbb{C})$.

$\mathcal{A}$ denotes the vector space over the field of complex numbers $\mathbb{C}$. For more detail on C^*-algebra theory and Operator theory, the reader is referred to [**35**] and [**36**].

Definition 1.1.1. (i) An algebra $\mathcal{A}$ is a vector space which is closed under the binary multiplication, given by the following map:

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{A} & \rightarrow & \mathcal{A} \\ (x, y) & \mapsto & x \cdot y \end{array}.$$

such that

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad \text{for every} \quad x, y, z \in \mathcal{A}. \tag{1.2}$$

(ii) A norm on an algebra $\mathcal{A}$ is a function

$$||.|| : \mathcal{A} \rightarrow \mathbb{R}_+$$

such that it satisfies the following properties:

(a) (Positivity): $||x|| = 0 \Leftrightarrow x = 0$.
(b) (Absolute scalable): $||cx|| = |c| \, ||x||$ for every $c \in \mathbb{F}$ and $x \in \mathcal{A}$.
(c) (Triangle inequality): $||x + y|| \leq ||x|| + ||y||$ for every $x, y \in \mathcal{A}$

Definition 1.1.2. A norm $||.||$ on $\mathcal{A}$ is said to be *sub-multiplicative* if

$$||xy|| \leq ||x|| \, ||y||$$

for every $x, y \in \mathcal{A}$

Definition 1.1.3. An algebra $\mathcal{A}$ equipped with sub-multiplicative norm is called a *normed algebra*. If $\mathcal{A}$ has a unit then it is called a unital normed algebra.

Definition 1.1.4. A subalgebra S of an algebra $\mathcal{A}$ is a vector subspace which is closed under the operation of multiplication.

Definition 1.1.5. A normed algebra $\mathcal{A}$ is *complete* if every cauchy sequence in $\mathcal{A}$ converges in $\mathcal{A}$, where the metric is the metric induced by the norm.

Definition 1.1.6. A complete normed algebra is called a *Banach algebra*. A complete unital normed algebra is called a *unital Banach algebra*.
A subalgebra of a normed algebra, is itself a normed algebra, with the induced norm.

Definition 1.1.7. An *involution* on an algebra $\mathcal{A}$ is a conjugate linear map $a \mapsto a^*$ such that following conditions hold:

(i) $a^{**} = a$
(ii) $(ab)^* = b^*a^*$

An algebra $\mathcal{A}$ equipped with the involution map $*$ is called a $*$-*algebra*. An algebra $\mathcal{A}$ is called a *Banach $*$-algebra* if it is a $*$-algebra and it is complete with respect to the sub-multiplicative norm.

Definition 1.1.8. A C^*-algebra $\mathcal{A}$ is a Banach $*$-algebra such that

$$||a^*a|| = ||a||^2$$

for every $a \in \mathcal{A}$.

1.1.1. ***Positive elements of a C^*-algebra.*** We give the definition of Positive elements of a C^*-algebra and study their properties.

Definition 1.1.9. An element b of a C^*-algebra $\mathcal{A}$ is called *positive* if b is hermitian and $\sigma_{\mathcal{A}}(b) \subseteq \mathbb{R}_+$, where $\sigma_{\mathcal{A}}(b)$ denotes the spectrum of $b \in \mathcal{A}$ in $\mathbb{C}$. We write $b \geq 0$ to mean b is positive and $\mathcal{A}^+$ to denote the space of all positive elements of $\mathcal{A}$.

Next we state the properties of positive elements of $\mathcal{A}$ and later focus on the convex cone of all positive elements of $M_n(\mathbb{C})$.

Theorem 1.1.10. *Let $\mathcal{A}$ be a C^*-algebra. Then,*

(i) *The sum of two positive elements of $\mathcal{A}$ is positive.*
(ii) *For an element $b \in \mathcal{A}^+$, there exists a unique element $a \in \mathcal{A}^+$ such that $b = a^2$.*
(iii) *An element b^*b is positive for any $b \in \mathcal{A}$. Moreover, the set $\mathcal{A}^+$ is equal to $\{b^*b \mid b \in \mathcal{A}\}$.*
(iv) *If $0 \leq a \leq b$, Then $||a|| \leq ||b||$.*
(v) *If $\mathcal{A}$ is unital and a, b are positive invertible elements of $\mathcal{A}$. Then $a \leq b \implies 0 \leq b^{-1} \leq a^{-1}$.*
(vi) *If $a, b \in \mathcal{A}^+$. Then, $a \leq b \implies a^{1/2} \leq b^{1/2}$.*

Definition 1.1.11. Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras. A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is called *positive* if it maps the convex cone of positive elements of $\mathcal{A}$ to the convex cone of positive elements of $\mathcal{B}$, that is, $\Phi(\mathcal{A}^+) \subseteq \mathcal{B}^+$. We write $\Phi \geq 0$ to denote positive linear maps.

Notation 1.1.12. $\mathbb{1}_n$ and i_n denote the identity element of $M_n(\mathbb{C})$ and the identity map on $M_n(\mathbb{C})$ respectively. $\text{Tr}(A)$ and A^t denote the canonical trace and the transpose of an element A in $M_n(\mathbb{C})$. Let θ_d denote the transpose map on $M_d(\mathbb{C})$ *i.e* $\theta_d(A) = A^t$ for $A \in M_d(\mathbb{C})$.
Let (e_{ij}) be the *matrix units* of $M_d(\mathbb{C})$, that is, $e_{ij}e_{kl} = \delta_{jk}e_{il}$ for $1 \leq i, j, k, l \leq d$, $\quad e_{ij}^* = e_{ji}$ and $\sum_{i=1}^{d} e_{ii} = \mathbb{1}_d$. For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denotes the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and $\mathcal{B}(\mathcal{H})$ denotes the space of all bounded linear operators on $\mathcal{H}$.

Positive linear maps are extremely important to study entanglement in Quantum Information Theory, yet the structure of positive linear maps is not completely understood. Given a linear map, it is difficult to say something about its positivity even in low dimensional matrix algebras. Hence, it becomes important to study the

various concepts of positivity.
For $k \in \mathbb{N}$, we give the definitions of k-positive and k-copositive linear maps.

Definition 1.1.13. A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is called *k-positive* (where k is a natural number) if $i_k \otimes \Phi : M_k(\mathcal{A}) \to M_k(\mathcal{B})$ is positive. It is called *k-copositive* if $\theta_k \otimes \Phi : M_k(\mathcal{A}) \to M_k(\mathcal{B})$ is positive. where θ_k is the transpose map on $M_k(\mathbb{C})$.

Note that the concept of positivity, 1-positivity and 1-copositive coincides. The following theorem is immediate from the definition of k-positive linear maps.

Theorem 1.1.14. *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a k-positive linear map. Then Φ is l-positive for every $l \leq k$, but converse is not necessarily true as it will be shown below.*

The simplest example is the transpose map $\theta_2 : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ given by $A \mapsto A^t$, Then, θ_2 is positive and not 2-positive.
Indeed,

$$\sum_{i,j=1}^{2} e_{ij} \otimes e_{ij} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

is positive.

$$(i_2 \otimes \theta_2)(\sum_{i,j=1}^{2} e_{ij} \otimes e_{ij}) = \sum_{i,j=1}^{2} e_{ij} \otimes e_{ji} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It can be easily seen that above matrix is not positive.
Hence, θ_2 is positive and not 2-positive. We will also show this by using graphical calculus in Section 1.2.

Definition 1.1.15. Φ is called *completely positive (CP)* if Φ is k-positive for every natural number k. It is called *completely copositive (CoCP)* if Φ is k-copositive for every natural number k.

Notation 1.1.16. We write $P_k[\mathcal{A}, \mathcal{B}]$ and $P^k[\mathcal{A}, \mathcal{B}]$ to denote the convex cones of k-positive and k-copositive linear maps from $\mathcal{A}$ to $\mathcal{B}$. $P_1[\mathcal{A}, \mathcal{B}]$ and $P^1[\mathcal{A}, \mathcal{B}]$ denote the convex cones of positive and copositive linear maps from $\mathcal{A}$ to $\mathcal{B}$.

Given a positive linear map, whether it is completely positive or not depends upon the underlying C^*-algebras. In the following, we state the cases when it is possible.

Theorem 1.1.17. [**37**] *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with $\mathcal{A}$ abelian. Then every positive map from $\mathcal{A}$ to $\mathcal{B}$ is completely positive, that is, $P_1[\mathcal{A}, \mathcal{B}] = P_\infty[\mathcal{A}, \mathcal{B}]$.*

Stinespring [**7**] gave the following characterization of completely positive maps which is an extension of GNS construction of states to completely positive maps [**35**]. The Stinespring dilation theorem for CP maps can be easily formulated for other classes of maps. Below we will state the Stinespring dilation theorem for CP and CoCP maps.

Theorem 1.1.18. *Let $\mathcal{A}$ be a unital C^*-algebra and $\Phi : \mathcal{A} \to B(\mathcal{H})$.*

(i) Φ *is completely positive if and only if there exist a Hilbert space* $\mathcal{K}$, *a bounded linear operator* $V : \mathcal{H} \to \mathcal{K}$ *and a* $*$*-homomorphism* $\pi : \mathcal{A} \to B(\mathcal{K})$ *such that*

$$\Phi(a) = V^*\pi(a)V \quad \text{for all} \quad a \in \mathcal{A}.$$

Furthermore, $||V||^2 \leq ||\Phi(1)||$.

(ii) Φ *is completely copositive if and only if there exist a Hilbert space* $\mathcal{K}$, *a bounded linear operator* $V : \mathcal{H} \to \mathcal{K}$ *and an* $*$*-anti-homomorphism* $\pi : \mathcal{A} \to B(\mathcal{K})$ *such that*

$$\Phi(a) = V^*\pi(a)V \quad \text{for all} \quad a \in \mathcal{A}.$$

In the study of completely positive maps, the Choi matrix is very important. We define the Choi matrix of a linear map and study the relation between linear maps and their Choi matrices.

Note 1.1.19. Very little is known about the structure of positive linear maps, even on low dimensional matrix algebras. In this thesis, we restrict our attention to linear maps on finite-dimensional matrix algebras.

Notation 1.1.20. We give the notation used for the algebra $M_n(\mathbb{C})$. $P_k[n,m]$ and $P^k[n,m]$ denotes the space of all k-positive and k-co-positive linear maps in $\mathcal{B}(M_n(\mathbb{C}), M_m(\mathbb{C}))$.

Definition 1.1.21. Let $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a linear map. Then *the Choi matrix* of Φ is defined as:

$$C_\Phi := \sum_{i,j=1}^{n} e_{ij} \otimes \Phi(e_{ij}).$$

With the help of Choi matrix, we can define a correspondence between the spaces $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$ and $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$.
Define $\Sigma : B(M_n(\mathbb{C}), M_m(\mathbb{C})) \to M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ given by

$$\Phi \mapsto C_\Phi.$$

This map is obviously linear and bijective. This isomorphism is known as the Choi-Jamiolkowski isomorphism [**8, 38**].

Notation 1.1.22. Let $n \wedge m$ and $n \vee m$ denote the minimum and maximum of n and m respectively.

Choi [**8**] gave a criterion to characterize all completely positive linear maps defined on finite-dimensional matrix algebras, given by the following theorem.

Theorem 1.1.23. *Let* $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ *be a linear map. Then following are equivalent:*

(i) Φ *is completely positive.*
(ii) Φ *is* $n \wedge m$*- positive.*
(iii) C_Φ *is positive.*

Theorem 1.1.23 can easily be formulated for completely copositive maps. Choi's condition of completely copositive maps is given by the following theorem.

Theorem 1.1.24. *Let* $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ *be a linear map. Then following are equivalent:*

(i) Φ *is completely copositive.*
(ii) Φ *is* $n \wedge m$*- Copositive.*
(iii) $C'_{\Phi} = \sum_{i,j=1}^{m} e_{ij} \otimes \Phi(e_{ji})$ *is positive.*

Note 1.1.25. *Following inequality can be easily seen from Theorems 1.1.23 and 1.1.24.*

$$P_1[n,m] \supset P_2[n,m] \supset P_3[n,m] ... \supset CP[n,m] \tag{1.3}$$

$$P^1[n,m] \supset P^2[n,m] \supset P^3[n,m] ... \supset CoCP[n,m] \tag{1.4}$$

From Equations (1.3) and (1.4), It can be easily seen that completely positivity and completely copositivity of linear maps saturates at $n \wedge m$, in the case of linear maps defined on finite-dimensional matrix algebras. To distinguish n-positivity from $(n-1)$-positivity of linear maps, Choi [**39**] gave the first example of a linear map on $M_n(\mathbb{C})$ which is $(n-1)$-positive but not n-positive. If we take $n = 3$, then the significance of Choi's map stems from the fact that it is not completely positive even though it is 2-positive unlike the case of transpose map on $M_n(\mathbb{C})$ which is not completely positive because it is not even 2-positive. This map is defined as follows:

Definition 1.1.26.

$$\begin{array}{rccc} \Phi: & M_n(\mathbb{C}) & \to & M_n(\mathbb{C}) \\ & A & \mapsto & \{(n-1)\mathrm{Tr}(A)\}\mathbb{1}_n - A \end{array}.$$

In 1983, Takasaki and Tomiyama [**9**] gave a method to construct any number of linear maps on finite-dimensional matrix algebras $M_n(\mathbb{C})$, which are $(k-1)$-positive and not k-positive. They consider the segment between the map Φ (Definition 1.1.26) and the transpose map θ_n on $M_n(\mathbb{C})$ [**40**] and give the structure of linear maps on the segment. They proved the following:

Theorem 1.1.27. *For a nonnegative number* α *with* $0 \leq \alpha \leq 1$, *the map*

$$\begin{array}{rccc} \Psi: & M_n(\mathbb{C}) & \to & M_n(\mathbb{C}) \\ & A & \mapsto & [(n-1)\mathrm{Tr}(A)\mathbb{1}_n - A] + (1-\alpha)A^t \end{array}$$

Then, Ψ *is*

(i) $(n-1)$*-positive but not* n*-positive when* $\frac{1}{2} < \alpha \leq 1$
(ii) *completely positive when* $\frac{1}{n} \leq \alpha \leq \frac{1}{2}$
(iii) *only positive when* $0 \leq \alpha < \frac{1}{n}$

In 1985, Tomiyama [**10**] generalized the above notion and studied the structure of extended segment between two linear maps. In particular, he proved the following result:

Theorem 1.1.28. *The linear map*

$$\begin{array}{rccc} \tau: & M_n(\mathbb{C}) & \to & M_n(\mathbb{C}) \\ & A & \mapsto & \frac{\lambda}{n}\mathrm{Tr}(A)\mathbb{1}_n + (1-\lambda)A \end{array}.$$

where $\lambda \in \mathbb{R}$ *and* $\mathrm{Tr}(A)$ *is the usual trace (not normalized trace) on* $M_n(\mathbb{C})$, *is* k*-positive for any* $1 \leq k \leq n$ *if and only if* $0 \leq \lambda \leq 1 + \frac{1}{nk-1}$.

He also proved an important result which is a necessary and sufficient condition for a linear map to be k-positive.

Theorem 1.1.29. *A linear map τ in $M_n(\mathbb{C})$ is k-positive for $1 \le k \le n$ if(and only if)*

$$[\tau(f_{ij})]_{i,j=1}^{k} \ge 0$$

whenever $f_{ij} \in M_n(\mathbb{C})$ for $1 \le i, j \le k$ and satisfy the following conditions $rank(f_{ij}) = 1, \quad f_{ij} = f_{ji}^* \quad$ *and* $\quad f_{ip}f_{qj} = \delta_{pq}f_{ij} \quad$ *for* $\quad 1 \le i, j, p, q \le k.$

Note 1.1.30. In [**41**], we gave a criterion to detect k-positivity of equivariant linear maps which makes it to important to study further this class of linear maps. We will continue this discussion in Chapters 2 and 3.

Dual cone relation. Let V and W be two finite-dimensional normed real spaces and $\langle ., . \rangle$ is a bilinear form defined on $V \times W$. Then for subsets $A \subseteq V$ and $B \subseteq W$. The dual of A and B is defined as follows:

$$A^\circ := \{w \in W : \langle v, w \rangle \ge 0 \quad \forall \quad v \in A\} \tag{1.5}$$

$$B^\circ := \{v \in V : \langle v, w \rangle \ge 0 \quad \forall \quad w \in B\} \tag{1.6}$$

For $A \subseteq V$ and $B \subseteq W$, the pair (A, B) is called a dual pair if $A^\circ = B$. In this case, B is called the dual of A and A is called the pre-dual of B. $A^{\circ\circ}$ is the closed convex cone generated by A.
In the following, we study a duality between the spaces $\mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ of all bounded linear maps from C^*-algebra $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ and projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{T}(H)$, where $\mathcal{T}(H)$ denotes the space of all trace class operatros in $\mathcal{B}(\mathcal{H})$. We recall the definition of a trace-class operator.

Definition 1.1.31. An operator $A \in \mathcal{B}(\mathcal{H})$ is called trace-class operator if its trace-class norm $||A||_1 := \sum_{x \in E} \langle |A|(x), x \rangle < \infty$, where E is an orthonormal basis for $\mathcal{H}$.

The duality between $\mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))$ and $\mathcal{A}\hat{\otimes}\mathcal{T}(H)$ is given by the following map:

$$\langle x \otimes y, \phi \rangle = \mathrm{Tr}(\phi(x)y^t) \quad \text{for} \quad x \in \mathcal{A}, \quad y \in \mathcal{T}(H) \quad \text{and} \quad \phi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H})) \tag{1.7}$$

This duality has many applications in the theory of states and linear maps. We restrict ourselves to the case of finite-dimensional matrix algebras. With this assumption, it gives the duality between the spaces $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ and $\mathcal{B}(M_n(\mathbb{C}), M_m(\mathbb{C}))$. The bilinear map is given by the following:

$$\begin{array}{cccc} \langle ., . \rangle : & M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \times \mathcal{B}(M_n(\mathbb{C}), M_m(\mathbb{C})) & \to & \mathbb{C} \\ & \langle A, \phi \rangle & \mapsto & \sum_{i,j=1}^{n} \langle a_{ij}, \phi(e_{ij}) \rangle . \end{array} \tag{1.8}$$

where $A = \sum_{i,j=1}^{n} e_{ij} \otimes a_{ij} \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ and $\phi \in \mathcal{B}(M_n(\mathbb{C}), M_m(\mathbb{C}))$ and the bilinear form on the right hand side is given by $\langle A, B \rangle = \mathrm{Tr}(BA^t)$.

From Equation (1.8), we have:

$$\begin{aligned}
\langle A, \phi \rangle &= \sum_{i,j=1}^{n} \langle e_{ij} \otimes a_{ij}, \phi \rangle \\
&= \sum_{i,j=1}^{n} \mathrm{Tr}(\phi(e_{ij}) a_{ij}^t) \\
&= \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \mathrm{Tr}(e_{lk} e_{ji} \otimes \phi(e_{lk}) a_{ij}^t) \\
&= \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \mathrm{Tr}(e_{lk} \otimes \phi(e_{lk})(e_{ji} \otimes a_{ij}^t)) \\
&= \mathrm{Tr}(C_\phi A^t) = \mathrm{Tr}(A^t C_\phi)
\end{aligned}$$

Where C_ϕ is the Choi matrix of ϕ defined in Definition 1.1.21.

Definition 1.1.32. Every vector $z \in (\mathbb{C}^n \otimes \mathbb{C}^m)$ may be written in a unique way as $z = \sum_{i=1}^{n} e_i \otimes z_i$ with $z_1, z_2, ..., z_n \in \mathbb{C}^m$ where $\{e_i : 1 \leq i \leq n\}$ denotes the canonical orthonormal basis of $\mathbb{C}^n$. We say that z is an s-simple vector if $\dim(span\{z_1, z_2, ..., z_n\}) \leq s$.

Define the convex cones $V_k(n, m)$ and $V^k(n, m)$ of $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ as follows:

$$V_k(n, m) := \{zz^* : z \text{ is a } k\text{-simple vector in } (\mathbb{C}^n \otimes \mathbb{C}^m)\}^{\circ\circ}$$

$$V^k(n, m) := \{(zz^*)^{\Gamma_n} : z \text{ is a } k\text{-simple vector in } (\mathbb{C}^n \otimes \mathbb{C}^m)\}^{\circ\circ},$$

Where $(zz^*)^{\Gamma_n}$ is called the partial transpose of zz^*. It acts as a transpose map on $M_n(\mathbb{C})$ (For more information on partial transpose, the reader is referred to Definition 1.1.44).
With above notation, we study the relation between s-positive (resp.s-copositive) linear maps in $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$ and the cones $V_k(n, m)$(resp. $V^k(n, m)$). The relation between two convex cones is given by the following theorem.

