

A note on the D -affinity of the flag variety in positive characteristic

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Let G be a simply connected simple algebraic group over an algebraically closed field $\mathbb{k}$ and let B be a Borel subgroup of G . Let $\mathfrak{X} = G/B$, $\mathcal{D}_{\mathfrak{X}}$ the sheaf of $\mathbb{k}$ -algebras of differential operators on $\mathfrak{X}$, $\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ the category of left $\mathcal{D}_{\mathfrak{X}}$ -modules that are quasi-coherent over the structure sheaf $\mathcal{O}_{\mathfrak{X}}$ of $\mathfrak{X}$, $\mathcal{D}(\mathfrak{X}) = \Gamma(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}})$ the $\mathbb{k}$ -algebra of differential operators on $\mathfrak{X}$, and $\mathcal{D}(\mathfrak{X})\mathbf{Mod}$ the category of left $\mathcal{D}(\mathfrak{X})$ -modules. We say $\mathfrak{X}$ is D -affine iff for each $\mathcal{M} \in \mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ (i) the natural morphism $\mathcal{D}_{\mathfrak{X}} \otimes_{\mathcal{D}(\mathfrak{X})} \Gamma(\mathfrak{X}, \mathcal{M}) \rightarrow \mathcal{M}$ is epic, and (ii) $H^i(\mathfrak{X}, \mathcal{M}) = 0 \ \forall i > 0$; equivalently, the functor $\Gamma(\mathfrak{X}, ?) : \mathcal{D}_{\mathfrak{X}}\mathbf{qc} \rightarrow \mathcal{D}(\mathfrak{X})\mathbf{Mod}$ gives an equivalence of categories with quasi-inverse $\mathcal{D}_{\mathfrak{X}} \otimes_{\mathcal{D}(\mathfrak{X})} ?$ (cf. [K98a, 1.6]).

In characteristic 0 a celebrated theorem of Beilinson and Bernstein [BB] affirms that $\mathfrak{X}$ is D -affine. In positive characteristic B. Haastert [H87, 4.4.1] shows that in (i) even the natural morphism

$$(1) \quad \mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{k}} \Gamma(\mathfrak{X}, \mathcal{M}) \rightarrow \mathcal{M} \text{ is epic.}$$

Then by Grothendieck's vanishing theorem (ii) will hold if $H^i(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}}) = 0 \ \forall i > 0$. If $(\text{Diff}_m)_{m \in \mathbb{N}}$ is the standard filtration of $\mathcal{D}_{\mathfrak{X}}$, however, [H87, 4.2.7] shows that if $p = \text{ch } \mathbb{k} > h$ the Coxeter number of G and if G is not of type A_1 , then

$$(2) \quad H^i(\mathfrak{X}, \text{Diff}_p) \neq 0 \text{ for some } i \neq 0.$$

And yet there is another filtration, called the p -filtration, on $\mathcal{D}_{\mathfrak{X}}$. If $\mathcal{O}_{\mathfrak{X}}^{(r)}$ is the sheaf of $\mathbb{k}$ -algebras such that $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}^{(r)}) = \{a^{p^r} | a \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})\}$ for each open $\mathfrak{U}$ of $\mathfrak{X}$ and if $\mathcal{D}_r = \text{Mod}_{\mathcal{O}_{\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}})$, then $\mathcal{D}_{\mathfrak{X}} = \bigcup_{r \in \mathbb{N}} \mathcal{D}_r$. As $\mathfrak{X}$ is noetherian,

$$(3) \quad H(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}}) \simeq \varinjlim_r H(\mathfrak{X}, \mathcal{D}_r).$$

Let $G_r = \ker F^r$ with $F^r : G \rightarrow G^{(r)}$ the r -th Frobenius morphism [J, I.9], $\hat{Z}_r$ the induction functor from the category $B\mathbf{Mod}$ of B -modules to the category $G_r B\mathbf{Mod}$ of $G_r B$ -modules [J, I.3], and let $\mathcal{L}$ be the functor from $B\mathbf{Mod}$ to the category of G -equivariant $\mathcal{O}_{\mathfrak{X}}$ -modules [J, I.5]. Then by [H87, 4.3.3]

$$(4) \quad \mathcal{D}_r \simeq \mathcal{L}(\hat{Z}_r(\mathbb{k})^*) \simeq \mathcal{L}(\hat{Z}_r(2(p^r - 1)\rho)),$$

where $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ with R^+ the positive system of roots of G such that the roots of B are $-R^+$. If $G = SL_2$ or SL_3 , then the composition factors of $\hat{Z}_r(2(p^r - 1)\rho)$ in $G_r B\text{Mod}$ have all dominant highest weights [H87, 4.5.4], hence $H^i(\mathfrak{X}, \mathcal{D}_r) = 0 \ \forall i > 0$ by Kempf's vanishing theorem, showing $\mathfrak{X}$ is D -affine in those cases. The argument unfortunately does not generalize.