Theorem 1.1.33. [**42**, Theorem 2.1] *For a linear map $\phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$, we have the following:*

(i) *The map ϕ is s-positive if and only if $\langle zz^*, \phi \rangle \geq 0$ for each s-simple vector $z \in \mathbb{C}^n \otimes \mathbb{C}^m$.*
(ii) *The map ϕ is s-copositive if and only if $\langle (zz^*)^{\Gamma_n}, \phi \rangle \geq 0$ for each s-simple vector $z \in \mathbb{C}^n \otimes \mathbb{C}^m$.*

Let ϕ be s-positive and $z \in V_s[m, n]$, that is, $z = \sum_i \alpha_i z_i z_i^*$, $0 \leq \alpha_i \leq 1, \sum_i \alpha_i = 1$ and z_i is s-simple for each i.

$$\begin{aligned}
\langle z, \phi \rangle &= \left\langle \sum_i \alpha_i z_i z_i^*, \phi \right\rangle \\
&= \sum_i \alpha_i \langle z_i z_i^*, \phi \rangle
\end{aligned}$$

By Theorem 1.1.33,

$$\langle z_i z_i^*, \phi \rangle \geq 0 \text{ for each } i.$$

Therefore, $\langle z, \Phi\rangle \geq 0$. We have

$$V_s[m,n] \subseteq P_s^\circ[m,n] \tag{1.9}$$

Conversely, Let $\phi \in V_s^\circ[m,n]$. for each $z \in V_s[m,n]$

$$\langle z, \phi\rangle \geq 0$$

where $z = \sum_i \alpha_i z_i z_i^*$ and $0 \leq \alpha_i \leq 1, \sum_i \alpha_i = 1$ for each i.
In particular, $\langle zz^*, \phi\rangle \geq 0$ for each s-simple vector z. Therefore, by Theorem 1.1.33, ϕ is s-positive.

$$V_s^\circ[m,n] \subseteq P_s[m,n] \tag{1.10}$$

$$\implies P_s^\circ[m,n] \subseteq V_s^{\circ\circ}[m,n] = V_s[m,n] \tag{1.11}$$

Therefore, we have

$$P_s^\circ[m,n] \subseteq V_s[m,n] \tag{1.12}$$

By Equations (1.9) and (1.12),

$$P_s^\circ[m,n] = V_s[m,n]$$

By a similar calculation, we can prove that

$$P^s[n,m] = V^{s\circ}(n,m) \tag{1.13}$$

Hence, $(P_s[n,m], V_s(n,m))$ (resp. $(P_s[n,m], V_s(n,m))$ are the dual pair under the aforementioned bilinear mapping. This is called the dual cone relation between positive linear maps and quantum states, in the sense that they determine each other with respect to the given bilinear mapping. This duality can be explained by the following diagram:

$$\begin{array}{lllll} V_1 \subset & V_2 \subset & \cdots \subset & V_{n\wedge m} & = (M_n \otimes M_m)^+ \\ \updownarrow & \updownarrow & & \updownarrow & \\ P_1 \supset & P_2 \supset & \cdots \supset & P_{n\wedge m} & \cong (M_n \otimes M_m)^+ \end{array}$$

FIGURE 1.1. Duality between positive maps in $\mathcal{B}(M_n(\mathbb{C}), M_m(\mathbb{C}))$ and states in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$

Decomposability. We give the definition of decomposability and we apply the dual cone relation to the class of decomposable linear maps and deduce further results. In the following, we study the dual cone of $(P_s + P^t)$ for some $1 \leq s,t \leq n \wedge m$. Following definitions are inspired by the dual cone relation.

Definition 1.1.34. Let p, q be two natural numbers.

(i) A linear map $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is called (p,q)-*decomposable* if it can be written as a sum of a p-positive and a q-copositive linear map. Otherwise, it is called (p,q)-indecomposable map.
(ii) Φ is called *decomposable* if it can be written as a sum of a completely positive and a completely copositive linear map. Otherwise, it is called an indecomposable map.
(iii) There is a special class of maps which are atomic maps. A linear map is called atomic if it is not $(2,2)$-*decomposable* that is if it can not be written as a sum of 2-positive and 2-Copositive linear maps.

$$\begin{array}{llll} V_1 \subset V_{n\wedge m} \cap V^{n\wedge m} & \subset V_{n\wedge m} = (M_n \otimes M_m)^+ \\ \updownarrow \quad\quad \updownarrow & \quad \updownarrow \\ P_1 \supset \quad D_{n,m} & \supset P_{n\wedge m} \cong (M_n \otimes M_m)^+ \end{array}$$

FIGURE 1.2. Duality between cone of CP maps and states in $(M_n \otimes M_m)^+$

$$\begin{array}{llll} V^1 \subset V_{n\wedge m} \cap V^{n\wedge m} & \subset V^{n\wedge m} = (M_n \otimes M_m)^+ \\ \updownarrow \quad\quad \updownarrow & \quad \updownarrow \\ P^1 \supset \quad D_{n,m} & \supset P^{n\wedge m} \cong (M_n \otimes M_m)^+ \end{array}$$

FIGURE 1.3. Duality between cone of CoCP maps and states in $(M_n \otimes M_m)^+$

Let $D_{n,m}$ denote the class of decomposable linear maps in $\mathcal{B}(M_n(\mathbb{C}), M_m(\mathbb{C}))$. By using the characterization of completely positive and completely copositive maps (By Theorems, 1.1.23 and 1.1.24) this class can be written as $P_{n\wedge m} + P^{n\wedge m}$. By Equation (1.8), $(P_{n\wedge m} + P^{n\wedge m}, V_{n\wedge m} \cap V^{n\wedge m})$ is a dual pair. By definition,

$$V_{n\wedge m} \cap V^{n\wedge m} = \{A \in (M_n \otimes M_m)^+ : A^\Gamma \geq 0\}$$

We have:

$$\phi \in D_{n,m} \iff \langle A, \phi \rangle \geq 0 \quad \forall \quad A \in V_{n\wedge m} \cap V^{n\wedge m} \tag{1.14}$$

$$A \in V_{n\wedge m} \cap V^{n\wedge m} \iff \langle A, \phi \rangle \geq 0 \quad \forall \quad \phi \in D_{n,m} \tag{1.15}$$

Elements of $V_{n\wedge m} \cap V^{n\wedge m}$ are called positive partial transpose or PPT elements.

we have the following diagrams from dual cone relationship between the elements of $V_{n\wedge m}$ (resp. $V^{n\wedge m}$) and the class of completely positive maps (resp. completely copositive maps).

For $s = 1, n\wedge m$, we get a nice inclusion relation between the states in $M_n \otimes M_m$ and completely positive linear maps in $\mathcal{B}(M_n(\mathbb{C}), M_m(\mathbb{C}))$. But we do not have any inclusion for other values of $1 < s < n \wedge m$. To understand the structure of other classes of positive linear maps, we ask the question whether these classes of linear maps are included in the class of decomposable linear maps.

It is clear from the dual cone relationship diagram for decomposable maps, that there is a strict inclusion between the cone of positive linear maps and decomposable maps that is, $P_1 \supset D_{n,m}$. Hence the inclusion is strict. It is intriguing to inquire about the converse inclusion that is, does there exist a linear map $\phi \in P_1$ such that $\phi \notin D_{n,m}$.

Woronowicz [**11**] showed that every positive linear map $\Phi : M_2(\mathbb{C}) \to M_n(\mathbb{C})$ is decomposable if and only if $n \leq 3$.

Theorem 1.1.35. [**11**] *Let $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a positive linear map. Then Φ is decomposable if $nm \leq 6$.*

The first example of an indecomposable map $\Phi : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ was given by Choi [**14**] and he reproduced the example in [**43**].

Example 1.1.36. *Let α be a natural number and*

$$\Phi_\alpha : M_3(\mathbb{C}) \to M_3(\mathbb{C})$$

given by

$$\Phi_\alpha(A) = \begin{bmatrix} a_{11} + \alpha a_{33} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} + \alpha a_{11} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} + \alpha a_{22} \end{bmatrix}$$

where $A = (a_{ij})$ for $1 \leq i, j \leq 3$.

The linear map Φ_1 is called the Choi map and it is indecomposable.

In 1982, Størmer [**13**] gave a significant result, which is a necessary and sufficient condition for a linear map to be decomposable that is (n, n)-decomposable and also proved the indecomposability of Φ_α by using this result.

Theorem 1.1.37. *Let $\mathcal{A}$ be a C^*-algebra and Φ be a linear map from $\mathcal{A}$ to $B(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space. Then, Φ is decomposable if and only if for all $n \in \mathbb{N}$ whenever (x_{ij}) and (x_{ji}) belong to $M_n(\mathcal{A})^+$, then $(\Phi(x_{ij})) \in M_n(B(\mathcal{H}))^+$.*

Later in 1988, Tanahashi and Tomiyama [**15**] constructed a series of linear maps. It can be considered as the extension of Choi's map [**14**]. They studied the positivity and decomposable properties of these maps.

Let ϵ be the norm-one projection of $M_n(\mathbb{C})$ onto the diagonal part and S be the shift unitary in $M_n(\mathbb{C})$ such that $S = [\delta_{i,j+1}]$.
With above setting in mind, Choi's map can be written as

$$\Phi : M_3(\mathbb{C}) \to M_3(\mathbb{C})$$

given by

$$\Phi(A) = 2\epsilon(A) + \epsilon(S^* AS) - A$$

for $A \in M_3(\mathbb{C})$

Generalizing the above methodology and technique, Tanahashi and Tomiyama[**15**] introduced the following extension of Choi maps to the matrix algebra $M_n(\mathbb{C})$ for $n \geq 3$:

$$\tau_{n,k} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

given by

$$\tau_{n,k}(A) = (n-k)\epsilon(A) + \sum_{i=1}^{k} \epsilon(S^* AS) - A$$

where S is the shift unitary in $M_n(\mathbb{C})$, that is, $S = [\delta_{i,j+1}]$. $A \in M_n(\mathbb{C})$ and $1 \leq k \leq n-1$.

The positivity of $\tau_{n,1}$ and $\tau_{n,n-2}$ was proved by Tanahashi and Tomiyama and Ando. Later in 1991, Osaka [**20**] showed the positivity of $\tau_{5,2}$. Finally, S. Yamagami [**21**] proved that all $\tau_{n,k}$ are positive.
Moreover, Tanahashi and Tomiyama proved that $\tau_{n,0}$ is completely positive and

$\tau_{n,n-1}$ is completely copositive. $\tau_{n,1}$ can not be decomposed into the sum of n-positive and 2-copositive map. They also proved that the Choi's map $\tau_{3,1}$ is atomic that is it can not be decomposed into the sum of 2-positive and 2-copositive map. Osaka [**20**] showed that for $n \geq 4$, $\tau_{n,n-2}$ can not be decomposed into a sum of 3-positive and 3-copositive map. Osaka [**22**] showed that all $\tau_{n,1}$ are atomic maps for $n \geq 4$.

S.J Cho, S.-H Kye, and S.G Lee [**24**] generalized the idea of Choi maps and constructed a class of parametric linear maps (also called generalized Choi maps) on $M_3(\mathbb{C})$ and study the question of positivity, completely positivity and decomposability etc. The family of parametric linear maps is given as follows:

Definition 1.1.38. Let a, b, c be three nonnegative real numbers. Define

$$\Phi[a,b,c] : M_3(\mathbb{C}) \to M_3(\mathbb{C})$$

given by
$\Phi[a,b,c](A)$ is given by the following matrix:

$$\begin{bmatrix} (a-1)a_{11} + ba_{22} + ca_{33} & -a_{12} & -a_{13} \\ -a_{21} & (a-1)a_{22} + ba_{33} + ca_{11} & -a_{23} \\ -a_{31} & -a_{32} & (a-1)a_{33} + ba_{11} + ca_{22} \end{bmatrix}$$

where $A = (a_{ij})$ for $1 \leq i, j \leq 3$.

Theorem 1.1.39. [**24**] *The linear map* $\Phi[a,b,c]$ *is*

(i) *positive if and only if the following three conditions hold*

$$\begin{aligned} &a \geq 1 \\ &a + b + c \geq 3 \\ &bc \geq (2-a)^2 \quad \text{if} \quad 1 \leq a \leq 2. \end{aligned}$$

(ii) *completely positive if and only if* $a \geq 3$.

(iii) *decomposable if and only if the following two conditions hold*

$$\begin{aligned} &a \geq 1 \\ &bc \geq \left(\frac{3-a}{2}\right)^2 \quad \text{if} \quad 1 \leq a \leq 3. \end{aligned}$$

(iv) 2-*positive if and only if*

$$\begin{aligned} &a \geq 3 \\ &\text{or} \\ &2 \leq a < 3 \text{ and } bc = (3-a)(b+c) > 0 \end{aligned}$$

(v) *is completely copositive if and only if it is* 2-*copositive if and only if the following two conditions hold*

$$\begin{aligned} &a \geq 1 \\ &bc \geq 1 \end{aligned}$$

Given a positive linear map, we try to study the question whether it is (p,q)-decomposable for some values of p and q, or if a linear map is 2-positive then whether it is $(3,3)$-decomposable, that is, given a linear map we enquire about whether it can be written as a sum of higher order of positive and copositive linear map than the given order of positivity.

S.J Cho, S.-H Kye and S.G Lee [**24**, Corollary 4.3] proved that if $\Phi[a,b,c]$ is 2-positive or 2-copositive then it is decomposable. Later in 2016, Yang, Leung and Tang [**25**] generalizes this concept and proved that this is even true for every 2-positive or 2-copositive linear map in $M_3(\mathbb{C})$.

Theorem 1.1.40. *Every 2-positive or 2-copositive map Φ in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ is decomposable.*

Mathematics of Quantum Information Theory. Every quantum system is associated to a Hilbert space $\mathcal{H}$, and it contains all the states of the system, also known as state space. First, we introduce the bra-ket notation, which is standard in quantum information theory. We will consider the state space as an n-dimensional complex Hilbert space $\mathbb{C}^n$.

A ket vector is a column vector in a Hilbert space denoted as $|\psi\rangle$. A bra vector, written as $\langle\phi|$ is the row vector obtained by taking the complex conjugation and transpose of the ket vector $|\psi\rangle$, that is,

$$\langle\phi| = (|\phi\rangle)^*$$

The bra vector can also be realised as the vector in dual space $\mathcal{H}^*$, that is, $\langle\phi| \in B(\mathcal{H}, \mathbb{C})$ given by

$$|x\rangle \mapsto \langle\phi|x\rangle_{\mathcal{H}} \quad \text{for every} \quad x \in \mathcal{H}$$

With the above bra-ket notation, the inner product $\langle\psi|\phi\rangle$ corresponds to $\langle\psi| \quad |\phi\rangle$ the multiplication of bra and ket vector respectively.
Bra-ket notation was introduced by Paul Dirac in 1939 [**44**], also known as the Dirac notation.

Definition 1.1.41. Let $\mathcal{H}$ be a finite-dimensional Hilbert space.

(i) A density operator ρ on $\mathcal{H}$ is defined as follows:

$$\{\rho \in \mathcal{B}(\mathcal{H})^+ \mid \mathrm{Tr}(\rho) = 1\}$$

(ii) A pure state of a system is a column vector, written as ket $|\psi\rangle \in \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space associated to the quantum system. The density operator corresponding to this state is $|\psi\rangle\langle\psi|$.

(iii) A mixed state ρ is given by a collection of pure states $\{p_i, |\psi_i\rangle\}$, where i is an index. Generally, a mixed state is represented by its density operator. A density operator ρ describing a mixed state takes the following form:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

where $\sum_i p_i = 1$ and $p_i \geq 0$ for all i.

$\mathcal{S}(\mathcal{H})$ denotes the set of all density operators of $\mathcal{H}$ and the set of all pure states is denoted by $P(\mathcal{H})$. A state $\rho \in \mathcal{S}(\mathcal{H})$ is called pure if $\rho = xx^*$ for some unit vector $x \in \mathcal{H}$, more precisely, it is a rank-one projection. All other states are called mixed quantum states. Before proceeding, we give few remarks about the set $\mathcal{S}(\mathcal{H})$.

(i) The set $\mathcal{S}(\mathcal{H})$ is convex.
(ii) The extreme points of $\mathcal{S}(\mathcal{H})$ are pure states.
(iii) The set $\mathcal{S}(\mathcal{H})$ is compact.

The state of a two dimensional Hilbert space is called quantum bit or qubit, in short. Ket vectors $|e_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|e_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are orthonormal basis vectors of this space. Therefore state of this space takes the following form

$$\rho = a|e_1\rangle + b|e_2\rangle \quad \text{such that} \quad a, b \in \mathbb{C} \quad \text{and} \quad |a|^2 + |b|^2 = 1$$

Entanglement. Entanglement is one of the striking phenomena in quantum information theory, which was first observed by Einstein, Podolsky, and Rosen [**45**]. Einstein referred it as spooky action at a distance [**46**]. Quantum entanglement occurs when pairs or groups of particles interact in such a way that the state of each particle cannot be described individually, and instead the state must be described for the entire system.

The intrigue of entanglement lies in the fact that two quantum particles are intimately correlated that any measurement on one particle instantly affect the other particle, even when they are separated by a long distance. Quantum entanglement plays an essential role in quantum technologies such as quantum teleportation, quantum cryptography and, superdense coding.

Before giving the mathematical definition of entanglement, we give a brief introduction of the composite quantum system.

Composite system 1.1.42. A composite quantum system is made up of two or more quantum systems. If it involves two quantum systems, then it is called a bipartite system otherwise it is called a multipartite system. The state space of the composite quantum system is the tensor product of the component systems and the joint state of the composite quantum system is the tensor product of the states of the component systems, that is, if $\rho_i \in \mathcal{S}(\mathcal{H}_i)$ for $1 \leq i \leq n$ are the states of systems, then $\rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n)$. This state is called the product state. This is the case when $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_n$ are mutually independent.

Definition 1.1.43. Let $\rho_i \in \mathcal{S}(\mathcal{H}_i)$ for $1 \leq i \leq n$ be the states of n-quantum systems.

(i) The state $\rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n)$ is called a product state of the composite quantum system.

(ii) A state ρ is separable if it is a convex combination of product state of $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n$. Otherwise, it is called entangled state. A composite system that has an entangled state is called an entangled system.

In the following, we deal with the case of a *bipartite* quantum system. The state of such a quantum system is called bipartite state. Before proceeding, we give two operations on the bipartite system $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$, partial trace and partial transpose.

Definition 1.1.44. Let ρ be the joint state of the bipartite system $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ and $a \otimes b$ denote an arbitrary element of $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$. Partial trace is defined as follows:

(i) The partial trace over systems $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ is defined by the following maps respectively.
$\mathrm{Tr}_\mathcal{A} : \mathcal{B}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}) \to \mathcal{B}(\mathcal{H}_\mathcal{B})$ given by $a \otimes b \mapsto \mathrm{Tr}(a)b$ and extend it linearly.
$\mathrm{Tr}_\mathcal{B} : \mathcal{B}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}) \to \mathcal{B}(\mathcal{H}_\mathcal{A})$ given by $a \otimes b \mapsto \mathrm{Tr}(b)a$ and extend it linearly.
Operators $\rho_\mathcal{A} = \mathrm{Tr}_\mathcal{B}(\rho^{\mathcal{AB}})$ and $\rho_\mathcal{B} = \mathrm{Tr}_\mathcal{A}(\rho^{\mathcal{AB}})$ are called reduced density operators of systems $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$.

(ii) The partial transpose over systems $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ is defined as follows:
$\Gamma_\mathcal{A} : \mathcal{B}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{B}(\mathcal{H}_\mathcal{B}) \to \mathcal{B}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{B}(\mathcal{H}_\mathcal{B})$ defined by $a \otimes b \mapsto a^t \otimes b$ and extend it linearly.
$\Gamma_\mathcal{B} : \mathcal{B}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{B}(\mathcal{H}_\mathcal{B}) \to \mathcal{B}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{B}(\mathcal{H}_\mathcal{B})$ defined by $a \otimes b \mapsto a \otimes b^t$ and extend it linearly.

States which remain positive under partial transpose of $\mathcal{H}_\mathcal{A}$(resp. $\mathcal{H}_\mathcal{B}$) are called positive partial transpose or PPT, in short.

For simplicity, we shall write ρ^{Γ_A} and ρ^{Γ_B} to denote $\Gamma_\mathcal{A}(\rho)$ and $\Gamma_\mathcal{B}(\rho)$ for $\rho \in \mathcal{B}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{B}(\mathcal{H}_\mathcal{B})$ respectively.
Let $\mathcal{S}_n$ denote the set of density matrices on $\mathbb{C}^n$. A density matrix $\rho \in \mathcal{S}_{n\times m}$ is called separable if it can be written as a convex hull of product density matrices, that is,

$$\rho = \sum_i \lambda_i \rho_A^i \otimes \rho_B^i \,;\, \rho_A^i \in \mathcal{S}_n, \rho_B^i \in \mathcal{S}_m \,,\, \lambda_i \geq 0, \sum_i \lambda_i = 1$$

The set of all separable density matrices on $\mathbb{C}^n \otimes \mathbb{C}^m$ is denoted by $\mathrm{SEP}(n,m)$. The elements of $\mathcal{S}_{n\times m} \setminus \mathrm{SEP}(n,m)$ are called entangled.

Several criteria for separability of quantum states have been given. First criterion of separability is given by A. Peres [**27**] which shows that positive partial transpose (PPT) is a necessary condition for separability.

First operational characterization for separability of mixed states was given by [**33**], which is given by the following theorem:

Theorem 1.1.45. [**33**, Theorem 2] *Let ρ act on a Hilbert space $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$. Then ρ is separable iff for any positive map $\Lambda : \mathcal{B}(\mathcal{H}_\mathcal{B}) \to \mathcal{B}(\mathcal{H}_\mathcal{A})$, the operator $(I \otimes \Lambda)\rho$ is positive.*

Note that $(I \otimes \Lambda)\rho$ is positive for every completely positive linear map Λ, irrespective of the separability of the state ρ. Thus, completely positive maps do not feel inseparability. Hence, positive and not completely positive maps lead to entanglement detectors.
PPT criterion is only a necessary condition for separability but not sufficient, which assures the existence of PPT entangled states, PPTES. However, it is sufficient in some cases and we formulate it in the following theorem.

Proposition 1.1.46. [**33**, Theorem 3] *A state ρ acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^2 \otimes \mathbb{C}^3$ is separable if and only if its partial transposition is a positive operator.*

The first example of PPT entangled state for $n = m = 3$ was given by Choi [**32**].

Note that whether a state is entangled or not, depends upon the tensor decomposition of Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Ha [**47**] gave an example of a state, which is separable in $M_2(\mathbb{C}) \otimes M_3(\mathbb{C})$ but entangled in $M_3(\mathbb{C}) \otimes M_2(\mathbb{C})$.

From the notation given in dual cone relation, we see that the set of all separable states coincides with the cone V_1 if we ignore the normalization condition. Then we can apply the duality between the quantum states in V_1 and the convex cone of all positive linear maps in $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$ that is P_1. which gives another separability criterion. We have:
A state $\rho \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ is separable iff $\langle \rho, \phi \rangle \geq 0$ for any positive linear map

$\phi \in \mathcal{B}(M_n(\mathbb{C}), M_m(\mathbb{C}))$. This criterion can be restated as:

Theorem 1.1.47. *A state $\rho \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ is entangled if and only if there exists a positive map $\phi \in (\mathcal{B}(M_n(\mathbb{C}), M_m(\mathbb{C}))$ such that $\langle \rho, \phi \rangle \ngeq 0$. In this case, we say that ϕ is an entanglement witness and ρ is detected by ϕ.*

1.2. Graphical calculus. The following table gives the graphical representation of Bra and Ket vectors, linear maps in $\mathcal{B}(\mathcal{H})$ and in bipartite tensor product $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. More detail about the graphical calculus can be found in [**48, 49**].

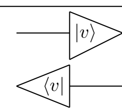
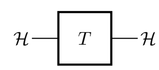
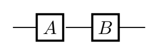
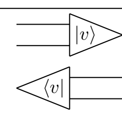
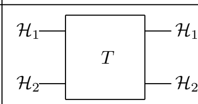
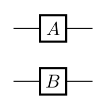
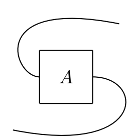
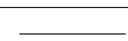
Vectors/Operators	Graphical representation
$\lvert v\rangle \in \mathbb{C}^n$ $\langle v\rvert \in (\mathbb{C}^n)^*$	
$T \in \mathcal{B}(\mathcal{H})$	
$AB \in M_n(\mathbb{C})$	
$\lvert v\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$ $\langle v\rvert \in (\mathbb{C}^n \otimes \mathbb{C}^n)^*$	
$T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$	
$A \otimes B \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$	
$A^t \in M_n(\mathbb{C})$	
$\mathbb{1}_n \in M_n(\mathbb{C})$	

FIGURE 1.4. Graphical representations of vectors and operators

In graphical calculus, transpose is represented by bending wires. Clearly,

$$(|v\rangle)^t = \langle \bar{v}| \tag{1.16}$$

$$(\langle v|)^t = |\bar{v}\rangle \tag{1.17}$$

Graphically, Equations (1.16) and (1.17) can be represented by the following diagram:

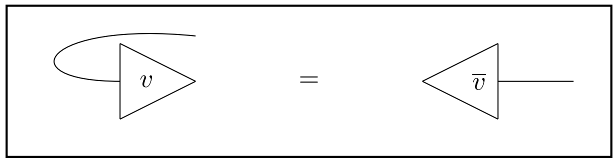


FIGURE 1.5. $(|v\rangle)^t = \langle\overline{v}|$

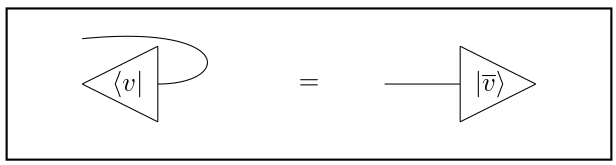


FIGURE 1.6. $(\langle v|)^t = |\overline{v}\rangle$

Figure 1.7 gives the graphical representation of partial transpose of a state $\rho \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ on first subsystem. ρ^{Γ_1} represents the partial transpose of ρ with respect to first subsystem.

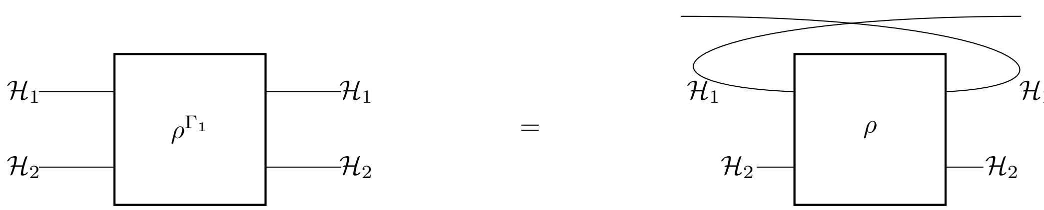


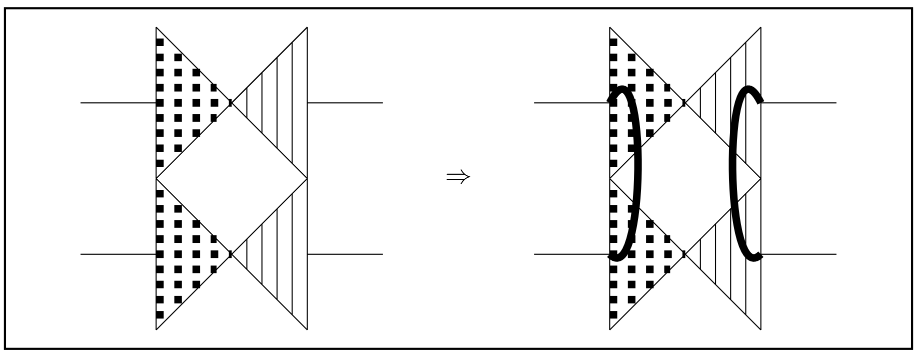
FIGURE 1.7. Partial transpose of ρ

Graphical representation of Operators. In the following, we give the graphical representation of Bell state and Swap operator. These graphical computations present a convenient method to construct any number of entanglement witness.

Bell state. Let $\mathcal{H}$ be a finite dimensional Hilbert space. Let $\{e_1, e_2, ..., e_n\}$ be its orthonormal basis and $e_{ij} = |e_i\rangle\langle e_j|$ be the matrix units for $1 \le i, j \le n$. Then unnormalized bell state can be represented as:

$$\begin{aligned}\text{Bell} &= \sum_{i,j=1}^{n} e_{ij} \otimes e_{ij} \\ &= \sum_{i,j=1}^{n} |e_i\rangle\langle e_j| \otimes |e_i\rangle\langle e_j| \\ &= \sum_{i,j=1}^{n} (|e_i\rangle \otimes |e_i\rangle)(\langle e_j| \otimes \langle e_j|)\end{aligned}$$

Following is the graphical representation of Bell state:



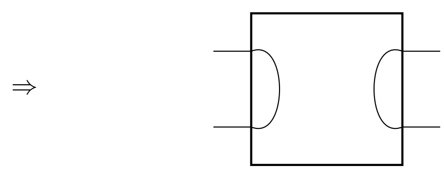


FIGURE 1.8. Bell State

Swap Operator. Let $\mathcal{H}$ and $\mathcal{K}$ be two finite dimensional Hilbert spaces with dimensions n and m respectively. Let $\{e_1, e_2, ..., e_n\}$ and $\{f_1, f_2, ..., f_m\}$ be orthonormal basis of $\mathcal{H}$ and $\mathcal{K}$ respectively. Then we define the swap operator as:

$$\sigma : \mathcal{H} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{H}$$

given by

$$x \otimes y \mapsto y \otimes x$$

for all $x \otimes y \in \mathcal{H} \otimes \mathcal{K}$. It can also be written as:

$$\sum_{i,j} (|f_j\rangle \otimes |e_i\rangle)(\langle e_i| \otimes \langle f_j|) \tag{1.18}$$

Figure 1.9 represents a construction of swap operator based on the formula given in Equation (1.18). Finally, we obtain the graphical representation of swap operator in Figure 1.10. This construction of swap operator is obtained from [**49**, Page 6].

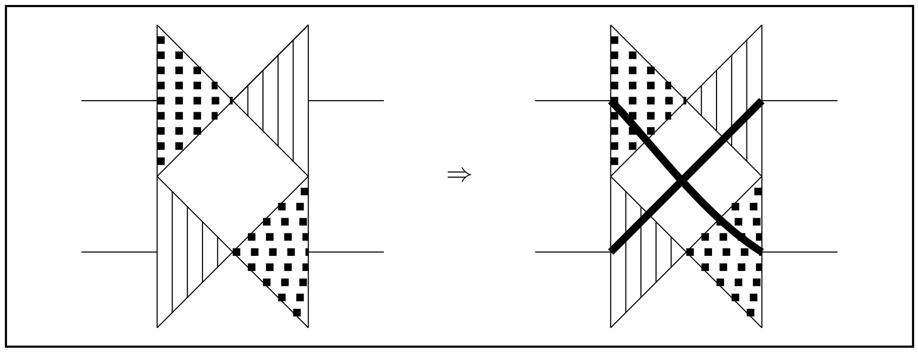


Figure 1.9. Construction of Swap Operator

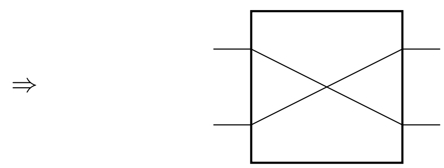


Figure 1.10. Swap Operator

Some Graphical computations. In this section, we will demonstrate how graphical calculus can be used to detect positivity and completely positivity of linear maps.

Example 1.2.1. We consider the transpose map θ_n on $M_n(\mathbb{C})$ given by $A \mapsto A^t$. Clearly, this map is positive. We will perform a graphical computation and show that this map is not completely positive.

$$\begin{aligned} C_{\theta_n} &= \sum_{i,j} e_{ij} \otimes \theta_n(e_{ij}) \\ &= (i_n \otimes \theta_n)(\sum_{i,j} e_{ij} \otimes e_{ij}) \\ &= (i_n \otimes \theta_n)\,\mathrm{Bell} \end{aligned}$$

Figure 1.11 gives the graphical representation of C_{θ_n}. It is evident from the graph that partial transpose of Bell state is swap operator, which is not positive. Hence, C_{θ_n} is not positive. Therefore, θ_n is not completely positive.

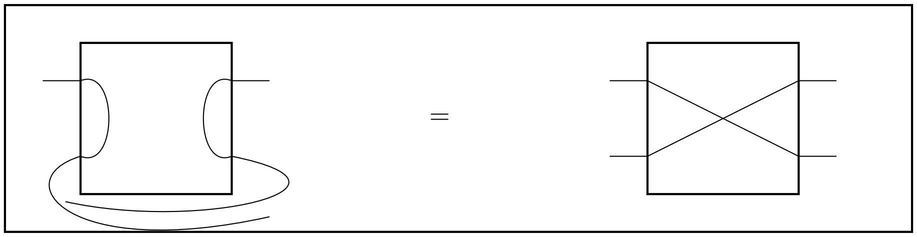


FIGURE 1.11. Partial Transpose of Bell state

1.3. Representation theory of finite groups. In this section, we present some basic definitions and theorems in representation theory which we need to understand an ingredient of Schur-Weyl duality. Throughout this text, we will deal with representations of finite groups over finite-dimensional vector spaces over $\mathbb{C}$. Let $GL(V)$ denote the space of all invertible linear transformations $T : V \to V$. More detail on representation theory of symmetric groups can be found in [**50**].

Definition 1.3.1. Let G be a finite group and V be a finite-dimensional vector space over $\mathbb{C}$. Representation (σ, V) of G is a group homomorphism

$$\sigma : G \to GL(V)$$

such that

(i) $\sigma(g_1 g_2) = \sigma(g_1)\sigma(g_1)$ for every $g_1, g_2 \in G$.
(ii) $\sigma(1_G) = I_V$.
(iii) $\sigma(g^{-1}) = \sigma(g)^{-1}$ for every $g \in G$.

The vector space V is called the representation space and its dimension is the dimension of the representation.
The pair (σ, V) is called a representation of G on V. If we consider V to be an n-dimensional vector space then representation of V is simply the group homomorphism $\tilde{\rho} : G \to GL(n, \mathbb{C})$.

Definition 1.3.2. A subspace W of V is called *invariant* under σ if $\sigma(g)(w) \in W$ for every $w \in W$ and $g \in G$, that is, $\sigma(g)|_W \subseteq W$.

Definition 1.3.3. Let W be an invariant subspace under σ. Set $\rho(g) = \sigma(g)|_W$. Then ρ is called the *sub-representation* of σ.

Every representation (σ, V) of G is always invariant under the zero subspace and the whole space V. They are called trivial invariant subspaces.

Definition 1.3.4. A representation is called *reducible* if it has a non-trivial invariant subspace. Otherwise, it is called *irreducible.*

Suppose that V is endowed with a Hermitian scalar product $\langle \cdot, \cdot \rangle_V$. A representation (σ, V) of G is unitary if $\sigma(g)$ is a unitary operator for all $g \in G$, that is, $\langle \sigma(g)v_1, \sigma(g)v_2 \rangle_V = \langle v_1, v_2 \rangle_V$ for every $v_1, v_2 \in V$.

Examples of representations.

(i) (Trivial representation:) Let G be a group and V be a vector space over a field $\mathbb{C}$. Define $\sigma : G \to GL(V)$ such that

$$\sigma(g) = \mathbb{1}_V \quad \text{for every} \quad g \in G$$

In this case, every subspace of V is invariant under σ.

(ii) Let X be a finite set and $L(X)$ denote the set of all complex valued functions on X. Then $L(X)$ is a vector space where the operation of addition and scalar multiplication is point-wise, that is,

$$(f+g)(x) = f(x) + g(x)$$

$$(cf)(x) = c(f)x$$

for every $f, g \in L(X), x \in X$ and $c \in \mathbb{C}$.
Suppose G acts on a finite set X. We define a representation λ of G on $L(X)$,

$$\lambda : G \to GL(L(X))$$

as

$$g \mapsto \lambda(g)$$

and $[\lambda(g)f](x) = f(g^{-1}x)$, where $(g^{-1}x)$ is the action of G on X.
Clearly λ is a representation on $L(X)$.
Define a scalar product $\langle \cdot, \cdot \rangle_{L(X)}$ on $L(X)$ given by

$$\langle f, g \rangle_{L(X)} = \sum_{x \in X} f(x)\overline{g(x)}.$$

λ is a unitary representation with respect to above scalar product.

(iii) (Regular representation:) Let G be a group acting on itself by left translation, given by the following:

$$\begin{array}{cccc} \cdot : & G \times G & \to & G \\ & (g, a) & \mapsto & ga \end{array} \tag{1.19}$$

The representation defined by this action is called *left regular representation.*

(iv) (The alternating representation:) Let S_n denote the group of all permutations of degree n. Define

$$\pi : S_n \to \mathbb{C}$$

$$\pi(\sigma) = \begin{cases} 1 & \text{if} \quad \sigma \quad \text{is even,} \\ -1 & \text{if} \quad \sigma \quad \text{is odd} \end{cases}$$

Definition 1.3.5. Let V and W be two finite dimensional vector spaces. We write $\mathrm{Hom}(V, W)$ to denote the space of all linear maps from V to W.
Let (σ, V) and (ρ, W) be two representations of G. Then, $T \in \mathrm{Hom}(V, W)$ is called *intertwining operator* if it satisfies:

$$T\sigma(g) = \rho(g)T \ \text{ for all } g \in G.$$

We denote by $\mathrm{Hom}_G(V, W)$ or by $\mathrm{Hom}_G(\sigma, \rho)$, the vector space of all linear maps that intertwine σ and ρ.

Definition 1.3.6. Two representations (σ, V) and (ρ, W) are said to be *equivalent* if there exists a $T \in \mathrm{Hom}_G(V, W)$ which is bijective. If this is the case, we say that T is an isomorphism and we write $\sigma \sim \rho$ and $V \cong W$. If not, we write $\sigma \nsim \rho$. Moreover, If σ and ρ are unitary representations and T is a unitary operator, then we say σ and ρ are *unitarily equivalent* to each other.

In the following, we enquire about the structure of $\mathrm{Hom}_G(\sigma, \rho)$, given that σ and ρ are irreducible representations on V_1 and V_2 respectively.

Lemma 1.3.7. *(Schur's Lemma) Let (σ, V) and (ρ, W) be two irreducible representations of G.*

(i) *If $T \in \mathrm{Hom}_G(\sigma, \rho)$ then either $T = 0$ or T is an isomorphism.*
(ii) $\mathrm{Hom}_G(\sigma, \sigma) = \mathbb{C}I_V$

Corollary 1.3.8. *If σ and ρ are two irreducible representations of G then*

$$\dim \mathrm{Hom}_G(\sigma, \rho) = \begin{cases} 1 & \text{if} \quad \sigma \sim \rho, \\ 0 & \text{if} \quad \sigma \nsim \rho \end{cases}$$

Construction of new representations from given representations.

The adjoint representation. To study the adjoint representation, we will first give the statement of Riesz Representation theorem.

Theorem 1.3.9. *Let ϕ be a bounded linear functional on a Hilbert space $\mathcal{H}$, then there is a unique vector $y \in \mathcal{H}$ such that*

$$\phi(x) = \langle y, x \rangle \text{ for all } x \in \mathcal{H}$$

Let V be a finite dimensional vector space over $\mathbb{C}$. Let $V^{'}$ be the algebraic dual of V, that is,

$$V^{'} = \{f : V \to \mathbb{C} \mid f \text{ is linear}\}.$$

In addition, assume that V is endow with a scalar product $\langle \cdot, \cdot \rangle_V$.
Define an anti-linear map $\Sigma : V \to V^{'}$ given by

$$y \mapsto \phi_y$$

where $\phi_y(x) = \langle x, y \rangle_V$.
By the Riesz Representation, it can be easily shown that the map Σ is an bijective.
We can identify $V^{'}$ with the help of Σ.
The dual scalar product on $V^{'}$ is defined by $\langle \phi_x, \phi_y \rangle_{V'} = \langle y, x \rangle_V$.
Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V. Then $\{v_1^{'}, v_2^{'}, \ldots, v_n^{'}\}$ be the corresponding dual basis of $V^{'}$.
Let (σ, V) be a representation of G. Then its adjoint representation $(\sigma^{'}, V^{'})$ is defined as

$$[\sigma^{'}(g)f](x) = f[\sigma(g^{-1})x]$$

for every $f \in V^{'}, g \in G$ and $x \in V$.
Note that σ is irreducible iff $\sigma^{'}$ is irreducible.

Direct sum of two representations. Let V_1 and V_2 be two n-dimensional vector spaces over $\mathbb{C}$ and (σ_1, V_1) and (σ_2, V_2) be two representations of G. Their direct sum is the representation (σ, V) where $V = V_1 \oplus V_2$ such that $\sigma(g)(v_1, v_2) = \sigma_1(g)(v_1) + \sigma_2(g)(v_2)$.
If β_1 and β_2 are bases of V_1 and V_2 respectively. Then, $\sigma(g)$ can also be realised as a matrix with respect to the basis of $V_1 \oplus V_2$ and is given by,

$$\sigma(g) = \begin{bmatrix} \sigma_1(g) & 0_n \\ 0_n & \sigma_2(g) \end{bmatrix}$$

Tensor product of representations. Let G_1 and G_2 be two finite groups. Let (σ_1, V_1) and (σ_2, V_2) be representations of G_1 and G_2 respectively. The outer tensor product of σ_1 and σ_2 be the representation $\sigma_1 \boxtimes \sigma_2$ of $G_1 \times G_2$ on $V_1 \otimes V_2$ defined by

$$[\sigma_1 \boxtimes \sigma_2](g_1, g_2) = \sigma_1(g_1) \otimes \sigma_2(g_2)$$

for all $g_1 \in G_1$ and $g_2 \in G_2$.
If $G_1 = G_2 = G$, we define a representation $\sigma_1 \otimes \sigma_2$ of G on $V_1 \otimes V_2$ by

$$[\sigma_1 \otimes \sigma_2](g) = \sigma_1(g) \otimes \sigma_2(g)$$

for all $g \in G$. This is called the internal tensor product of σ_1 and σ_2.

Decomposition of a representation.

Theorem 1.3.10. *Let (σ, V) be a finite dimensional representation of a finite group G. Then it can be decomposed into a direct sum of irreducible representations, that is,*

$$V = \bigoplus_{j=1}^{m} V_j$$

Where V_j is irreducible for every $1 \leq j \leq m$. Moreover, If σ is unitary the above decomposition can be chosen to be orthogonal.

Multiplicities and isotypic components. Let (σ, V) be a finite dimensional representation of a finite group G. Then it can be decomposed as a direct sum of irreducible representations. That is,

$$V = \bigoplus_{j=1}^{m} V_j$$

Let J be the set of inequivalent irreducible representations. Then for every representation $\rho \in J$, let m_ρ be the number of invariant subspaces such that their corresponding representations are equivalent to each other. Then V can be written as:

$$V = \bigoplus_{\rho \in J} m_\rho W_\rho$$

So there exist operators $I_{\rho,1}, I_{\rho,2}, ..., I_{\rho,m_\rho} \in \mathrm{Hom}_G(W_\rho, V)$, necessarily linearly independent such that

$$m_\rho W_\rho = \bigoplus_{j=1}^{m_\rho} I_{\rho,j} W_\rho$$

and

$$V = \bigoplus_{\rho \in J} \bigoplus_{j=1}^{m_\rho} I_{\rho,j} W_\rho$$

Hence, any $v \in V$ can be written as:

$$v = \sum_{\rho \in J} \sum_{j=1}^{m_\rho} v_{\rho,j} \quad \text{for} \quad v_{\rho,j} \in I_{\rho,j} W_\rho$$

Define

$$E_{\rho,j} : V \to V$$

such that

$$v \mapsto v_{\rho,j}$$

Clearly, $E_{\rho,j}$ is the orthogonal projection of V onto $I_{\rho,j} W_\rho$.
Note that $I_V = \sum_{\rho \in J} \sum_{j=1}^{m_\rho} E_{\rho,j}$
We will prove that, $E_{\rho,j} \in \mathrm{Hom}_G(\sigma, \sigma)$.
Let $v \in V$. then v can be written in the following form:

$$v = \sum_{\rho \in J} \sum_{j=1}^{m_\rho} v_{\rho,j} \quad \text{for} \quad v_{\rho,j} \in I_{\rho,j} W_\rho$$

Then $\sigma(g)v$ can be represented as follows:

$$\sigma(g)v = \sum_{\rho \in J} \sum_{j=1}^{m_\rho} \sigma(g) v_{\rho,j}$$

Therefore,

$$\begin{aligned} E_{\rho,j}\sigma(g)(v) &= \sigma(g) v_{\rho,j} \\ &= \sigma(g) E_{\rho,j}(v) \end{aligned}$$

Hence, $E_{\rho,j} \in \mathrm{Hom}_G(\sigma, \sigma)$

Definition 1.3.11. The positive integer m_ρ is called the *multiplicity* of ρ in σ and the invariant subspace $m_\rho W_\rho$ is called the *ρ-isotypic component of ρ in σ*.

Representation theory of finite dimensional $*$-algebras. Throughout the text, we will assume that V is a finite dimensional vector space over $\mathbb{C}$ and $\mathcal{A}$ is a subalgebra of $\mathrm{End}(V)$ where $\mathrm{End}(V)$ denotes the algebra of all linear operators $T : V \to V$. We start the theory with some definitions and theorems.