There is another criterion for $\mathfrak{X}$ to be D -affine [Ka, Th. 1.4.1]: $\mathfrak{X}$ is D -affine iff there is a dominant weight λ such that for all $r \gg 0$ the natural morphism

$$(1) \quad \mathcal{D}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}(-r\lambda) \otimes_{\mathfrak{k}} H^0(r\lambda) \rightarrow \mathcal{D}_{\mathfrak{X}}$$

splits as a morphism of sheaves of abelian groups, where $H^0(?) = H^0(\mathfrak{X}, \mathcal{L}(?)) = \Gamma(\mathfrak{X}, \mathcal{L}(?))$. If $\text{Dist}(G)$ (resp. $\text{Dist}(B)$) is the algebra of distributions on G (resp. B), the natural morphism (5) can be described by the commutative diagram

$$(2) \quad \begin{array}{ccc} \mathcal{D}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}(-r\lambda) \otimes_{\mathfrak{k}} H^0(r\lambda) & \xrightarrow{\hspace{10em}} & \mathcal{D}_{\mathfrak{X}} \\ \sim \downarrow & & \downarrow \sim \\ \mathcal{L}(\text{Dist}(G) \otimes_{\text{Dist}(B)} (-r\lambda) \otimes_{\mathfrak{k}} H^0(r\lambda)) & \xrightarrow{\mathcal{L}(\text{Dist}(G) \otimes_{\text{Dist}(B)} (-r\lambda) \otimes_{\mathfrak{k}} \text{ev}_{r\lambda})} & \mathcal{L}(\text{Dist}(G)), \end{array}$$

where $\text{ev}_{r\lambda} : H^0(r\lambda) \rightarrow r\lambda$ is the evaluation at the identity element of G . In characteristic 0 the map $\text{Dist}(G) \otimes_{\text{Dist}(B)} (-r\lambda) \otimes_{\mathfrak{k}} \text{ev}_{r\lambda}$ has been proved to split in $B\text{Mod}$ so that $\mathcal{L}(\text{Dist}(G) \otimes_{\text{Dist}(B)} (-r\lambda) \otimes_{\mathfrak{k}} \text{ev}_{r\lambda})$ splits as a morphism of G -equivariant $\mathcal{O}_{\mathfrak{X}}$ -modules to show the D -affinity of $\mathfrak{X}$ [BB].

Assume in the following that $\text{ch } \mathfrak{k} = p > 0$. If $\mathfrak{X}$ is D -affine, in view of $1 \in \mathcal{D}(\mathfrak{X})$ we must have for a given r the morphism

$$(3) \quad \mathcal{D}_s \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}(-r\lambda) \otimes_{\mathfrak{k}} H^0(r\lambda) \rightarrow \mathcal{D}_s$$

splits as a morphism of sheaves of abelian groups for $s \gg 0$. By (4) the morphism (7) reads as

$$\mathcal{L}(\widehat{\text{ev} \otimes_{\mathfrak{k}} \text{ev}}) : \mathcal{L}(\hat{Z}_s(2(p^s - 1)\rho - r\lambda) \otimes_{\mathfrak{k}} H^0(r\lambda)) \rightarrow \mathcal{L}(\hat{Z}_s(2(p^s - 1)\rho)),$$

where $\widehat{\text{ev} \otimes_{\mathfrak{k}} \text{ev}} \in G_s B\text{Mod}(\hat{Z}_s(2(p^s - 1)\rho - r\lambda) \otimes_{\mathfrak{k}} H^0(r\lambda), \hat{Z}_s(2(p^s - 1)\rho))$ is induced by the Frobenius reciprocity from $\text{ev} \otimes_{\mathfrak{k}} \text{ev} \in B\text{Mod}(\hat{Z}_s(2(p^s - 1)\rho - r\lambda) \otimes_{\mathfrak{k}} H^0(r\lambda), 2(p^s - 1)\rho)$ the tensor product of evaluations $\text{ev}_{2(p^s - 1)\rho - r\lambda} : \hat{Z}_s(2(p^s - 1)\rho - r\lambda) \rightarrow 2(p^s - 1)\rho - r\lambda$ and $\text{ev}_{r\lambda} : H^0(r\lambda) \rightarrow r\lambda$.