Definition 1.3.12. Let $\mathcal{A} \subseteq \mathrm{End}(V)$ be a subalgebra and $W \subseteq V$ be a subspace. An element $w \in W$ is *cyclic* if $\mathcal{A}w := \{Tw : T \in \mathcal{A}\} = W$.
The subspace W is *$\mathcal{A}$-invariant* if $Tw \in W$ for all $T \in \mathcal{A}$ and $w \in W$.
An $\mathcal{A}$-invariant subspace W is *trivial* if $W = V$ or $W = \{0\}$. An $\mathcal{A}$-invariant nontrivial subspace W is called *irreducible* if it does not have any non-trivial $\mathcal{A}$-invariant subspace, that is, $\mathcal{A}$-invariant subspace are $\{0\}$ and W only.

The next lemma gives an orthogonal decomposition of a finite dimensional unitary space V into irreducible $\mathcal{A}$-invariant subspaces of V

Lemma 1.3.13. *Let $\mathcal{A} \subseteq \mathrm{End}(V)$ be a $*$-algebra of operators on V. Then there exists an orthogonal decomposition*

$$V = \bigoplus_{j \in J} W_j$$

where J is a finite set of indices such that for every $j \in J$, W_j is $\mathcal{A}$-invariant and irreducible.

Next we will state Burnside's theorem. We start with a lemma which we need to prove the theorem.

Lemma 1.3.14. *Let $\mathcal{A}$ be a subalgebra of $\mathrm{End}(V)$ and $W \subseteq V$ be an invariant subspace. Suppose that $\mathcal{A}W \neq \{0\}$. Then W is irreducible if and only if every nontrivial vector in W is cyclic.*

The next theorem is a crucial theorem which characterizes the irreducible representations of non-trivial subalgebras of $\mathrm{End}(V)$.

Theorem 1.3.15 (Burnside's theorem)**.** *Let $\mathcal{A}$ be a subalgebra of $\mathrm{End}(V)$ and $\mathcal{A} \neq \{0\}$. Then the space V is irreducible if and only if $\mathcal{A} = \mathrm{End}(V)$.*

Schur's Lemma. In this section, we will present Schur's lemma in the context of representation theory of finite dimesnional $*$-algebras. We will start with defining representations on finite dimensional $*$-algebras and some of the concepts which are already defined in the context of finite groups.

Definition 1.3.16. Let V and U be two finite-dimensional complex vector spaces and $\mathcal{A}$ be a subalgebra of $\mathrm{End}(V)$. A *representation* ρ of $\mathcal{A}$ on U is a linear map

$$\rho : \mathcal{A} \to \mathrm{End}(U)$$

such that

$$\rho(TS) = \rho(T)\rho(S) \ \ \forall\, T, S \in \mathcal{A}.$$

We can also say that a representation of $\mathcal{A}$ on U is an algebra homomorphism. If $\mathcal{A}$ is unital, we also require $\rho(\mathbb{1}_V) = \mathbb{1}_U$. We denote the representation of $\mathcal{A}$ on U by (ρ, U) or simply by U.
The representation ρ is trivial if $\rho(T) = 0$ for every $T \in \mathcal{A}$.

Definition 1.3.17. Let (ρ, U) be a representation of $\mathcal{A}$.

(i) We define the *kernel of* ρ as the subspace $Ker(\rho) = \{T \in \mathcal{A} : \rho(T) = 0\}$. $Ker(\rho)$ is a subalgebra of $\mathcal{A}$.
(ii) We say that ρ is *faithful* if $Ker(\rho) = \{0\}$.
(iii) A subspace W of U is called $\mathcal{A}$-*invariant* if $\rho(T)w \in W$ for all $w \in W$ and $T \in \mathcal{A}$.

Definition 1.3.18. A non-trivial representation (ρ, U) of $\mathcal{A}$ is called *irreducible* if U does not have any nontrivial $\mathcal{A}$-invariant subspace, that is, if $\{0\}$ and U are the only $\mathcal{A}$-invariant subspaces of U.

Note 1.3.19. A subspace $W \subseteq U$ is $\mathcal{A}$-invariant if and only if it is $\rho(\mathcal{A})$-invariant. Therefore, W does not have a nontrivial $\mathcal{A}$-invariant subspace if and only if W does not have a $\rho(\mathcal{A})$-invariant subspace.

Following proposition is a consequence of Note 1.3.19 and Burnside's theorem 1.3.15.

Proposition 1.3.20. *Let $\mathcal{A}$ be a subalgebra of* $\mathrm{End}(V)$ *and (ρ, U) be a representation of $\mathcal{A}$. Then ρ is irreducible if and only if $\rho(\mathcal{A}) = \mathrm{End}(U)$.*

Now we define the intertwining operator for representations of algebra $\mathcal{A}$.

Definition 1.3.21. Let (ρ, U) and (σ, W) be two representations of an algebra $\mathcal{A}$. An *intertwining operator* $T : U \to W$ is a linear map such that

$$T\rho(A) = \sigma(A)T$$

for all $A \in \mathcal{A}$. We denote by $\mathrm{Hom}_{\mathcal{A}}(\rho, \sigma)$ or $\mathrm{Hom}_{\mathcal{A}}(U, W)$, the space of all operators that intertwine ρ and σ.

Definition 1.3.22. Two representations (ρ, U) and (σ, W) of $\mathcal{A}$ are called *equivalent* if there exist a bijective operator $T \in \mathrm{Hom}_{\mathcal{A}}(\rho, \sigma)$.

Now we give Schur's lemma for representations of an algebra $\mathcal{A}$.

Theorem 1.3.23. *[Schur's Lemma] Let $\mathcal{A}$ be a subalgebra of* $\mathrm{End}(V)$, *(ρ, U) and (σ, W) be two irreducible representations of $\mathcal{A}$.*

(i) *If $T \in \mathrm{Hom}_{\mathcal{A}}(\rho, \sigma)$, then either $T = 0$ or T is an isomorphism.*
(ii) $\mathrm{Hom}_{\mathcal{A}}(\rho, \rho) = \mathbb{C}I_U$.

Next, we define the $*$-representation of a $*$-subalgebra $\mathcal{A}$ of End(V).

Definition 1.3.24. Let $\mathcal{A} \subseteq \mathrm{End}(V)$ be a $*$-subalgebra. A $*$-*representation* of $\mathcal{A}$ on a unitary space W is a representation (σ, W) of $\mathcal{A}$ which satisfy

$$\sigma(A^*) = (\sigma(A))^*$$

for all $A \in \mathcal{A}$.

From now on, we will assume that $\mathcal{A}$ is a $*$-subalgebra of $\mathrm{End}(V)$. Next, we study about the commutant of a $*$-algebra $\mathcal{A}$ and the double commutant theorem. We start with the definition of commutant. We also define the isotypic component of a representation of a $*$-algebra. Much of the theory about isotypic component is similar to what we have already studied in Section 1.3.

Definition 1.3.25. Let $\mathcal{A} \subseteq \mathrm{End}(V)$ be a finite-dimensional $*$-algebra. The *commutant of* $\mathcal{A}$ is the $*$-algebra $\mathcal{A}'$ given by

$$\mathcal{A}' = \{S \in \mathrm{End}(V) : ST = TS \quad \text{for all} \quad T \in \mathcal{A}\}.$$

Now we study about the isotypic decomposition of a representation of a $*$-algebra $\mathcal{A}$. The proofs of most of the propositions and theorems are same as for the representation theory of groups defined in Section 1.3.
Let (σ, V) be a representation of a $*$-algebra $\mathcal{A}$. By Lemma 1.3.13, V can be decomposed as

$$V = \bigoplus_{j \in J} W_j \tag{1.20}$$

$$= \bigoplus_{\lambda \in \Lambda} m_\lambda W_\lambda \tag{1.21}$$

Where Equation (1.21) can be written by grouping all irreducible σ-invariant subspaces of V. Λ is a set of all pairwise inequivalent representations of $\mathcal{A}$ on W_λ and m_λ is the multiplicity of W_λ in V.
Therefore, for every $\lambda \in \Lambda$, there exist injective operators $I_{\lambda,1}, I_{\lambda,2}, ..., I_{\lambda,m_\lambda} \in \mathrm{Hom}_{\mathcal{A}}(W_\lambda, V)$, necessarily linearly independent such that

$$m_\lambda W_\lambda = \bigoplus_{j=1}^{m_\lambda} I_{\lambda,j} W_\lambda$$

$$V = \bigoplus_{\lambda\in\Lambda} \bigoplus_{j=1}^{m_\lambda} I_{\lambda,j} W_\lambda$$

As in Section 1.3, we define the orthogonal projection $E_{\lambda,j}$ of V onto $m_\lambda W_\lambda$. It can be easily seen that $E_{\lambda,j} \in \mathcal{A}'$.

Definition 1.3.26. The positive integer m_λ is called the *multiplicity of* λ for $\mathcal{A}$. The invariant subspace $m_\lambda W_\lambda$ is called the λ-*isotypic component of* λ *for* $\mathcal{A}$. $V = \bigoplus_{\lambda\in\Lambda} m_\lambda W_\lambda$ is called the $\mathcal{A}$-*isotypic decompodition of* V.

The double commutant theorem. We first recall the double commutant theorem. Our aim is to prove a theorem which is the main ingredients of the classical Schur-Weyl-duality. For more detail on Schur-Weyl-duality Theorem, the reader is referred to [**50**, Theorem 8.2.10].

Theorem 1.3.27. *Let $A \subseteq \mathrm{End}(V)$ be a $*$-algebra of operators on a finite dimensional unitary space V. Then*

$$\mathcal{A}'' = (\mathcal{A}')' = \mathcal{A}$$

that is the double commutant of $\mathcal{A}$ is same as $\mathcal{A}$.

Let V be a finite-dimensional vector space over $\mathbb{C}$. In the following text, we identify $\mathrm{End}(V^{\otimes k})$ with $\mathrm{End}(V)^{\otimes k}$. We define two representations σ_k and ρ_k of S_k and $\mathrm{GL}(V)$ respectively on $V^{\otimes k}$. We give a characterization of the commutant $\mathrm{End}_{S_k}(V^{\otimes k})$ of $V^{\otimes k}$ with respect to σ_k in terms of the representation ρ_k of $\mathrm{GL}(V)$.

Definition 1.3.28. Let G be a group. We define an associative algebra $\mathcal{A}[G]$, called the *group algebra of G*

$$\mathcal{A}[G] := \{f : G \to \mathbb{C} |\ Supp(f) \text{ is finite}\}$$

$\mathcal{A}[G]$ is a vector space with respect to the point-wise addition and scalar multiplication. Moreover, the set of Dirac functions $\{\delta_g : g \in G\}$ constitute an orthonormal basis for this vector space.
An element $g \in G$ can be identified as $\delta_g \in \mathcal{A}[G]$. Thus, $f \in \mathcal{A}[G]$ can be written as

$$f = \sum_{g\in G} f(g)\delta_g,$$

where only finite number of coefficients are non-zero. Multiplication of two elements in this vector space is given by their convolution. Multiplication operation is associative by the associativity of group multiplication. The element δ_e corresponding to the identity e of group G becomes the identity element of this algebra.

Note 1.3.29. Every representation ρ of G on V can be extended uniquely to a representation of $\mathcal{A}[G]$ on V, defined as follows:

$$\rho : \mathcal{A}[G] \to \mathrm{End}(V)$$

given by $f \mapsto \sum_{g\in G} f(g)\rho(g)$.

Definition 1.3.30. Let k be a natural number. We define two representations ρ_k and σ_k of $\mathrm{GL}(V)$ and symmetric group S_k on $V^{\otimes k}$ respectively.

(i)
$$\rho_k : \mathrm{GL}(V) \to \mathrm{GL}(V^{\otimes k})$$
given by
$$\rho_k(g)(v_1 \otimes v_2 \otimes ... \otimes v_k) = gv_1 \otimes gv_2 \otimes ... \otimes gv_k$$

(ii)
$$\sigma_k : S_k \to \mathrm{GL}(V^{\otimes k})$$
given by
$$\sigma_k(\pi)(v_1 \otimes v_2 \otimes ... \otimes v_k) = v_{\pi^{-1}(1)} \otimes v_{\pi^{-1}(2)} \otimes ... \otimes v_{\pi^{-1}(k)}$$

We give the duality between S_k and $\mathrm{GL}(V)$, which is the main ingredient in the characterization of (a, b)-unitarily equivariant maps (See Chapter 3).

Theorem 1.3.31. [**50**, Theorem 8.2.8] *We have,*

$$\mathrm{End}_{S_k}(V^{\otimes k}) = \rho_k(\mathcal{A}[\mathrm{GL}(V)])$$

where $\mathcal{A}[\mathrm{GL}(V)]$ *is the group algebra of* $\mathrm{GL}(V)$.

In the above theorem, $\rho_k(\mathcal{A}[\mathrm{GL}(V)]) = \{\sum_{g\in G} f(g)\rho_k(g) \text{ for } f \in \mathcal{A}[\mathrm{GL}(V)]\}$. The above theorem can also be stated as follows: the commutant of $\sigma_k(\mathcal{A}[S_k])$ in $\mathrm{End}(V^{\otimes k})$ coincides with $\rho_k(\mathcal{A}[\mathrm{GL}(V)])$. Since,

$$T \in \sigma_k(\mathcal{A}[S_k])' \iff T\sigma_k(f) = \sigma_k(f)T \text{ for every } f \in \mathcal{A}[S_k].$$

In above equation, σ_k is the representation of group algebra $A[S_k]$. We have

$$\sum_{\pi\in S_k} f(\pi)T\sigma_k(\pi) = \sum_{\pi\in S_k} f(\pi)\sigma_k(\pi)T$$

It is possible only when $T \in \mathrm{End}_{S_k}(V^{\otimes k})$. By Theorem 1.3.31 [**50**, Theorem 8.2.8], $T \in \rho_k(\mathcal{A}[\mathrm{GL}(V)])$. We refer to [**50**, Theorem 8.2.8] for instance for a presentation of this Theorem.

CHAPTER 2

Equivariant maps: Definitions and Examples

Introduction

In [**1**] Bhat studied a family of linear maps which respects unitary conjugation. He gave a characterization of this class of linear maps. The study of equivariant maps began with an example of $(1,1)$-unitarily equivariant maps in $\mathcal{B}(M_n(\mathbb{C}), M_{n^2}(\mathbb{C}))$. These maps satisfy an algebraic property which helps to detect their k-positivity.
Theorem 2.1.3 gives a necessary and sufficient condition for an equivariant map to be k-positive. We give a parametric family of $(1,1)$-unitarily equivariant linear maps in $\mathcal{B}(M_n(\mathbb{C}), M_{n^2}(\mathbb{C}))$ and show that this family of linear maps act as entanglement witness which shows the significance of studying this class of linear maps. Part of this chapter is taken from our paper [**41**], which is a joint work with Prof. Collins and Prof. Osaka.

The main new results of this chapter

- A criterion to detect k-positivity of equivariant linear maps (Theorem 2.1.3).
- New examples of entanglement witness (Section 2.2).

2.1. Definition and k-positivity criterion. We start with a definition of $V(U)$-equivariant maps.

Definition 2.1.1. A self-adjoint linear map $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is called $V(U)$-*equivariant*, if for every unitary matrix $U \in M_n(\mathbb{C})$ there exists $V = V(U) \in M_m(\mathbb{C})$ such that

$$\Phi(UXU^*) = V\,\Phi(X)\,V^* \qquad \forall \quad X \in M_n(\mathbb{C}); \tag{2.1}$$

k-positivity criterion. Establishing k-positivity of a linear map is a difficult task, even on low dimensional matrix algebras. We study a criterion which is a necessary and sufficient condition for an equivariant map to be k-positive. In this regard, we expect that such maps may play an important role in quantum information theory.
We first state the criterion for $k = 1$ and later generalize it for any $k \in \mathbb{N}$.

1-positivity.

Proposition 2.1.2. *Let $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a $V(U)$-equivariant map. Then the following are equivalent.*

(i) *Φ is positive.*
(ii) *$\Phi(e_{11}) \geq 0$, where e_{11} is a matrix unit in $M_n(\mathbb{C})$.*

PROOF. (1) $\implies$ (2): Since Φ is positive, therefore $\Phi(e_{11}) \geq 0$.
(2) $\implies$ (1):
It is enough to prove that $\Phi(p) \in M_m(\mathbb{C})^+$ for any rank-one projection $p \in M_n(\mathbb{C})$. We will use the fact that two hermitian matrices are unitarily equivalent if and only if they have the same eigenvalues (counting the multiplicity). Hence, p and e_{11} are unitarily equivalent.
Therefore, there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that $p = Ue_{11}U^*$.

$$\begin{aligned}\Phi(p) &= \Phi(Ue_{11}U^*)\\ &= V\Phi(e_{11})V^* \quad \text{(by equivariance property of } \Phi).\end{aligned}$$

Given that $\Phi(e_{11}) \geq 0$ which implies $V\Phi(e_{11})V^* \geq 0$.
Therefore, $\Phi(p) \geq 0$ and Φ is positive. □

k-positivity.

Theorem 2.1.3. *Let $k \leq min\{n, m\}$ and $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a $V(U)$-equivariant linear map. Then the following are equivalent.*

(i) *Φ is k-positive .*
(ii) *The block matrix $[\Phi(e_{ij})]_{i,j=1}^k$ is positive, where (e_{ij}) are matrix units in $M_n(\mathbb{C})$.*

PROOF. (1) $\implies$ (2):
Let Φ be k-positive. Then the map

$$i_k \otimes \Phi : M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) \to M_k(\mathbb{C}) \otimes M_m(\mathbb{C})$$

is positive. Since the matrix $[e_{ij}]_{i,j=1}^k$ is positive, where (e_{ij}) are matrix units in $M_n(\mathbb{C})$. Hence,

$$(i_k \otimes \Phi)[e_{ij}]_{i,j=1}^k = [\Phi(e_{ij})]_{i,j=1}^k$$

is positive.
(2) $\implies$ (1):
We prove that the map $i_k \otimes \Phi$ is positive. It is enough to prove that $(i_k \otimes \Phi)p$ is positive for any rank-one projection $p \in M_k(\mathbb{C}) \otimes M_n(\mathbb{C})$. We prove it in two steps:

(i) $p = |x\rangle\langle x|$ for $|x\rangle \in \mathbb{C}^k \otimes \mathbb{C}^n, |||x\rangle|| = 1$.
Consider the Schmidt decomposition of $|x\rangle$, we get

$$|x\rangle = (U \otimes V^*)\big(\sum_{i=1}^{k} c_i|e_i\rangle \otimes |e_i\rangle\big), \tag{2.2}$$

where c_i's are positive real numbers with $\sum_{i=1}^k c_i^2 = 1$.
In expression (2.2), $U \in M_k(\mathbb{C})$ and $V \in M_n(\mathbb{C})$ are unitaries arising from singular value decomposition of the matrix $C \in M_{k,n}(\mathbb{C})$, where C is the coefficient matrix of $|x\rangle$, when written in the canonical basis of $(\mathbb{C}^k \otimes \mathbb{C}^n)$.

$$p = |x\rangle\langle x| = (U \otimes V^*)(\sum_{i,j=1}^{k} c_i c_j e_{ij} \otimes e_{ij})(U \otimes V^*)^*. \tag{2.3}$$

In expression (2.3), first component e_{ij} of tensor product are matrix units in $M_k(\mathbb{C})$ and second component are matrix units in $M_n(\mathbb{C})$.

$$p = (U \otimes V^*)X(U \otimes V^*)^*$$

where $X = \sum_{i,j=1}^{k} c_i c_j e_{ij} \otimes e_{ij}$. It can be easily seen that if Φ satisfy the equivariance property on $M_n(\mathbb{C})$, then $(i_k \otimes \Phi)$ also satisfy the equivariance property on $M_k(\mathbb{C}) \otimes M_n(\mathbb{C})$, if unitaries are coming from $U_k(\mathbb{C}) \otimes U_n(\mathbb{C})$, where $U_d(\mathbb{C})$ denotes the set of all unitaries in $M_d(\mathbb{C})$ for any natural number d. By equivariance property of $(i_k \otimes \Phi)$, there exist matrices V_1 and V_2 in $M_k(\mathbb{C})$ and $M_m(\mathbb{C})$ respectively such that

$$(i_k \otimes \Phi)p = (i_k \otimes \Phi)[(U \otimes V^*)X(U \otimes V^*)^*] \tag{2.4}$$

$$= (V_1 \otimes V_2)(i_k \otimes \Phi)X(V_1 \otimes V_2)^*. \tag{2.5}$$

In fact $V_1 = U$.

(ii) In this step, we prove that $(i_k \otimes \Phi)p$ is positive if the block matrix $[\Phi(e_{ij})]_{i,j=1}^k$ is positive. Let the block matrix $[\Phi(e_{ij})]_{i,j=1}^k$ be positive. First, we prove that $(i_k \otimes \Phi)X$ is positive. Indeed,

$$(i_k \otimes \Phi)X = (i_k \otimes \Phi)\left(\sum_{i,j=1}^{k} c_i c_j e_{ij} \otimes e_{ij}\right) \tag{2.6}$$

$$= [\Phi(c_i c_j e_{ij})]_{i,j=1}^k \tag{2.7}$$

$$= [c_i c_j \Phi(e_{ij})]_{i,j=1}^k \tag{2.8}$$

$$= [c_i c_j]_{i,j=1}^k \circ [\Phi(e_{ij})]_{i,j=1}^k. \tag{2.9}$$

The last term being the Schur product of two positive matrices is positive. Hence, $(i_k \otimes \Phi)X$ is positive.
By expressions (2.5) and (2.9), $(i_k \otimes \Phi)p$ is positive if the block matrix $[\Phi(e_{ij})]_{i,j=1}^k$ is positive.
Therefore, we conclude that Φ is k-positive.

□

2.2. New examples of equivariant maps. In this section, we define a parametric family of $(1,1)$-unitarily equivariant linear maps $\Phi_{\alpha,\beta,n}$ and study the properties of positivity, completely positivity. Moreover, the values of parameters α and β for which the family of maps $\Phi_{\alpha,\beta,n}$ is $(k-1)$-positive and not k-positive for natural numbers $n > 3$ and $2 \le k \le n$ can easily be deduced from Theorem 2.1.3.

Definition 2.2.1. Let α and β be two real numbers and $n \ge 3$. Then the family of maps is defined as follows:

$$\begin{array}{llll} \Phi_{\alpha,\beta,n}: & M_n(\mathbb{C}) & \to & M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \\ & A & \mapsto & A^t \otimes \mathbb{1}_n + \mathbb{1}_n \otimes A + \mathrm{Tr}(A)(\alpha \mathbb{1}_{n^2} + \beta\, \mathrm{Bell}). \end{array} \tag{2.10}$$

Properties of linear map $\Phi_{\alpha,\beta,n}$

Positivity.

Proposition 2.2.2. *The linear map $\Phi_{\alpha,\beta,n}$ $(n \geq 3)$ is positive if and only if it is copositive if and only if one of the following conditions holds.*

(i) $\alpha \in [0, \infty)$ *if* $\beta \geq 0$.

(ii) $\alpha \in \left[\frac{-(2+n\beta)+\sqrt{n^2\beta^2-4(n-2)\beta+4}}{2}, \infty\right)$ *if* $\beta \leq 0$.

Proof. First, assume that $\Phi_{\alpha,\beta,n}$ is positive. By a calculation, It is easy to see that $(\lambda-\alpha)^{n(n-2)}(\lambda-(1+\alpha))^{2(n-1)}\left(\lambda-\left(\alpha-\frac{-(2+n\beta)+\sqrt{n^2\beta^2-4(n-2)\beta+4}}{2}\right)\right)$ $\left(\lambda-\left(\alpha-\frac{-(2+n\beta)-\sqrt{n^2\beta^2-4(n-2)\beta+4}}{2}\right)\right)$ is the characteristic polynomial of $\Phi_{\alpha,\beta,n}(e_{11})$, which is positive. It is possible only when

(i) $\alpha \geq 0$,

(ii) $\alpha \geq \frac{-(2+n\beta)+\sqrt{n^2\beta^2-4(n-2)\beta+4}}{2}$.

We observe that above two conditions hold if and only if

$$\begin{cases} \alpha \geq 0 & \text{if} \quad \beta \geq 0, \\ \alpha \geq \frac{-(2+n\beta)+\sqrt{n^2\beta^2-4(n-2)\beta+4}}{2} & \text{if} \quad \beta \leq 0. \end{cases}$$

For the sufficient part, we apply Proposition 2.1.2. We will show that $\Phi_{\alpha,\beta,n}$ has the equivariance property.
Let $U \in M_n(\mathbb{C})$ be a unitary matrix and $X \in M_n(\mathbb{C})$ be any matrix. Then we have,

$$\Phi_{\alpha,\beta,n}(UXU^*) = (\overline{U}X^tU^t \otimes \mathbb{1}_n) + (\mathbb{1}_n \otimes UXU^*) + \operatorname{Tr}(X)(\alpha\mathbb{1}_{n^2} + \beta\,\mathrm{Bell}).$$

Choose $V = (\overline{U} \otimes U)$ and use the fact that Bell commutes with $(\overline{U} \otimes U)$. We have,

$$\begin{aligned} V\Phi_{\alpha,\beta,n}(X)V^* &= (\overline{U} \otimes U)(X^t \otimes \mathbb{1}_n + \mathbb{1}_n \otimes X + \operatorname{Tr}(X)(\alpha I_{n^2} + \beta\,\mathrm{Bell}))(\overline{U} \otimes U)^* \\ &= (\overline{U} \otimes U)(X^t \otimes \mathbb{1}_n)(U^t \otimes U^*) + (\overline{U} \otimes U)(\mathbb{1}_n \otimes X)(U^t \otimes U)^* + \\ &\quad \operatorname{Tr}(X)(\overline{U} \otimes U)(\alpha\mathbb{1}_{n^2} + \beta\,\mathrm{Bell})(U^t \otimes U^*) \\ &= (\overline{U}X^tU^t \otimes \mathbb{1}_n) + (\mathbb{1}_n \otimes UXU^*) + \operatorname{Tr}(X)(\alpha\mathbb{1}_{n^2} + \beta\,\mathrm{Bell}) \\ &= \Phi_{\alpha,\beta,n}(UXU^*). \end{aligned}$$

Next, we need to show that the matrix $\Phi_{\alpha,\beta,n}(e_{11})$ is positive. It is clear from the reasoning given in the proof of necessary part, where values of parameters satisfy conditions (1) and (2) of the statement. Hence, $\Phi_{\alpha,\beta,n}$ is positive. □

Graphically, the region where $\Phi_{\alpha,\beta,3}$ is positive is shown in Figure 2.1. The shaded region with blue colour represents the values of parameters α and β where $\Phi_{\alpha,\beta,3}$ is positive.

Remark 2.2.3. Let σ be the swap operator on $(\mathbb{C}^n \otimes \mathbb{C}^n)$ as defined in Section 1.2. We have

$$\Phi_{\alpha,\beta,n} \circ \theta_n = Ad\sigma \circ \Phi_{\alpha,\beta,n}$$

Therefore, $\Phi_{\alpha,\beta,n} \circ \theta_n(A) = \sigma\Phi_{\alpha,\beta,n}(A)\sigma$. We obtain that the linear map $\Phi_{\alpha,\beta,n}$ is k-positive if and only if it is k-copositive.

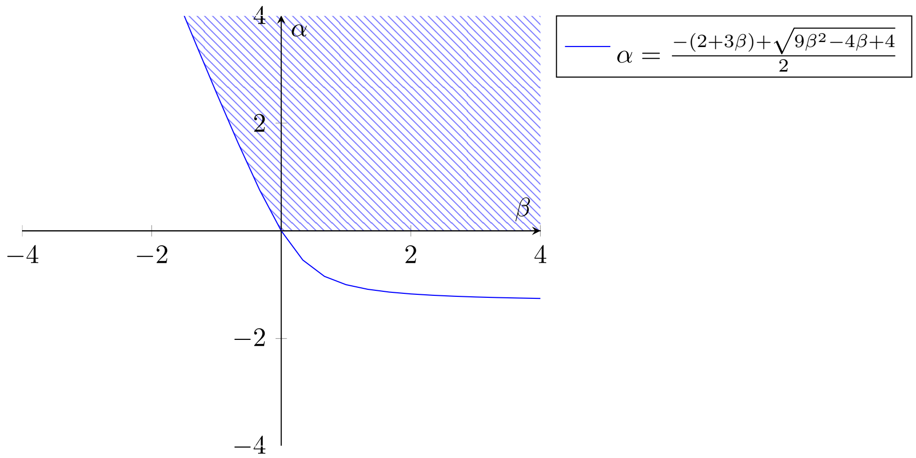


FIGURE 2.1. Region of Positivity

Note 2.2.4. *From now on, we focus on the case of $n = 3$. The family of maps $\Phi_{\alpha,\beta,3}$ is denoted as $\Phi[\alpha, \beta]$.*

Completely Positivity.

Proposition 2.2.5. *The map $\Phi[\alpha, \beta]$ is completely positive if and only if it is completely copositive if and only if one of the following conditions holds.*

(i) $\alpha \in \left[\frac{-(3+3\beta)+\sqrt{9\beta^2-10\beta+17}}{2}, \infty\right)$ *if* $\beta \leq -0.2$.

(ii) $\alpha \in \left[\frac{-(3+3\beta)+\sqrt{9\beta^2-10\beta+17}}{2}, \infty\right) \cap [1, \infty)$ *if* $\beta \geq -0.2$.

PROOF. To prove the above statement, we work on the eigenvalues of the Choi matrix $C_{\Phi[\alpha,\beta]}$ of $\Phi[\alpha, \beta]$. By a calculation, the characteristic polynomial of $C_{\Phi[\alpha,\beta]}$ is given as: $(\lambda-(-1+\alpha))^3(\lambda-(1+\alpha))^{12}(\lambda-\alpha)^6\left(\lambda-\left(\alpha+\frac{(3+3\beta)+\sqrt{9\beta^2-10\beta+17}}{2}\right)\right)^3$
$\left(\lambda-\left(\alpha+\frac{(3+3\beta)-\sqrt{9\beta^2-10\beta+17}}{2}\right)\right)^3$.

To prove the necessary part, assume that $\Phi[\alpha, \beta]$ is completely positive. Hence, all the eigenvalues of the Choi matrix $C_{\Phi[\alpha,\beta]}$ are positive. That is,

(i) $\alpha \geq 1$,

(ii) $\alpha \geq \frac{-(3+3\beta)+\sqrt{9\beta^2-10\beta+17}}{2}$.

It is evident that if $\beta \leq -0.2$, then $\alpha \geq \frac{-(3+3\beta)+\sqrt{9\beta^2-10\beta+17}}{2} \geq 1$. Hence, it is enough to take,

(i) $\alpha \in \left[\frac{-(3+3\beta)+\sqrt{9\beta^2-10\beta+17}}{2}, \infty\right)$ when $\beta \leq -0.2$.

(ii) $\alpha \in \left[\frac{-(3+3\beta)+\sqrt{9\beta^2-10\beta+17}}{2}, \infty\right) \cap [1, \infty)$ when $\beta \geq -0.2$.

For the sufficient part, let $\Phi[\alpha, \beta]$ satisfy conditions (1) and (2) of above Proposition. From the above reasoning, it can be easily seen that the Choi matrix of $\Phi[\alpha, \beta]$ is

positive. Hence, $\Phi[\alpha,\beta]$ is completely positive.
By Remark 2.2.3, we have

$$C_{\Phi_{[\alpha,\beta]\circ\theta_3}} = (\mathbb{1}_3 \otimes \sigma) C_{\Phi[\alpha,\beta]} (\mathbb{1}_3 \otimes \sigma)$$

σ is the swap operator defined on $(\mathbb{C}^3 \otimes \mathbb{C}^3)$. Therefore, the Choi matrix of $\Phi[\alpha,\beta]$ is positive iff the Choi matrix of $\Phi[\alpha,\beta] \circ \theta_3$ is positive.
Hence, $\Phi[\alpha,\beta]$ is completely positive if and only if it is completely copositive if and only if $\Phi[\alpha,\beta]$ satisfy conditions (i) and (ii) of the statement.

□

Note 2.2.6. In Proposition 2.2.5, we see that $\Phi[\alpha,\beta]$ (In fact, $\Phi_{\alpha,\beta,n}$) is completely positive if and only if it is completely copositive. This is not always the case for a linear map satisfying the equivariance property. For example, consider the transpose map on finite-dimensional matrix algebras. Clearly, it satisfies equivariance property (Take $V = \overline{U}$). It is completely copositive and not completely positive.
Hence, this property is not necessarily true for any map satisfying equivariance property.

Remark 2.2.7. Graphically, the portion where $\Phi[\alpha,\beta]$ is positive and completely positive can be represented in Figure 2.2. The dotted region in red colour describes the values of parameters α and β for which the family of maps $\Phi[\alpha,\beta]$ is completely positive.

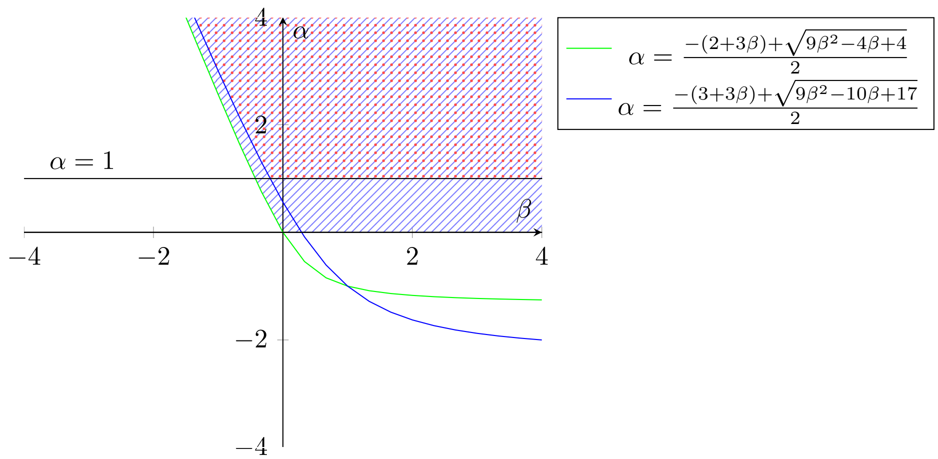


FIGURE 2.2. Region of Completely Positivity

Propositions 2.2.2 and 2.2.5 give two different sets of values of parameters for which the family of maps $\Phi[\alpha,\beta]$ is positive and completely positive. The next theorem gives the values of parameters α and β for which the map $\Phi[\alpha,\beta]$ is positive and not completely positive.

Positive and not completely positive.

Theorem 2.2.8. *The linear map $\Phi[\alpha,\beta]$ is positive and not completely positive if and only if one of the following conditions holds.*

(i) $\alpha \in \left[\frac{-(2+3\beta)+\sqrt{9\beta^2-4\beta+4}}{2}, \frac{-(3+3\beta)+\sqrt{9\beta^2-10\beta+17}}{2}\right)$ *if* $\beta \leq -0.2$.

(ii) $\alpha \in \left[\frac{-(2+3\beta)+\sqrt{9\beta^2-4\beta+4}}{2}, 1\right)$ *if* $-0.2 \leq \beta \leq 0$.

(iii) $\alpha \in [0,1)$ *if* $\beta \geq 0$.

Proof. It is clear from Propositions 2.2.2 and 2.2.5. □

Remark 2.2.9. Figure 2.3 represents the portion where the family of linear maps $\Phi[\alpha,\beta]$ is positive, completely positive and not completely positive. The starred region in green colour gives the values of parameters α and β for which the family of linear maps $\Phi[\alpha,\beta]$ is positive and not completely positive.

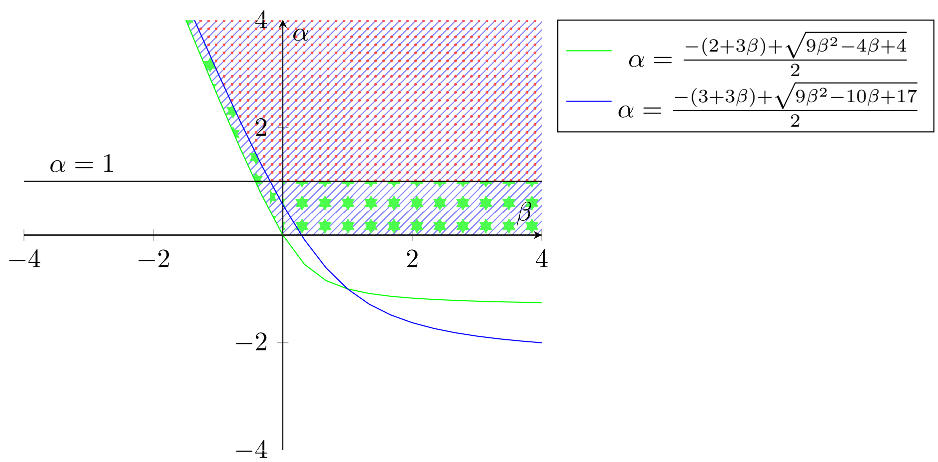


Figure 2.3. Region of Positivity and not Completely positivity

2-**positivity.** Choi [**39**] gave the first example of a linear map defined on $M_3(\mathbb{C})$, which is 2-positive and not completely positive. We show that there exist values of parameters α and β for which the family of maps $\Phi[\alpha,\beta]$ is 2-positive and not completely positive.

Theorem 2.2.10. *The linear map* $\Phi[\alpha,\beta]$ *is* 2*-positive if and only if* $\beta \in \mathbb{R}$ *and* $\alpha \in [1,\infty) \cap [-S_\beta,\infty)$*, where* S_β *is the smallest root of the polynomial* $\lambda^3 + (-2-3\beta)\lambda^2 + (-2+4\beta)\lambda + 2\beta$.

Proof. Let $\Phi[\alpha,\beta]$ be 2-positive. By the definition of 2-positive linear maps, $(i_2 \otimes \Phi[\alpha,\beta])$ is positive. Hence, $(i_2 \otimes \Phi[\alpha,\beta])[e_{ij}]_{i,j=1}^2 \geq 0$ where (e_{ij}) are matrix units in $M_3(\mathbb{C})$.
By a calculation, the characteristic polynomial of $[\Phi[\alpha,\beta](e_{ij})]_{i,j=1}^2$ is
$(\lambda-(2+\alpha))(\lambda-(-1+\alpha))(\lambda-(1+\alpha))^7(\lambda-\alpha)^3((\lambda-\alpha)^3+(-2-3\beta)(\lambda-\alpha)^2+(-2+4\beta)(\lambda-\alpha)+2\beta)^2$.
Therefore, we have the required bound on the values of parameters α and β.
To prove the sufficient part, we use Proposition 2.1.3. It is clear from the proof of Proposition 2.2.2 that $\Phi[\alpha,\beta]$ has the equivariance property. Given the conditions

on the values of parameters α and β, it follows that the matrix $[\Phi[\alpha,\beta](e_{ij})]_{i,j=1}^{2}$ is positive. Hence, the proof is completed. $\square$

Theorem 2.2.11. *The linear map $\Phi[\alpha,\beta]$ is 2-positive and not completely positive if and only if $\beta \le -0.2$ and*

$$\alpha \in \left[\frac{-(2+3\beta)+\sqrt{9\beta^2-4\beta+4}}{2}, \frac{-(3+3\beta)+\sqrt{9\beta^2-10\beta+17}}{2}\right) \cap [1,\infty) \cap [-S_\beta,\infty),$$

where S_β is the smallest root of the polynomial $\lambda^3+(-2-3\beta)\lambda^2+(-2+4\beta)\lambda+2\beta$.

Proof. It is clear from Theorems 2.2.8 and 2.2.10 that the conditions for α and β are sufficient for $\Phi[\alpha,\beta]$ to be 2-positive and not completely positive. We need to consider those values of parameters α and β for which the map $\Phi[\alpha,\beta]$ is 2-positive and not completely positive. If we consider $\beta > -0.2$, then by Theorem 2.2.8, $\alpha < 1$ which destroys the 2-positivity of linear maps $\Phi[\alpha,\beta]$.
Hence, $\beta \le -0.2$ and $\alpha \in \left[\frac{-(2+3\beta)+\sqrt{9\beta^2-4\beta+4}}{2}, \frac{-(3+3\beta)+\sqrt{9\beta^2-10\beta+17}}{2}\right)$.
Therefore, we conclude that
$\alpha \in \left[\frac{-(2+3\beta)+\sqrt{9\beta^2-4\beta+4}}{2}, \frac{-(3+3\beta)+\sqrt{9\beta^2-10\beta+17}}{2}\right) \cap [1,\infty) \cap [-S_\beta,\infty)$ if $\beta \le -0.2$. $\square$

Remark 2.2.12. The conditions of being $(k-1)$-positive and not k-positive can easily be generalized for any natural numbers $n > 3$ and $2 \le k \le n$. From Proposition 2.1.3, it boils down to calculate the eigenvalues of block matrix $[\Phi[\alpha,\beta](e_{ij})]_{i,j=1}^{k}$ for k-positivity. Here, (e_{ij}) are matrix units in $M_n(\mathbb{C})$.

Conclusion. In this chapter, we studied a necessary and sufficient condition for a $V(U)$-equivariant map to be k-positive. Positive and not completely positive maps are important to detect entangled states in quantum information theory. Since the class of positive maps is far from being understood, and there is no simple technique available to check the positivity of a linear map. Therefore, their construction is not easy. On the other hand, It is rather easy to conclude about completely positivity, because of Choi's theorem [**8**]. We deduce in Theorem 2.1.3, that the k-positivity of an equivariant linear map depends upon the positivity of k-block submatrix of the Choi matrix, that is, detecting about the k-positivity of an equivariant linear map is as easy as it's completely positivity.
We presented a family of $(1,1)$-unitarily equivariant maps and showed that this family can act as entanglement witness.
This chapter serves two purposes: it gives a criterion to detect k-positivity of equivariant maps. Secondly, It answers the question why one should be interseted in the study of equivariant maps and shows the worth of studying this class of equivariant maps to solve open problems in quantum information theory.
This provides enough evidence to dig further about the class of $V(U)$-equivariant maps, which makes the foundation of the next chapter. Where we define certain subclasses of equivariant linear maps and study about their characterization.

CHAPTER 3

Characterization of Equivariant maps

Introduction

In this chapter, we define a subclass of $V(U)$-equivariant maps, called unitarily equivariant maps. We further study a subclass of unitarily-equivariant maps, which are (a,b)-unitarily equivariant maps in $\mathcal{B}(M_n(\mathbb{C}), M_n(\mathbb{C})^{\otimes a} \otimes M_n(\mathbb{C})^{\otimes b})$ for $a,b \in \mathbb{N}$ and unitary $U \in M_n(\mathbb{C})$.
This chapter deals with the characterization of these two subclasses of $V(U)$-equivariant linear maps. Theorem 3.1.4 gives a correspondence between equivariant maps and their Choi matrices. We prove in Theorem 3.2.1 that every unitarily equivariant map comes from a unitary representation $U \mapsto V(U)$. Consequently, we prove in Corollary 3.2.3 that every unitarily equivariant map can be realized as a (a,b)-unitarily equivariant map for some $a,b \in \mathbb{N}$. Theorem 3.3.1 gives the characterization of all (a,b)-unitarily equivariant maps and we study their graphical representation.
Group representation theory (More precisely, The Schur-Weyl duality Theorem) plays an important role in the characterization of (a,b)-unitarily equivariant maps. Part of this chapter is taken from our paper [**51**], which is a joint work with Prof. Collins and Ivan Bardet.

The main results of this chapter

- Characterization of unitarily equivariant maps in terms of unitary representations (Theorem 3.2.1).
- Characterization of (a,b)-unitarily equivariant maps (Theorem 3.3.1).
- A correspondence between above two classes (Corollary 3.2.3).