Now $1 \in \mathcal{D}_s$ belongs to $\mathcal{O}_{\mathfrak{X}}$ and $\mathcal{O}_{\mathfrak{X}}$ is a direct summand of $\mathcal{D}_s$ as an $\mathcal{O}_{\mathfrak{X}}$ -module, in fact, as a G -equivariant $\mathcal{O}_{\mathfrak{X}}$ -module, corresponding to the splitting of the quotient $\pi : \hat{Z}_s(2(p^s - 1)\rho) \rightarrow \text{hd}_{G_s B} \hat{Z}_s(2(p^s - 1)\rho) = \mathfrak{k}$ in $B\text{Mod}$. Then we should have at least the composite

$$\begin{array}{ccc} H^0(\hat{Z}_s(2(p^s - 1)\rho - r\lambda) \otimes_{\mathfrak{k}} H^0(r\lambda)) & \dashrightarrow & \mathfrak{k} \\ & \searrow \text{H}^0(\widehat{\text{ev} \otimes_{\mathfrak{k}} \text{ev}}) & \uparrow \text{H}^0(\pi) \\ & & H^0(\hat{Z}_s(2(p^s - 1)\rho)) \end{array}$$

to be surjective, that we will verify in what follows.

We will suppress $\mathfrak{k}$ in $\otimes_{\mathfrak{k}}$. By the tensor identity we have a commutative diagram

$$\begin{array}{ccc} \hat{Z}_s(2(p^s - 1)\rho - r\lambda) \otimes_{\mathfrak{k}} H^0(r\lambda) & \xrightarrow{\widehat{\text{ev} \otimes \text{ev}}} & \hat{Z}_s(2(p^s - 1)\rho) \\ \sim \downarrow & \nearrow \hat{Z}_s((2(p^s - 1)\rho - r\lambda) \otimes \text{ev}) & \\ \hat{Z}_s((2(p^s - 1)\rho - r\lambda) \otimes H^0(r\lambda)) & & \end{array}$$

As $\text{ev} : H^0(r\lambda) \rightarrow r\lambda$ is surjective and as $\hat{Z}_s$ is exact, $\widehat{\text{ev} \otimes \text{ev}}$ is surjective, hence $\pi \circ \widehat{\text{ev} \otimes \text{ev}}$ is surjective. On the other hand,

$$\begin{aligned} G_s B\text{Mod}(\hat{Z}_s(2(p^s - 1)\rho - r\lambda) \otimes H^0(r\lambda), \mathfrak{k}) &\simeq G_s B\text{Mod}(\hat{Z}_s(r\lambda)^* \otimes H^0(r\lambda), \mathfrak{k}) \\ &\simeq G_s B\text{Mod}(H^0(r\lambda), \hat{Z}_s(r\lambda)) \\ &\simeq B\text{Mod}(H^0(r\lambda), r\lambda) \quad \text{by the Frobenius reciprocity} \\ &\simeq \mathfrak{k}. \end{aligned}$$

If $\text{Tr} : \text{Mod}_{\mathfrak{k}}(\hat{Z}_s(r\lambda), \hat{Z}_s(r\lambda)) \rightarrow \mathfrak{k}$ is the trace map, the composite

$$\begin{array}{ccc} \hat{Z}_s(\lambda)^* \otimes H^0(r\lambda) & \dashrightarrow & \mathfrak{k} \\ \hat{Z}_s(\lambda)^* \otimes \text{res}_{r\lambda} \downarrow & & \uparrow \text{Tr} \\ \hat{Z}_s(r\lambda)^* \otimes \hat{Z}_s(r\lambda) & \xrightarrow{\sim} & \text{Mod}_{\mathfrak{k}}(\hat{Z}_s(r\lambda), \hat{Z}_s(r\lambda)) \end{array}$$

also belongs to $G_s B\text{Mod}(\hat{Z}_s(r\lambda)^* \otimes H^0(r\lambda), \mathfrak{k})$, where $\text{res}_{r\lambda}$ is the restriction from G to $G_s B$. Take s so large that $\langle r\lambda, \alpha^\vee \rangle < p^s$ for all simple root α . Then $\text{res}_{r\lambda} : H^0(r\lambda) \rightarrow \hat{Z}_s(r\lambda)$ is injective, hence $\text{Tr} \circ (\hat{Z}_s(r\lambda)^* \otimes \text{res}_{r\lambda}) \neq 0$. It follows that

$$\pi \circ \widehat{\text{ev} \otimes \text{ev}} = \text{Tr} \circ (\hat{Z}_s(r\lambda)^* \otimes \text{res}_{r\lambda}) \quad \text{up to } \mathfrak{k}^\times.$$

Proposition. Assume $p \geq 2(h-1)$. If $0 \leq \langle \nu + \rho, \alpha^\vee \rangle < p^s$ for each simple root α , then $H^0(\pi \circ \widehat{\text{ev} \otimes \text{ev}}) : H^0(\hat{Z}_s(2(p^s - 1)\rho - \nu) \otimes H^0(\nu)) \rightarrow \mathfrak{k}$ is surjective.