Introduction to equivariant maps

In the following, we define two sub-classes of $V(U)$-equivariant maps (Definition 2.1.1)

Definition 3.0.1. Let $n, N \geq 2$ be two natural numbers. A self-adjoint linear map $\Phi : M_n(\mathbb{C}) \to M_N(\mathbb{C})$ is called:

(i) *unitarily equivariant*, if for every unitary matrix $U \in M_n(\mathbb{C})$ there exists a unitary $V = V(U) \in M_N(\mathbb{C})$ such that

$$\Phi(UXU^*) = V(U)\,\Phi(X)\,V(U)^* \qquad \forall X \in M_n(\mathbb{C}); \tag{3.1}$$

(ii) *(a,b)-unitarily equivariant*, if there are a,b natural numbers such that $N = n^{a+b}$, and $M_N(\mathbb{C}) \equiv M_n(\mathbb{C})^{\otimes a} \otimes M_n(\mathbb{C})^{\otimes b}$, and such that for every unitary $U \in M_n(\mathbb{C})$,

$$\Phi(UXU^*) = (\overline{U}^{\otimes a} \otimes U^{\otimes b})\, \Phi(X)\, (\overline{U}^{\otimes a} \otimes U^{\otimes b})^* \quad \forall X \in M_n(\mathbb{C}) \tag{3.2}$$

Thus, (a,b)-unitarily equivariant maps are a subfamily of unitary equivariant maps, that is itself a subclass of equivariant maps. To our knowledge, we do not know of any equivariant map which is not a unitary equivariant map. All the examples, which have been studied so far are unitary equivariant or more precisely, (a,b)-unitarily equivariant.

3.1. First examples and properties of equivariant maps. Incidentally, some well-known examples of k-positive but not completely positive linear maps are actually examples of unitarily equivariant maps, even though this was not explicitly stated when they were introduced. We recall these examples and give an alternative proof of their k-positivity, based on Theorem 2.1.3.

(i) Every $*$-homomorphism or $*$-anti-homomorphism on a finite-dimensional matrix algebras is equivariant. Such maps are always completely positive or co-completely positive.

(ii) Choi [**39**, Theorem 1] gave the first example of a linear map on $M_n(\mathbb{C})$ which is $(n-1)$-positive but not n-positive, given by

$$\begin{array}{rccc} \Phi: & M_n(\mathbb{C}) & \to & M_n(\mathbb{C}) \\ & A & \mapsto & \{(n-1)\mathrm{Tr}(A)\}\mathbb{1}_n - A\,. \end{array}$$

The above map is unitarily equivariant (take $V(U) = U$). We apply Theorem 2.1.3 to prove that Φ is $(n-1)$-positive. The eigenvalues of $[\Phi(e_{ij})]_{i,j=1}^{n-1}$ are 0 with multiplicity 1 and $(n-1)$ with multiplicities $n(n-1)-1$, which are positive. Hence, Φ is $(n-1)$-positive.

(iii) Tomiyama [**10**, Theorem 2] gave an example of a parametric family of linear maps and studied the conditions of its k-positivity. The map is defined by:

$$\begin{array}{rccc} \Psi: & M_n(\mathbb{C}) & \to & M_n(\mathbb{C}) \\ & A & \mapsto & \frac{\lambda}{n}\mathrm{Tr}(A)\mathbb{1}_n + (1-\lambda)A\,. \end{array}$$

This map is k-positive for any $1 \le k \le n$ if and only if $0 \le \lambda \le 1 + \frac{1}{nk-1}$. We give an alternative proof based on Theorem 2.1.3. Indeed, Ψ is unitarily equivariant (take $V(U) = U$) and to find the conditions of k-positivity on Ψ, we only need to find the values of λ such that $[\Psi(e_{ij})]_{i,j=1}^{k}$ is positive. It can be easily seen that $\frac{\lambda}{n}$ and $\frac{\lambda}{n} + (1-\lambda)k$ are two distinct eigenvalues of $[\Psi(e_{ij})]_{i,j=1}^{k}$ with different multiplicities. Therefore, the map Ψ is k-positive if and only if $\lambda \ge 0$ and $\lambda \le 1 + \frac{1}{nk-1}$.

Example 3.1.1. Bhat characterized in [**1**] all $(0,1)$-equivariant maps on $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, not necessarily finite-dimensional. More precisely, he proved that a linear map Φ acting on $\mathcal{B}(\mathcal{H})$ satisfies $\Phi(UXU^*) = U\Phi(X)U^*$ for all $X \in \mathcal{B}(\mathcal{H})$ iff there exist $\alpha, \beta \in \mathbb{C}$ such that

$$\Phi(X) = \alpha X + \beta\, \mathrm{Tr}\,[X]\mathbb{1}_{\mathcal{H}}\,.$$

Direclty from this, we get that any $(1,0)$-equivariant map Ψ on $\mathcal{B}(\mathcal{H})$ is of the form $\Psi(X) = \alpha\theta_n(X) + \beta\,\mathrm{Tr}\,[X]\mathbb{1}_{\mathcal{H}}$, where θ_n is the transpose map. Indeed, we can check that $\theta_n \circ \Psi$ is $(0,1)$-equivariant and apply

Bhat's result. We shall similarly characterize all (a,b)-unitarily equivariant maps in Theorem 3.3.1, thus generalizing these two case, when $\mathcal{H}$ is finite-dimensional.

Properties of equivariant linear maps. In this section, we study some basic properties of equivariant linear maps, which will help us to give a characterization of these maps.

Lemma 3.1.2. *Let $a, b \in \mathbb{N}$. Then,*

(i) *The set of all (a,b)-unitarily equivariant maps is a vector space.*
(ii) *The set of all unitarily equivariant maps is not a vector space. Neither is the set of equivariant maps.*

PROOF. It is an easy calculation to show that the set of all (a,b)-unitarily equivariant maps is closed under addition and scalar multiplication. Hence, this set is a vector space.
We give an example of two unitarily equivariant maps such that their sum is not even equivariant, which is enough to conclude that both sets are not vector spaces.

Let $\Phi_1, \Phi_2 : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be given by $\Phi_1 = i_2$ and $\Phi_2 = \theta_2$. It is straightforward that both maps are unitarily equivariant. We prove that the map $(\Phi_1 + \Phi_2)$ is not equivariant, that is, there exists a unitary $U \in M_2(\mathbb{C})$ such that there is no $V \in M_2(\mathbb{C})$ with

$$(\Phi_1 + \Phi_2)(UXU^*) = V(\Phi_1 + \Phi_2)(X)V^* \ \ \forall \ \ X \in M_2(\mathbb{C})\,.$$

Let

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-\iota}{\sqrt{2}} & \frac{\iota}{\sqrt{2}} \end{bmatrix}.$$

Where $\iota^2 = -1$. Take $X = e_{11}$, where e_{11} is a matrix unit in $M_2(\mathbb{C})$. We prove that there is no $V \in M_2(\mathbb{C})$ such that,

$$(\Phi_1 + \Phi_2)(Ue_{11}U^*) = V(\Phi_1 + \Phi_2)(e_{11})V^*\,. \tag{3.3}$$

$$(\Phi_1 + \Phi_2)(Ue_{11}U^*) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, $rank((\Phi_1 + \Phi_2)(Ue_{11}U^*)) = 2$.
Since, $rank(\Phi_1 + \Phi_2)(e_{11}) = 1$. Therefore there exists no $V \in M_2(\mathbb{C})$ such that,

$$(\Phi_1 + \Phi_2)(Ue_{11}U^*) = V(\Phi_1 + \Phi_2)(e_{11})V^*\,. \tag{3.4}$$

□

Example 3.1.3. To conclude this section, we give an example of a completely positive linear map on $M_2(\mathbb{C})$ which is not equivariant. Consider the linear map

$$\begin{array}{cccc} \Phi : & M_2(\mathbb{C}) & \to & M_2(\mathbb{C}) \\ & A & \mapsto & A - A^t + \mathrm{Tr}(A)\mathbb{1}_2 \end{array}.$$

The map Φ is completely positive, but it is not equivariant. Let

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-\iota}{\sqrt{2}} & \frac{\iota}{\sqrt{2}} \end{bmatrix} \text{ and } \quad X = -\iota e_{11} + \iota e_{22}$$

Where $\iota^2 = -1$. Then

$$\Phi(UXU^*) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \textit{ and } \quad \Phi(X) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, there exists no $V \in M_2(\mathbb{C})$ such that $\Phi(UXU^*) = V\Phi(X)V^*$ for unitary U and $X = e_{12} + e_{21}$. Hence, Φ is not equivariant.

First, we give the characterization of equivariant maps in terms of their Choi matrices. It will be helpful to characterize certain subclasses of equivariant maps.

Theorem 3.1.4. *Let $\Phi : M_n(\mathbb{C}) \to M_N(\mathbb{C})$ be a linear map. Then the following two assertions are equivalent:*

(i) *Φ is an equivariant map.*
(ii) *For every unitary matrix U in $M_n(\mathbb{C})$, there exists a matrix $V \in M_N(\mathbb{C})$ such that $[C_\Phi, \overline{U} \otimes V] = 0$.*

Proof. To prove (i) $\implies$ (ii), we use the fact that $(\overline{U} \otimes U)$ commutes with Bell for every unitary matrix $U \in M_n(\mathbb{C})$. We have the following

$$\begin{aligned} C_\Phi &= (i_n \otimes \Phi)(\text{Bell}) \\ &= (i_n \otimes \Phi)(\overline{U} \otimes U)(\sum_{i,j} e_{ij} \otimes e_{ij})(\overline{U} \otimes U)^* \\ &= \sum_{i,j} \overline{U} e_{ij} \overline{U}^* \otimes \Phi(U e_{ij} U^*) \end{aligned} \tag{3.5}$$

Since Φ is equivariant, for every unitary $U \in M_n(\mathbb{C})$ there exists $V \in M_N(\mathbb{C})$ such that Φ satisfies Equation (3.1). From Equation (3.5), C_Φ becomes,

$$\begin{aligned} C_\Phi &= \sum_{i,j} \overline{U} e_{ij} \overline{U}^* \otimes V\Phi(e_{ij})V^* \\ &= (\overline{U} \otimes V) C_\Phi (\overline{U} \otimes V)^* . \end{aligned}$$

Hence, C_Φ commutes with $(\overline{U} \otimes V)$.
We now prove the converse implication (ii) $\implies$ (i).
We have,

$$\Phi(UXU^*) = \text{Tr}_{M_n(\mathbb{C})} \left[((UXU^*)^t \otimes \mathbb{1}_m) C_\Phi \right] \tag{3.6}$$

Since C_Φ commutes with $(\overline{U} \otimes V)$. Equation (3.6) becomes,

$$\Phi(UXU^*) = \text{Tr}_{M_n(\mathbb{C})}((\overline{U} X^t U^t) \otimes \mathbb{1}_N)(\overline{U} \otimes V) C_\Phi (\overline{U} \otimes V)^* \tag{3.7}$$

$$= \text{Tr}_{M_n(\mathbb{C})} \left[(\overline{U} X^t U^t \overline{U} \otimes V) C_\Phi (\overline{U} \otimes V)^* \right] \tag{3.8}$$

$$= \text{Tr}_{M_n(\mathbb{C})} \left[\sum_{i,j} (\overline{U} X^t \otimes V) e_{ij} \otimes \Phi(e_{ij}) (\overline{U} \otimes V)^* \right] \tag{3.9}$$

Let $X = (x_{kl}) \in M_n(\mathbb{C})$. Then equation (3.9) becomes

$$\begin{aligned}\Phi(UXU^*) &= \mathrm{Tr}_{M_n(\mathbb{C})}\left[\sum_{i,j}(\overline{U}x_{lk}e_{kl}e_{ij}U^t)\otimes V\Phi(e_{ij})V^*\right]\\ &= \mathrm{Tr}_{M_n(\mathbb{C})}\left[\sum_{i,j}(\overline{U}x_{lk}e_{kj}\delta_{il}U^t)\otimes V\Phi(e_{ij})V^*\right]\\ &= \sum_{i,j,k}\mathrm{Tr}(\overline{U}x_{ik}e_{kj}U^t)V\Phi(e_{ij})V^*\\ &= \sum_{i,j}x_{ij}V\Phi(e_{ij})V^*\\ &= V\Phi(X)V^*\end{aligned}$$

Hence, the proof is complete.

□

3.2. Characterization of unitarily equivariant maps. It is still an open question whether the map $U \mapsto V(U)$ in Equation (3.1) must be a unitary representation. This would be a crucial step in the understanding of equivariant maps, as it shall allow using group representation theory to study them. However, we can answer this question by the affirmative whenever the algebra generated by the image of the map is the whole algebra. More precisely, for any $*$-preserving map $\Phi \in \mathcal{B}(M_n(\mathbb{C}), M_N(\mathbb{C}))$, we denote by $\mathcal{A}[\Im(\Phi)]$ the C^*-algebra generated by the image of Φ: it is the norm-closure of the $*$-subalgebra generated by $\Im(\Phi)$, where $\Im(\Phi)$ denotes the image of Φ.

Theorem 3.2.1. *Let $\Phi : M_n(\mathbb{C}) \to M_N(\mathbb{C})$ be a unitarily equivariant linear map. Let $\mathcal{A}[\Im(\Phi)] = M_N(\mathbb{C})$ and the map $U \mapsto V(U)$ in Equation* (3.1) *is continuous. Then there exists a* unitary representation $U \mapsto \tilde{V}(U)$ *of the special unitary group $\mathcal{SU}_n$ on $\mathbb{C}^N$ such that for all $X \in M_n(\mathbb{C})$ and $U \in \mathcal{SU}_n$,*

$$\Phi(UXU^*) = \tilde{V}(U)\Phi(X)\,\tilde{V}(U)^* \tag{3.10}$$

Proof. Let $U_1, U_2 \in \mathcal{U}_n$ be two unitaries. We have:

$$\Phi(U_1XU_1^*) = V(U_1)\Phi(X)V(U_1)^*$$
$$\Phi(U_2XU_2^*) = V(U_2)\Phi(X)V(U_2)^*$$
$$\Phi(U_1U_2X(U_1U_2)^*) = V(U_1U_2)\Phi(X)V(U_1U_2)^*\,. \tag{3.11}$$

Then, considering that $\Phi(U_1U_2X(U_1U_2)^*) = \Phi(U_1(U_2XU_2^*)U_1^*)$ and using Equation (3.11), it becomes

$$V(U_1U_2)\,\Phi(X)\,V(U_1U_2)^* = V(U_1)V(U_2)\,\Phi(X)\,V(U_2)^*V(U_1)^*\,.$$
$$\implies V(U_2)^*V(U_1)^*V(U_1U_2) \in \Im(\Phi)^{'} = \mathbb{C}\mathbb{1}_N$$

We have,

$$V(U_2)^*V(U_1)^*V(U_1U_2) = \lambda(U_1,U_2)\mathbb{1}_N \text{ for some } \lambda(U_1,U_2) \in \mathbb{C}.$$
$$V(U_1U_2) = \lambda(U_1,U_2)V(U_1)V(U_2) \tag{3.12}$$

In other words, there exists $\theta(U_1,U_2) \in \mathbb{R}$ such that

$$V(U_1)V(U_2) = e^{\iota\theta(U_1,U_2)}V(U_1U_2) \tag{3.13}$$

Equation (3.13) looks very similar to that of a representation, up to the phase. Actually, under the mild assumption that V and the phase $e^{\iota\theta(U_1,U_2)}$ is a smooth function of U_1, U_2, we can prove that V can undergo a phase modification that will turn it into a representation of $\mathcal{SU}_n$. First of all, without loss of generality, we may assume that $\theta(I_n, I_n) = 0$, at the possible expense of replacing V by $e^{\iota\theta(I_n,I_n)}V$. Then, looking in a vicinity of the identity, taking the determinant, we get that

$$\det V(U_1) \det V(U_2) = e^{m\iota\theta(U_1,U_2)} \det V(U_1U_2),$$

in other words

$$e^{\iota\theta(U_1,U_2)} = (\det V(U_1) \det V(U_2)/ \det V(U_1U_2))^{1/m},$$

where the m-th root is obtained with the prinicipal branch of the logarithm, which is well defined in a neighbourhood of 0.

So, we may rewrite Equation (3.13) as

$$V(U_1)V(U_2) = (\det V(U_1) \det V(U_2)/ \det V(U_1U_2))^{1/m} V(U_1U_2),$$

or equivalently,

$$V(U_1) \det V(U_1)^{-1/m} V(U_2) \det V(U_2)^{-1/m} = V(U_1U_2)(\det V(U_1U_2))^{-1/m}$$

Therefore, setting

$$\tilde{V}(U_1) = V(U_1) \det V(U_1)^{-1/m},$$

we see that, in a neighbourhood of 0,

$$\tilde{V}(U_1)\tilde{V}(U_2) = \tilde{V}(U_1U_2).$$

It is known that a finite dimensional representation of a Lie group is in one to one correspondance with that of its Lie algebra, and therefore that it suffices to define it on a neighbourhood of identity, in the simply connected case. So, this proves that $\tilde{V}$ extends on $\mathcal{SU}_n$ to a representation of $\mathcal{SU}_n$. □

Remark 3.2.2. *In Theorem 3.2.1, we proved that for a unitarily equivariant map Φ, there exists a unitary representation $U \mapsto \tilde{V}(U)$ of $\mathcal{SU}_n$ of $\mathbb{C}^N$ such that it satisfies Equation* (3.10). *However, this unitary representation of $\mathcal{SU}_n$ does not affect our main results. Indeed, for a given unitary $U \in \mathcal{U}_n$, we can have an element of $u \in \mathcal{SU}_n$ and a phase factor $e^{\iota\phi} \in U(1)$ such that $U = ue^{\iota\phi}$. Here $\iota^2 = -1$.*

The next corollary builds a bridge between unitarily equivariant maps and (a, b)-unitarily equivariant maps. It justifies the computation of the latter in the next section.

Corollary 3.2.3. *Let $\Phi \in \mathcal{B}(M_n(\mathbb{C}), M_N(\mathbb{C}))$ be a unitarily equivariant map. Assume that the map $U \mapsto V(U)$ is an* irreducible representation *of the unitary group. Then there exist a, b natural numbers, a partial isometric map $W : \mathbb{C}^N \to (\mathbb{C}^n)^{\otimes a} \otimes (\mathbb{C}^n)^{\otimes b}$ and an (a, b)-unitarily equivariant map Ψ on $M_n(\mathbb{C})$ such that for all $X \in M_n(\mathbb{C})$,*

$$W\Phi(X)W^* = \Psi(X)\,. \tag{3.14}$$

Proof. This is a direct consequence of the representation theory of the unitary group. Indeed, for all irreducible representations, there exist a, b natural integers such that this irreducible representation appears in the unitary representation (see [**52**] for instance)

$$U \mapsto \overline{U}^{\otimes a} \otimes U^{\otimes b}\,.$$

It means that there exists a partial isometry W from $\mathbb{C}^N$ to $(\mathbb{C}^n)^{\otimes a} \otimes (\mathbb{C}^n)^{\otimes b}$ such that for all $U \in \mathcal{U}_n$,

$$W\,V(U)W^* = P_\lambda\,\overline{U}^{\otimes a} \otimes U^{\otimes b}\,P_\lambda\,,$$

where P_λ is the orthogonal projection on some irreducible supspace of $(\mathbb{C}^n)^{\otimes a} \otimes (\mathbb{C}^n)^{\otimes b}$ invariant by $(\overline{U}^{\otimes a} \otimes U^{\otimes b})$ for all $U \in \mathcal{U}_n$. We can then define the map $\Psi : X \mapsto W\,\Phi(X)\,W^*$ and it can be readily checked that Ψ is an (a,b)-unitarily equivariant map. $\square$

3.3. Characterization of (a, b)-unitarily equivariant maps. *We now deal with the question of characterizing (a,b)-unitarily equivariant maps. More explicitly, let $a, b \in \mathbb{N}$ and $n \geq 2$. Then, what are all the linear maps*

$$\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})^{\otimes a} \otimes M_n(\mathbb{C})^{\otimes b}$$

such that for every unitary $U \in M_n(\mathbb{C})$, Φ satisfy Equation (3.2)*. By Lemma 3.1.2(1) the set of all (a,b)-unitarily equivariant maps is a vector space. Characterizing this vector space is then equivalent to exhibiting one of its basis. In order to do that, we need some basic results from group representation theory and more precisely the Schur-Weyl duality Theorem* [**50**, Theorem 8.2.10]*. More detail can be found in* [**50**, Chapters 7,8].

Theorem 3.3.1. *Let $a, b \in \mathbb{N}$ and $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})^{\otimes a} \otimes M_n(\mathbb{C})^{\otimes b}$. Then the following assertions are equivalent.*

(i) *Φ is (a,b)-unitarily equivariant.*
(ii) *$[C_\Phi, \overline{U}^{\otimes a+1} \otimes U^{\otimes b}] = 0$ for all $U \in \mathcal{U}_n$, where C_Φ is the Choi matrix of Φ.*
(iii) *There exists $f : S_{k+1} \to \mathbb{C}$ such that*

$$C_\Phi = \sum_{\pi \in S_{k+1}} f(\pi)\,\left(\theta_n^{\otimes a+1} \otimes i_n^{\otimes b}\right)[\sigma_{k+1}(\pi)]\,, \tag{3.15}$$

where $k = a + b$ and θ_n is the transpose map on $M_n(\mathbb{C})$.

Proof. The equivalence of (i) and (ii) is clear from Theorem 3.1.4.
$(ii) \Rightarrow (iii)$.
Assume that C_Φ commutes with $(\overline{U}^{\otimes a+1} \otimes U^{\otimes b})$ for all $U \in \mathcal{U}_n$, that is,

$$(\overline{U}^{\otimes a+1} \otimes U^{\otimes b})C_\Phi(\overline{U}^{\otimes a+1} \otimes U^{\otimes b})^* = C_\Phi \tag{3.16}$$

Remark that the transpose map $\theta_n \in \mathcal{B}(M_n(\mathbb{C}))$ is $(1,0)$-unitarily equivariant, so that for all $U \in \mathcal{U}_n$ and all $C \in M_n(\mathbb{C})^{\otimes a+b+1}$,

(3.17)
$$(\theta_n^{\otimes a+1} \otimes i_n^{\otimes b})((\overline{U}^{\otimes a+1} \otimes U^{\otimes b})\,C\,(\overline{U}^{\otimes a+1} \otimes U^{\otimes b})^*) = U^{\otimes k+1}\,[(\theta_n^{\otimes a+1} \otimes i_n^{\otimes b})C]\,U^{*\otimes k+1}\,.$$

where $k = a + b$. Applying Equations (3.16) and (3.17), we get:

$$(\theta_n^{\otimes a+1} \otimes i_n^{\otimes b})[C_\Phi] = U^{\otimes k+1}\,(\theta_n^{\otimes a+1} \otimes i_n^{\otimes b})[C_\Phi]\,U^{*\otimes k+1}\,.$$

Consequently, $(\theta_n^{\otimes a+1} \otimes i_n^{\otimes b})[C_\Phi] \in \rho_k(\mathcal{U}_n)'$. By Theorem 1.3.31, we obtain that $(\theta_n^{\otimes a+1} \otimes i_n^{\otimes b})[C_\Phi] \in \sigma_{k+1}(S_{k+1})$, which implies (iii).
(iii) $\Rightarrow$ (ii) is straightforward using the converse part of Theorem 1.3.31 and the previous computation. $\square$

If $n \leq k-1$ then the representation σ_k of S_k is not faithful, but this is not the case here as $n \geq a+b$. Therefore, by point (iii) in Theorem 3.3.1, the set of all (a,b)-unitarily equivariant maps is isomorphic as a vector space to the space $\sigma_{k+1}(S_{k+1})$. Therefore we get the straightforward corollary:

Corollary 3.3.2. *Let $n \geq a+b$. Then the dimension of the vector space of (a,b)-unitarily equivariant maps is $(k+1)!$, where $k = a+b$. A basis for this vector space is given by the maps $\{\Phi_\pi : \pi \in S_{k+1}\}$, with the corresponding Choi matrices:*

$$C_{\Phi_\pi} = (\theta_n^{\otimes a+1} \otimes i_n^{\otimes b})[\sigma_{k+1}(\pi)]\,, \tag{3.18}$$

where the matrix $\sigma_{k+1}(\pi)$ acts on vectors $(v_1 \otimes \cdots \otimes v_{k+1})$ as

$$\sigma_{k+1}(\pi)\,(v_1 \otimes \cdots \otimes v_{k+1}) = v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(k+1)}\,. \tag{3.19}$$

3.4. Graphical representation of the Choi matrix of (a,b)-unitarily equivariant maps. *In this section we study the graphical representation of (a,b)-unitarily equivariant maps for any $a, b \in \mathbb{N}$. This gives a visual and very convenient method to compute them. We illustrate this with the $(1,1)$-unitarily equivariant maps studied in Chapter 2 (Definition 2.2.1).*
Let $W_{a,b}$ denote the vector space of all (a,b)-unitarily equivariant maps. Recall from Section 3.3 that the maps $C_{\Phi_\pi} = (\theta_n^{\otimes a+1} \otimes i_n^{\otimes b})(\sigma_{k+1}(\pi))$ form a basis of $W_{a,b}$ with π running through the permutation group S_{k+1}, where $\sigma_{k+1}(\pi)$ is the unitary representation of the permutation group defined in Equation (3.19). Based on the graphical representations given in Table 1.4, we give the graphical representations of the C_{Φ_π}. We first consider the case of $a=1$ and $b=1$ and make some observations. We discuss the general case in Theorem 3.4.1.

We compute the graphical representation of the Choi matrices C_{Φ_π} with $\pi \in S_3$, by first computing the one of $\sigma_3(\pi)$. We start with the representation of $\sigma_3(123)$ as an example. This map is defined as follows:

$$\begin{array}{cccc} \sigma_3(123): & (\mathbb{C}^n)^{\otimes 3} & \to & (\mathbb{C}^n)^{\otimes 3} \\ & v_1 \otimes v_2 \otimes v_3 & \mapsto & v_3 \otimes v_1 \otimes v_2 \end{array}.$$

Its graphical representation is given on the left in Figure 3.1. By Equation (3.18), $C_{\Phi_{(123)}} = (\theta_n^{\otimes 2} \otimes i_n)\sigma_3(123)$. Using the graphical representation of the transpose map, we obtain that $C_{\Phi_{(123)}}$ is represented by the right figure in Figure 3.1.

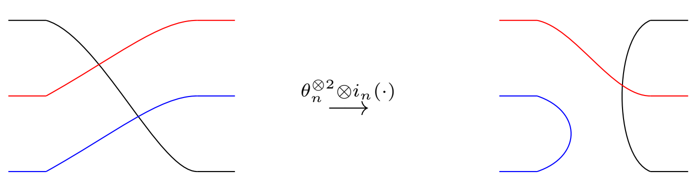


FIGURE 3.1. Graphical representation of $\sigma_3(123)$ (on the left) and $C_{\Phi_{(123)}}$ (on the right)

We take the following convention in order to represent the operators $\sigma_{k+1}(\pi)$: input on the left are represented by black (full) dots while output on the right are represented by white (empty) dots. The graphical representation of the operator $\sigma_{k+1}(\pi)$ then corresponds to tracing a wire from the ith (black) dot on the left to

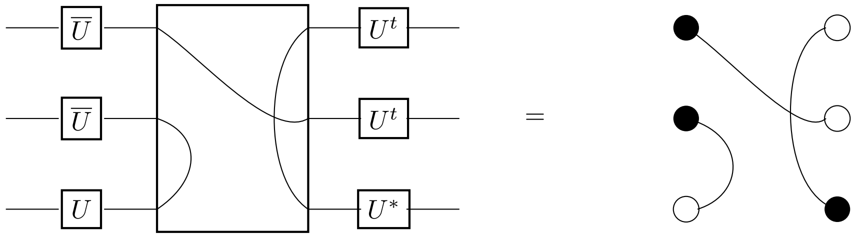


FIGURE 3.2. Graph of $C_{\Phi_{(123)}}$. Each black dot is connected to a white dot in a one-to-one way, according to the permutation (123).

the $\pi^{-1}(i)$ *(white) dot on the right. Using graphical computation, it can be directly checked that* $\sigma_{k+1}(\pi)$ *and* $U^{\otimes k+1}$ *commute for all unitary operator* $U \in \mathcal{U}_n$*.*
Then, applying the transpose map to the first $a+1$ *tensors corresponds graphically to interchange the black and the white dot at the first* $a+1$ *rows (starting from the highest). Again, it can be directly check from a graphical computation that* $C_{\Phi(\pi)}$ *and* $(\overline{U}^{\otimes a+1} \otimes U^{\otimes b})$ *commute. This is illustrated in the case of the permutation* (123) *in Figure 3.2.*

We can obtain similarly the graphical representations of the Choi matrices C_{Φ_π} *corresponding to all* $\pi \in S_3$*. There are* $3! = 6$ *different possibilities to trace a wire between black and white dots in a one-to-one way. Each of them corresponds to a different permutation* $\pi \in S_3$*. This is illustrated in Figure 3.3.*

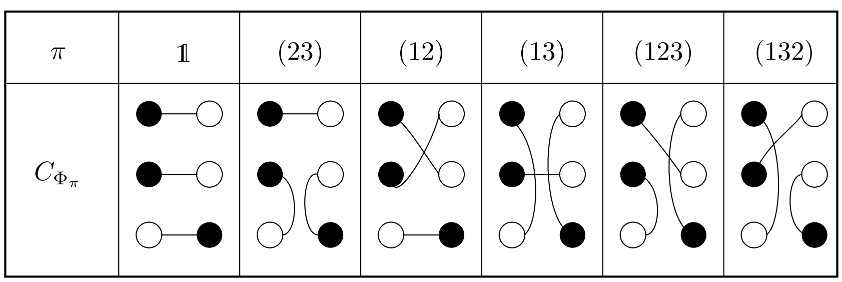


FIGURE 3.3. Graphical representations of the C_{Φ_π} for each $\pi \in S_3$

Figure 3.3 gives the basis elements of the vector space $W_{1,1}$*. Theorem 3.4.1 generalizes this fact to any* (a, b)*-unitarily equivariant map. We omit the proof, as it follows exactly the same idea as the example above.*

Theorem 3.4.1. *The graphical representation of the Choi matrices of the basis elements* $C_{\Phi(\pi)}$ *of the vector space* $W_{a,b}$ *is given by the following rule. Being a matrix on* $(\mathbb{C}^n)^{\otimes k+1}$*, its graphical representative has* $k+1$ *input (on the left) and* $k+1$ *output (on the right). The first* $a+1$ *inputs are symbolized by black (full) dots, the remaining* b *by white (empty) dots. In the same way, the first* $a+1$ *outputs are symbolized by white (empty) dots, the remaining* b *by black (full) dots. Then each element of this basis corresponds to possible wiring between a black and white dot, in a one-to-one way.*

Theorem 3.4.1 gives a graphical representation of the Choi matrices of the basis elements of $W_{a,b}$*, which completes the characterization of the vector space* $W_{a,b}$ *of all* (a, b)*-unitarily equivariant maps. As an illustration, we study the graphical representation of the Choi matrix of the family of linear maps* $\Phi_{\alpha,\beta,3}$ *studied in defined in Definition 2.2.1. It is given as follows:*

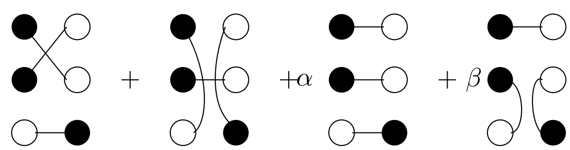


FIGURE 3.4. Graphical representation of $C_{\Phi_{\alpha,\beta,n}}$

Remark that it corresponds to taking linear combinations of the graphical representations of the permutations (12)*,* (13)*,* (1) *and* (23) *respectively and from left to right.*

Conclusion. In this chapter, we studied two sub-classes of $V(U)$-equivariant linear maps, which are unitarily equivariant maps and (a, b)-unitarily equivariant maps. We tried to understand the map $U \mapsto V(U)$ in Equation (3.1) and we deduced that if it becomes a unitary representation, then every unitarily equivariant map is "a corner" of (a, b)-unitarily equivariant map for some $a, b \in \mathbb{N}$. Finally, we give a characterization of (a, b)-unitarily equivariant maps and study their graphical representation.

This characterization may help to solve the question of decomposability of equivariant linear maps and it has many applications in quantum information theory since decomposable maps can be used to detect PPT entangled states. In the next chapter, we consider equivariant linear maps to entanglement detection.

CHAPTER 4

Further examples of equivariant maps and their applications

Introduction

This chapter deals with the applications of equivariant maps to detect entanglement. We prove that any entangled density matrix can be detected by a unitarily equivariant map with the image on infinite-dimensional Hilbert space. We conjecture that this latter requirement is not necessary and the image can be taken finite-dimensional. We give an example of $(2, 0)$-unitarily equivariant map and analyze its properties.

The main new results of this chapter

- Applications of equivariant maps to detect entanglement (Theorem 4.1.1).
- An example of $(2, 0)$-unitarily equivariant map (Section 4.4).

4.1. Application to entanglement detection. The main result of this section is the following.

Theorem 4.1.1. *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a positive - but not completely positive - map. Then there exists a unitarily equivariant map $\Phi : M_n(\mathbb{C}) \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ such that:*

$$\mathcal{S}_m(\Phi) \subset \mathcal{S}_m(\phi)\,. \tag{4.1}$$

where

$$\mathcal{S}_m(\psi) = \{\rho \in \mathcal{S}_{(m\times n)}\,;\,(i_m \otimes \psi)(\rho) \geq 0\}\,.$$

It means that all entangled density matrices that are detected by ϕ are also detected by the unitarily equivariant map Φ.

We formulated the previous theorem in terms of the set $\mathcal{S}_m(\phi)$. As a corollary we obtain the weaker statement:

Corollary 4.1.2. *Any entangled density matrix $\rho \in \mathcal{S}_{(m\times n)}$ can be detected thanks to a unitarily-equivariant positive map $\Phi : M_n \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

The fact that the Hilbert space $\mathcal{H}$ is infinite dimensional is a huge drawback in practice. To our knowledge, it is not clear if this can be improved. We shall however discuss in Section 4.3 applications of this conjecture, based on Theorem 3.2.1 and Corollary 3.2.3.

4.2. Proof of Theorem 4.1.1.

PROOF. Let $\rho \in \mathcal{S}_{(m\times n)} \setminus \mathrm{SEP}(m,n)$ be an entangled density matrix and let $\phi \in \mathcal{B}(M_n(\mathbb{C}), M_n(\mathbb{C}))$ be a positive map such that $(i_m \otimes \phi)(\rho)$ is not positive semi-definite, as provided by the Horodecki Theorem [**53**]. Define $\mathcal{H} = L_2(\mathcal{U}_n, \mathbb{C}^n)$, that is,

$$\mathcal{H} = \left\{ (\psi_U)_{U\in\mathcal{U}_n} \,;\, \psi_U \in \mathbb{C}^n \ \forall U \in \mathcal{U}_n \,, \quad \int_{\mathcal{U}_n} \|\psi_U\|^2 \mu(dU) < +\infty \right\}$$

where the integration is with respect to the Haar measure on $\mathcal{U}_n$. We can define a map $\Phi \in \mathcal{B}(M_n(\mathbb{C}), \mathcal{B}(\mathcal{H}))$ as

$$X \in M_n(\mathbb{C}) \mapsto \Phi(X) = (\phi(UXU^*))_{U\in\mathcal{U}_n} \ .$$

Thus for all $X \in M_n(\mathbb{C})$ and all $(\psi_U)_{U\in\mathcal{U}_n} \in \mathcal{H}$, $\Phi(X)(\psi_U)_{U\in\mathcal{U}_n} = (\phi(UXU^*)\psi_U)_{U\in\mathcal{U}_n}$. Define the following unitary representation of $\mathcal{U}_n$ on $\mathcal{H}$:

$$V : W \mapsto \big(V(W) : \psi \in \mathcal{H} \mapsto (\psi_{UW})_{U\in\mathcal{U}_n}\big) \,.$$

Then we check that the map Φ is unitarily equivariant with respect to $U \mapsto V(U)$. Indeed, for all $W \in \mathcal{U}_n$, $X \in M_n(\mathbb{C})$ and all $\psi \in \mathcal{H}$,

$$\begin{aligned} V(W)^*\, \Phi(WXW^*)\, V(W)\, \psi &= V(W)^*\, \Phi(WXW^*)\, (\psi_{UW})_{U\in\mathcal{U}_n} \\ &= V(W)^*\, (\phi(UW\, X\, (UW)^*)\psi_{UW})_{U\in\mathcal{U}_n} \\ &= \Phi(X)\psi \,. \end{aligned}$$

We now show that Φ "detects" ρ, that is, $(i_m \otimes \Phi)(\rho)$ is not positive semi-definite. Indeed, this is true as by continuity there exist a vector $\varphi \in \mathbb{C}^m \otimes \mathbb{C}^n$ and a small neighbourhood $\mathcal{U}$ of I_n in $\mathcal{U}_n$ such that for all $U \in \mathcal{U}$, $\langle \varphi \,, (i_m \otimes \phi)(I_m \otimes U\, \rho\, I_m \otimes U^*))\varphi \rangle < 0$. Then defining the vector $\psi \in \mathbb{C}^m \otimes \mathcal{H}$ as:

$$\psi_U = \varphi \quad \forall U \in \mathcal{U}\,, \qquad \psi_U = 0 \quad \text{elsewhere} \ ,$$

we get that $\langle \psi \,, (i_m \otimes \Phi)(\rho)\psi \rangle < 0$ which shows that $(i_m \otimes \Phi)(\rho)$ is not positive semi-definite. □

Remark 4.2.1. *In the previous proof, the construction of the Hilbert space $\mathcal{H}$ and the representation π does not depend on ϕ. So is also the case for the unitary representation $U \mapsto V(U)$: it means that its decomposition into irreducible representations does not depend on ϕ, as we should expect!*

4.3. Discussion on (a, b)-unitarily equivariant maps. We discuss in this section the applications to entanglement detection of the following conjecture:

Conjecture 4.3.1. *For any entangled density matrix ρ on $(\mathbb{C}^m \otimes \mathbb{C}^n)$, there exists a unitarily equivariant and a positive map $\Phi \in \mathcal{B}(M_n(\mathbb{C}), M_N(\mathbb{C}))$ for some natural integer N such that the map $U \mapsto V(U)$ in Equation (3.1) can be assumed to be an irreducible unitary representation, and such that $(i_m \otimes \Phi)(\rho)$ is not positive semi-definite.*

Remark 4.3.2. *Assume that Conjecture 4.3.1 is true.*

- *It allows to reduce the problem of entanglement detection to the case of (a, b)-unitarily equivariant positive maps.*
- *We have entirely characterized the (a, b)-unitarily equivariant maps in Section 3.3. As they are equivariant, it is a simple computation to know which of them are positive. Partial results in this direction were obtained in Chapter 2, in the case $a = b = 1$.*

To conclude, the conjecture implies that the study of entanglement detection can be entirely reduced to the study of (a, b)-unitarily equivariant maps studied in Section 3.3 and Section 3.4. This would give a new and explicit way to detect if a density matrix is entangled or not.

4.4. An example of $(2, 0)$-unitarily equivariant map. From Theorem 4.1.1, it becomes clear that the study of entanglement detectors (positive and not completely positive maps) can be completed by studying equivariant maps or more precisely, (a, b)-unitarily equivariant maps. Therefore, it becomes important to have an abundance of examples of equivariant linear maps. In Chapter 2, we studied a family of $(1, 1)$-unitarily equivariant maps. In this section, we present a family of $(2, 0)$-unitarily equivariant maps.

Definition 4.4.1.

$$\begin{array}{rccl} \Psi_n : & M_n(\mathbb{C}) & \to & M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \\ & A & \mapsto & [\sigma]_\beta (A^t \otimes \mathbb{1}_n) + (A^t \otimes \mathbb{1}_n)[\sigma]_\beta + \mathrm{Tr}(A)(\mathbb{1}_{n^2} - [\sigma]_\beta)\,. \end{array} \tag{4.2}$$

Here σ denotes the swap operator in $\mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^n)$ and β is a canonical basis of $(\mathbb{C}^n \otimes \mathbb{C}^n)$.

Lemma 4.4.2. *The linear map Ψ_n is $(2, 0)$-unitarily equivariant.*

Proof. Let $U \in M_n(\mathbb{C})$ be an arbitrary unitary matrix. Consider,

$$\begin{aligned} \Psi_n(UXU^*) &= [\sigma]_\beta ((UXU^*)^t \otimes \mathbb{1}_n) + ((UXU^*)^t \otimes \mathbb{1}_n)[\sigma]_\beta + \mathrm{Tr}(UXU^*)(\mathbb{1}_{n^2} - [\sigma]_\beta) \\ &= [\sigma]_\beta (\overline{U} X^t U^t \otimes \mathbb{1}_n) + (\overline{U} X^t U^t \otimes \mathbb{1}_n)[\sigma]_\beta + \mathrm{Tr}(X)(\mathbb{1}_{n^2} - [\sigma]_\beta) \end{aligned}$$

Chose $V = (\overline{U} \otimes \overline{U})$ and an easy calculation shows that :

$$\begin{aligned} (\overline{U} \otimes \overline{U})\Psi_n(X)(\overline{U} \otimes \overline{U})^* &= (\overline{U} \otimes \overline{U})([\sigma]_\beta (X^t \otimes \mathbb{1}_n) + (X^t \otimes \mathbb{1}_n)[\sigma]_\beta + (\mathbb{1}_{n^2} - [\sigma]_\beta)\mathrm{Tr}(X))) \\ &\quad (\overline{U} \otimes \overline{U})^* \\ &= [\sigma]_\beta (\overline{U} X^t U^t \otimes \mathbb{1}_n) + (\overline{U} X^t U^t \otimes \mathbb{1}_n)[\sigma]_\beta + (\mathbb{1}_{n^2} - [\sigma]_\beta)\mathrm{Tr}(X) \end{aligned}$$

Hence, $\Psi_n(UXU^*) = (\overline{U} \otimes \overline{U})\Psi_n(X)(\overline{U} \otimes \overline{U})^*$ for every unitary $U \in M_n(\mathbb{C})$. $\square$

Theorem 4.4.3. *The linear map Ψ_n is positive and not completely positive.*

Proof. We supply two proofs of this theorem. The first proof is based on the calculation and the second proof is more abstract.

First proof.

(i) To prove the positivity of Ψ_n, we will use Proposition 2.1.2. $\Psi_n(e_{11})$ is given by the following matrix.

$$\begin{bmatrix} 2e_{11} + \sum_{i=2}^n e_{ii} & 0_n & 0_n & 0_n & \dots & 0_n \\ 0_n & \sum_{\substack{i=1 \\ i\neq 2}}^n e_{ii} & -e_{32} & -e_{42} & \dots & -e_{n2} \\ 0_n & -e_{23} & \sum_{\substack{i=1 \\ i\neq 3}}^n e_{ii} & -e_{43} & \dots & -e_{n3} \\ 0_n & -e_{24} & -e_{34} & \sum_{\substack{i=1 \\ i\neq 4}}^n e_{ii} & \dots & -e_{n4} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0_n & -e_{2n} & -e_{3n} & -e_{4n} & \dots & \sum_{i=1}^{n-1} e_{ii} \end{bmatrix}$$

Above matrix is positive since all its minors are positive. For any $n \in \mathbb{N}$, $\Psi_n(e_{11})$ can be seen as a block diagonal matrix by interchanging two rows and the corresponding columns. Hence, Ψ_n is a positive.

(ii) We show that Ψ_n is not completely positive. We consider the Choi matrix of Ψ_n, C_{Ψ_n} and show that the Choi matrix is not positive definite.

For notational convenience, consider $\Psi_n(e_{ij}) = A_{ij} + B_{ij} + C_{ij}$. where $A_{ij} = [\sigma]_\beta(e_{ij}^t \otimes \mathbb{1}_n), B_{ij} = (e_{ij}^t \otimes \mathbb{1}_n)[\sigma]_\beta$ and $C_{ij} = (\mathbb{1}_{n^2} - [\sigma]_\beta)\mathrm{Tr}(e_{ij})$.
For $1 \le i, j \le n,\ A_{ij}, B_{ij}$ and C_{ij} are given by following matrices.

$$A_{ij} = \begin{array}{c} \begin{bmatrix} 0_n & 0_n & \cdots & 0_n & e_{j1} & 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n & e_{j2} & 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n & e_{j3} & 0_n & 0_n & \cdots & 0_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_n & 0_n & \cdots & 0_n & e_{j,n-1} & 0_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n & e_{jn} & 0_n & 0_n & \cdots & 0_n \end{bmatrix} \\ \qquad \underbrace{\hspace{9em}}_{i-1} \quad \underbrace{\hspace{2em}}_{i} \quad \underbrace{\hspace{9em}}_{n-i} \end{array}$$

$$B_{ij} = \begin{array}{cl} \begin{bmatrix} 0_n & 0_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & 0_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{1i} & e_{2i} & e_{3i} & \cdots & \cdots & \cdots & \cdots & \cdots & e_{ni} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_n & 0_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & 0_n \end{bmatrix} & \begin{array}{l} \left.\vphantom{\begin{matrix} 0 \\ \vdots \end{matrix}}\right\} j-1 \\ \left.\vphantom{e}\right\} j \\ \left.\vphantom{\begin{matrix} \vdots \\ 0 \end{matrix}}\right\} n-j \end{array} \end{array}$$

$$C_{ij} = \begin{cases} 0_{n^2}, & \text{if } i \neq j \\ C_{kk}, & \text{if } i = j = k \end{cases}$$

and C_{kk} is given by the following matrix

$$C_{kk} = \begin{bmatrix} \sum\limits_{t=2}^{n} e_{tt} & -e_{21} & -e_{31} & \cdots & -e_{n,1} \\ -e_{1,2} & \sum_{\substack{t=1 \\ t\neq 2}}^{n} e_{tt} & -e_{32} & \cdots & -e_{n,2} \\ -e_{1,3} & -e_{23} & \sum_{\substack{t=1 \\ t\neq 3}}^{n} e_{tt} & \cdots & -e_{n,3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -e_{1,n} & -e_{2n} & -e_{3n} & \cdots & \sum_{t=1}^{n-1} e_{tt} \end{bmatrix}$$

$$\Psi_n(e_{ij}) = \begin{cases} A_{ij} + B_{ij} & \text{if } i \neq j \\ A_{kk} + B_{kk} + C_{kk}, & \text{if } i = j = k \end{cases}$$

We show the existence of a vector $x \in (\mathbb{C}^n)^{\otimes 3}$ such that $\langle C_{\Psi_n} x, x \rangle \ngeq 0$. For any $n \in \mathbb{N},\ x$ is given by:

$$x = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -2 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} n^3-(n^2+1) \text{ times} \\ \left.\vphantom{0}\right\} \text{one time} \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} n^2-(n+1) \text{ times} \\ \left.\vphantom{0}\right\} \text{one time} \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} (n-2) \text{ times} \\ \left.\vphantom{0}\right\} \text{one time} \\ \left.\vphantom{0}\right\} \text{one time} \end{array}$$

A simple calculation shows that $C_{\Psi_n}(x) = -x$. Therefore, vector x is the eigenvector of the matrix C_{Ψ_n} corresponding to eigenvalue -1.

Therefore,

$$\begin{aligned} \langle C_{\Psi_n} x, x\rangle &= \langle -x, x\rangle \\ &\ngeq 0. \end{aligned}$$

Therefore, Ψ_n is not completely positive.

$\square$

Second proof. We now give a more abstract proof of positivity and completely positivity of the map Ψ_n. It has already been proved that linear map Ψ_n is $(2,0)$-unitarily equivariant (Lemma 4.4.2). All we need to prove is $\Psi_n(e_{11})$ is positive. Consider,

$$\begin{aligned} \Psi_n(e_{11}) &= \sum_{i,j=1}^{n} (e_{ij}\otimes e_{ji})(e_{11}\otimes \mathbb{1}_n) + \sum_{i,j=1}^{n} (e_{11}\otimes \mathbb{1}_n)(e_{ij}\otimes e_{ji}) + (\mathbb{1}_{n^2} - [\sigma]_\beta) \\ &= \sum_i e_{i1}\otimes e_{1i} + \sum_j e_{1j}\otimes e_{j1} + (\mathbb{1}_{n^2} - [\sigma]_\beta) \\ &= e_{11}\otimes e_{11} + (\mathbb{1}_{n^2} - \sum_{\substack{i,j \\ i,j\neq 1}} e_{ij}\otimes e_{ji}) \end{aligned}$$

We claim that the operator $e_{11}\otimes e_{11} + (\mathbb{1}_{n^2} - \sum_{\substack{i,j \\ i,j\neq 1}} e_{ij}\otimes e_{ji})$ is positive. By a calculation, the smallest eigenvalue of $(e_{11}\otimes e_{11} - \sum_{\substack{i,j \\ i,j\neq 1}} e_{ij}\otimes e_{ji})$ is -1. Therefore, all the eigenvalues of $(\mathbb{1}_{n^2} + e_{11}\otimes e_{11} - \sum_{\substack{i,j \\ i,j\neq 1}} e_{ij}\otimes e_{ji})$ are positive. Hence, $\Psi_n(e_{11})$ is positive.

We now prove that the linear map Ψ_n is not completely positive. We will show that its Choi matrix is not positive. The Choi matrix of Ψ_n is given by,

$$\begin{aligned}
C_{\Psi_n} &= (\mathbb{1}_n \otimes \Psi_n)(\sum_{i,j=1}^{n} e_{ij} \otimes e_{ij}) \\
&= \sum_{i,j} e_{ij} \otimes \Psi_n(e_{ij}) \\
&= \sum_{i,j,k} e_{ij} \otimes e_{ki} \otimes e_{jk} + \sum_{i,j,l} e_{ij} \otimes e_{jl} \otimes e_{li} + \sum_{i,j} e_{ij} \otimes \mathrm{Tr}(e_{ij})(\mathbb{1}_{n^2} - [\sigma]_\beta) \\
&= \sum_{i,j,k} e_{ij} \otimes e_{ki} \otimes e_{jk} + \sum_{i,j,l} e_{ij} \otimes e_{jl} \otimes e_{li} + \sum_{i} e_{ii} \otimes (\mathbb{1}_{n^2} - [\sigma]_\beta)
\end{aligned}$$

We will prove that -1 is a negative eigenvalue and the corresponding eigenvector is $x = -2e_{n-2} \otimes e_{n-1} \otimes e_{n-1} + e_{n-1} \otimes e_{n-2} \otimes e_{n-1} + e_{n-1} \otimes e_{n-1} \otimes e_{n-2}$.

By a simple calculation,

$$\begin{aligned}
C_{\Psi_n}(x) &= -2e_{n-1} \otimes e_{n-1} \otimes e_{n-2} + e_{n-2} \otimes e_{n-1} \otimes e_{n-1} + e_{n-1} \otimes e_{n-2} \otimes e_{n-1} \\
&\quad - 2e_{n-1} \otimes e_{n-2} \otimes e_{n-1} + e_{n-1} \otimes e_{n-1} \otimes e_{n-2} + e_{n-2} \otimes e_{n-1} \otimes e_{n-1} \\
&= -e_{n-1} \otimes e_{n-1} \otimes e_{n-2} - e_{n-1} \otimes e_{n-2} \otimes e_{n-1} + 2e_{n-2} \otimes e_{n-1} \otimes e_{n-1} \\
&= -(-2e_{n-2} \otimes e_{n-1} \otimes e_{n-1} + e_{n-1} \otimes e_{n-2} \otimes e_{n-1} + e_{n-1} \otimes e_{n-1} \otimes e_{n-2}) \\
&= -x
\end{aligned}$$

Hence, C_{Ψ_n} is not positive, proving that Ψ_n is not completely positive.

Theorem 4.4.4. *The map $\Psi_n (n = 2, 3)$ is completely copositive.*

PROOF. $(n = 2)$: By a calculation, the characteristic polynomial of the Choi matrix of $(\Psi_2 \circ \theta_2)$ is $\lambda^4(\lambda - 1)^2(\lambda - 3)^2$.
$(n = 3)$: By a calculation, the characteristic polynomial of the Choi matrix of $(\Psi_3 \circ \theta_3)$ is $\lambda^{18}(\lambda - 2)^6(\lambda - 4)^3$.
Therefore maps $(\Psi_2 \circ \theta_2)$ and $(\Psi_3 \circ \theta_3)$ is completely positive. Hence, Ψ_2 and Ψ_3 are completely copositive.

□

Conclusion. In this chapter, we applied unitarily equivariant maps to the study of entanglement detection. We conjectured that any entangled density matrix can be detected by a unitarily equivariant positive map where the map $U \mapsto V(U)$ comes from a unitary representation of the unitary group. Using representation theory, we showed that it implies that the (a, b)-unitarily equivariant positive maps are enough to detect all entangled density matrices. In order to support our conjecture, we proved it when the image of the equivariant map is infinite dimensional. The problem of detecting entanglement is an NP-hard problem [**2**]. Positive and not completely positive maps help to detect entanglement. Therefore, it is of significant interest to have families of positive linear maps. However, the $(2, 0)$-unitarily equivariant map presented in this chapter turns out to be completely copositive (for $n = 2, 3$) and hence decomposable.

CHAPTER 5

Conclusion and further directions

Introduction

In this chapter, we give a necessary condition of decomposability of (a, b)-unitarily equivariant maps (Lemma 5.0.1). We also provide a region of decomposability of the family of linear maps $\Phi_{\alpha,\beta,3}$ studied in chapter 2 (Proposition 5.0.3). We conclude this chapter by summarizing the problem of 2-positivity and decomposability.

The main new results of this chapter

- Decomposability of (a, b)-unitarily equivariant linear maps (Lemma 5.0.1).
- Decomposability of $\Phi_{\alpha,\beta,3}$ (Proposition 5.0.3).

A necessary condition of decomposability of equivariant linear maps

We give a lemma which shows that for an equivariant decomposable map its two components (that is two CP maps in its decomposition) can be chosen as equivariant.

Lemma 5.0.1. *Let Φ be an (a, b)-unitarily equivariant map. Assume that Φ is positive and decomposable. Then there exist two completely positive maps $\Psi_1, \Psi_2 : M_n(\mathbb{C}) \to M_n(\mathbb{C})^{a+b}$ such that Ψ_1 and $\Psi_2 \circ \theta_n$ are (a, b)-unitarily equivariant and*

$$\Phi = \Psi_1 + \Psi_2 \circ \theta_n \,.$$

Proof. If Φ is positive and decomposable, there exist two completely positive maps $\Psi_1, \Psi_2 : M_n(\mathbb{C}) \to M_n(\mathbb{C})^{\otimes a+b}$ such that $\Phi = \Psi_1 + \Psi_2 \circ \theta_n$. We shall work at the level of their Choi matrices: we thus have $C_\Phi = C_{\Psi_1} + C_{\Psi_2\circ\theta_n}$. It is a direct computation to check that $C_{\Psi_2\circ\theta_n} = (\theta_n \otimes i_{n^{a+b}})C_{\Psi_2}$.
Define $\mathcal{E}$ to be the conditional expectation on the commutant of the representation $U \mapsto \overline{U}^{\otimes a+1} \otimes U^{\otimes b}$. It is given by the explicit formula:

$$\mathcal{E}(X) = \int_U \overline{U}^{\otimes a+1} \otimes U^{\otimes b}\, X\, (\overline{U}^{\otimes a+1} \otimes U^{\otimes b})^*\, dU\,, \qquad \forall X \in M_n(\mathbb{C})^{\otimes 3}\,,$$

where the integration is with respect to the Haar measure on $\mathcal{U}_n$. Define $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ to be the two maps from $M_n(\mathbb{C})$ to $M_n(\mathbb{C})^{\otimes a+b}$ with respective Choi matrices

$$C_{\tilde{\Psi}_1} = \mathcal{E}(C_{\Psi_1})\,, \qquad C_{\tilde{\Psi}_2} = \int_U U \otimes \overline{U}^{\otimes a} \otimes U^{\otimes b}\, C_{\psi_2}\, (U \otimes \overline{U}^{\otimes a} \otimes U^{\otimes b})^*\, dU\,.$$

$C_{\tilde{\Psi}_1}$ and $C_{\tilde{\Psi}_2}$ are the image of positive semi-definite matrices by conditional expectations, therefore they are positive semi-definite. It shows that $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ are completely positive. Furthermore, by Theorem 3.1.4 we have that $\tilde{\Psi}_1$ is (a, b)-unitarily equivariant and a simple computation shows that $(\theta_n \otimes i_{n^{a+b}})C_{\tilde{\Psi}_2} = \mathcal{E}\left(C_{\tilde{\Psi}_2\circ\theta_n}\right)$, so

that the same theorem implies that $\tilde{\Psi}_2 \circ \theta_n$ is also (a,b)-unitarily equivariant.
By Theorem 3.1.4, C_Φ commutes with $(\overline{U}^{\otimes a+1} \otimes U^{\otimes b})$ for all $U \in \mathcal{U}_n$ and therefore $\mathcal{E}(C_\Phi) = C_\Phi$. We therefore get

$$\begin{aligned} C_\Phi &= \mathcal{E}(C_{\Psi_1}) + \mathcal{E}(C_{\Psi_2 \circ \theta_n}) \\ &= C_{\tilde{\Psi}_1} + \mathcal{E}\left((\theta_n \otimes i_{n^2}) C_{\Psi_2}\right) \\ &= C_{\tilde{\Psi}_1} + (\theta_n \otimes i_{n^2}) C_{\tilde{\Psi}_2} \\ &= C_{\tilde{\Psi}_1} + C_{\tilde{\Psi}_2 \circ \theta_n} \,. \end{aligned}$$

It shows that $\Phi = \tilde{\Psi}_1 + \tilde{\Psi}_2 \circ \theta_n$, which concludes the proof. $\square$

Decomposability of $\Phi[\alpha,\beta]$. In this section, we give sufficient conditions of decomposability of $(1,1)$-unitarily equivariant linear maps (studied in Chapter 2). We recall the definition:

Definition 5.0.2. Let α and β be two real numbers and $n \geq 3$. Then the family of maps,

$$\begin{array}{rccc} \Phi_{\alpha,\beta,n}: & M_n(\mathbb{C}) & \to & M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \\ & A & \mapsto & A^t \otimes \mathbb{1}_n + \mathbb{1}_n \otimes A + \operatorname{Tr}(A)(\alpha \mathbb{1}_{n^2} + \beta \, \mathrm{Bell}) \,. \end{array} \tag{5.1}$$

Proposition 5.0.3. *The linear map $\Phi_{\alpha,\beta,3}$ is decomposable if $\alpha, \beta \geq 0$ or $\beta < 0$ and $\alpha \geq \dfrac{-(6+3\beta)+\sqrt{9\beta^2-28\beta+36}}{2}$.*

PROOF. Define $\Phi_1[\alpha,\beta] : M_3(\mathbb{C}) \to M_3(\mathbb{C}) \otimes M_3(\mathbb{C})$ given by

$$A \mapsto A^t \otimes \mathbb{1}_3 + \frac{\operatorname{Tr}(A)}{2}(\alpha \mathbb{1}_9 + \beta \, \mathrm{Bell}).$$

And

$\Phi_2[\alpha,\beta] : M_3(\mathbb{C}) \to M_3(\mathbb{C}) \otimes M_3(\mathbb{C})$ given by

$$A \mapsto \mathbb{1}_3 \otimes A + \frac{\operatorname{Tr}(A)}{2}(\alpha \mathbb{1}_9 + \beta \, \mathrm{Bell}).$$

Since,

$$C_{\Phi_1[\alpha,\beta]\circ\theta_3} = (\mathbb{1}_3 \otimes \sigma) C_{\Phi_2[\alpha,\beta]} (\mathbb{1}_3 \otimes \sigma)$$

where σ is the swap operator on $(\mathbb{C}^3 \otimes \mathbb{C}^3)$.
Therefore, $\Phi_1[\alpha,\beta]$ is completely copositive if and only if $\Phi_2[\alpha,\beta]$ is completely positive. The characteristic polynomial of $C_{\Phi_2[\alpha,\beta]}$ is given by,

$$\left(\lambda - \frac{\alpha}{2}\right)^{21} \left(\lambda - \left(\alpha + \frac{6+3\beta+\sqrt{9\beta^2-28\beta+36}}{2}\right)\right)^3 \left(\lambda - \left(\alpha + \frac{6+3\beta-\sqrt{9\beta^2-28\beta+36}}{2}\right)\right)^3$$

Therefore, $\Phi_1[\alpha,\beta]$ is completely copositive if and only if $\Phi_2[\alpha,\beta]$ is completely positive if and only if the following conditions hold.

(i) $\alpha \geq 0$,
(ii) $\alpha \geq \frac{-(6+3\beta)+\sqrt{9\beta^2-28\beta+36}}{2}$.

Observe that

$$\Phi_{\alpha,\beta,3} = \Phi_1[\alpha,\beta] + \Phi_2[\alpha,\beta].$$

Hence, we conclude that $\Phi_{\alpha,\beta,3}$ is decomposable if above two conditions hold. $\square$

Graphical representation of the region of decomposability. Figure (5.1) gives values of parameters α and β such that the family of linear maps $\Phi_{\alpha,\beta,3}$ is decomposable. The region of decomposability is shown by red vertical lines. It is evident from the Figure 5.1,

$$\begin{cases} \alpha \geq 0 & \text{if } \beta \geq 0, \\ \alpha \geq \frac{-(2+n\beta)+\sqrt{n^2\beta^2-4(n-2)\beta+4}}{2} & \text{if } \beta \leq 0. \end{cases}$$

For above values of parameters α and β, $\Phi_{\alpha,\beta,3}$ is completely positive and hence decomposable. The region with vertical lines in red colour gives the values of those parameters.
The remaining portion is where $\Phi_{\alpha,\beta,3}$ is decomposable and not completely positive, which is shown by black stars.

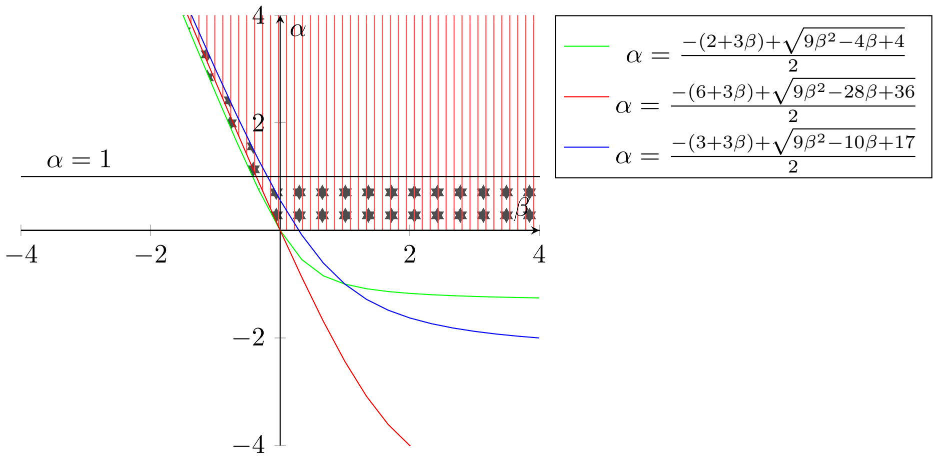


FIGURE 5.1. Region of decomposability

Further discussion of the necessary condition of decomposability and the problem of 2-positivity

We expect that the shaded region in Figure 5.1 is the only region of decomposability, that is, the condition given in Proposition 5.0.3 is necessary as well. In the following, we explain about the problem of 2-positivity and we analyse the difficulty in solving the problem.

Problem of 2-positivity. This question was raised in [**34**, Page 72] by Terhal. She showed the existence of indecomposable maps in $\mathcal{B}(M_m(\mathbb{C}), M_n(\mathbb{C}))$ which are not CP. It is natural to ask about the existence of a k-positive map in $\mathcal{B}(M_m(\mathbb{C}), M_n(\mathbb{C}))$ and $1 \leq k \leq m \wedge n$ but not decomposable. Obtaining an answer to this question may lead to getting a better understanding of the structure of positive linear maps. Some results are obtained for the case of $M_3(\mathbb{C})$. Yu Yang, Denny H. Leung, Wai-Shing Tang [**25**] proved that every 2-positive linear map from $M_3(\mathbb{C})$ to $M_3(\mathbb{C})$ is decomposable.
We hope to find a region where the map $\Phi_{\alpha,\beta,3}$ is 2-positive and indecomposable.

An attempt to solve the problem of 2-positivity and indecomposability: We believe that there exists an (a,b)-unitarily equivariant map which is 2-positive and not decomposable. By using the graphical representation of (a,b)-unitarily equivariant maps (Section 3.4), the general form of a $(1,1)$-unitarily equivariant map Φ (which is obtained by using Figure 3.3) is given as follows:

$$\begin{aligned}\Phi(A) =& a_1(A^t \otimes \mathbb{1}_3) + a_2(\mathbb{1}_3 \otimes A) + a_3 \mathrm{Tr}(A)\mathbb{1}_9 + a_4 \mathrm{Tr}(A)\,\mathrm{Bell} \\ &+ a_5\,\mathrm{Bell}(A^t \otimes \mathbb{1}_3) + a_6(A^t \otimes \mathbb{1}_3)\,\mathrm{Bell}\end{aligned} \tag{5.2}$$

for $\{a_1, a_2, a_3, a_4, a_5, a_6\} \subset \mathbb{R}$.

In Equation (5.2), Φ consists of 6 parameters. Therefore, it becomes difficult to analyse Φ.

As a suitable candidate, we may consider the family of parametric maps $\Phi_{\alpha,\beta,3}$ defined in Chapter 2. We have already proved that this family is an entanglement witness. By Theorem 2.2.10, we have the region of 2-positivity of this family of maps. To find the region of indecomposability of this family of maps, We may use Lemma 5.0.1. We need to obtain two $(1,1)$-unitarily equivariant CP maps Φ_1 and Φ_2 such that $\Phi_{\alpha,\beta,3} = \Phi_1 + \Phi_2 \circ \theta_3$. This condition of Φ_1 and Φ_2 helps to reduce the number of parameters they have. We hope to get a nice form of Φ_1 and Φ_2, which further reduces the complexity in obtaining the conditions of completely positivity.

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