Proof. By the argument above it is enough to show $H^0(\text{Tr} \circ (\hat{Z}_s(\nu)^* \otimes \text{res}_\nu)) : H^0(\hat{Z}_s(\nu)^* \otimes H^0(\nu)) \rightarrow \mathfrak{k}$ is surjective. By the hypothesis on ν we have from [J, II.11.13]

$$(4) \quad \text{hd}_G H^0(2(p^s - 1)\rho) \simeq \mathfrak{k} \simeq \text{hd}_{G_s} H^0(2(p^s - 1)\rho)$$

and that the restriction

$$\text{res}_{2(p^s - 1)\rho - \nu} : H^0(2(p^s - 1)\rho - \nu) \rightarrow \hat{Z}_s(2(p^s - 1)\rho - \nu) \text{ is surjective.}$$

On the other hand, $\text{res}_\nu : H^0(\nu) \rightarrow \hat{Z}_s(\nu)$ is injective. As $G_s B\text{Mod}(\hat{Z}_s(\nu)^* \otimes H^0(\nu), \mathfrak{k}) \simeq \mathfrak{k}$, there is a commutative diagram up to $\mathfrak{k}^\times$

$$\begin{array}{ccccc} H^0(\nu)^* \otimes H^0(\nu) & \xleftarrow{\text{res}_\nu^* \otimes H^0(\nu)} & \hat{Z}_s(\nu)^* \otimes H^0(\nu) & \xrightarrow{\hat{Z}_s(\nu)^* \otimes \text{res}_\nu} & \hat{Z}_s(\nu)^* \otimes \hat{Z}_s(\nu) \\ & \searrow \text{Tr} & & \swarrow \text{Tr} & \\ & & \mathfrak{k} & & \end{array}$$

Hence we have only to show that $H^0(\text{Tr} \circ (\text{res}_\nu^* \otimes H^0(\nu)))$ is surjective.

As $G_s B\text{Mod}(Z_s(\nu)^* \otimes H^0(\nu), \mathfrak{k}) \simeq \mathfrak{k}$ again, we have a commutative diagram in $G_s B\text{Mod}$

$$(5) \quad \begin{array}{ccc} H^0(\nu)^* \otimes H^0(\nu) & \xrightarrow{\text{Tr}} & \mathfrak{k} \\ \uparrow \text{res}^* \otimes H^0(\nu) & & \downarrow \sim \\ \hat{Z}_s(\nu)^* \otimes H^0(\nu) & & \text{hd}_{G_s B} \hat{Z}_s(2(p^s - 1)\rho) \\ \sim \downarrow & & \uparrow \pi \\ \hat{Z}_s(2(p^s - 1)\rho - \nu) \otimes H^0(\nu) & \xrightarrow{\widehat{\text{ev}} \otimes \widehat{\text{ev}}} & \hat{Z}_s(2(p^s - 1)\rho) \\ \uparrow \text{res}_{2(p^s - 1)\rho - \nu} \otimes H^0(\nu) & & \uparrow \text{res}_{2(p^s - 1)\rho} \\ H^0(2(p^s - 1)\rho - \nu) \otimes H^0(\nu) & \longrightarrow & H^0(2(p^s - 1)\rho), \end{array}$$

where the bottom horizontal map is the cup product surjective by Mathieu's theorem [M] (cf. also [K98b]). Moreover, if $\pi_G : H^0(2(p^s - 1)\rho) \rightarrow \text{hd}_G H^0(2(p^s - 1)\rho)$ is the quotient morphism, we have from (8) a commutative diagram

$$\begin{array}{ccc} H^0(2(p^s - 1)\rho) & \xrightarrow{\pi \circ \text{res}_{2(p^s - 1)\rho}} & \text{hd}_{G_s B} \hat{Z}_s(2(p^s - 1)\rho) \\ & \searrow \pi_G & \downarrow \sim \\ & & \text{hd}_G H^0(2(p^s - 1)\rho). \end{array}$$

Hence taking $H^0(?)$ of (9) yields a commutative diagram

$$\begin{array}{ccc} H^0(\hat{Z}_s(\nu)^* \otimes H^0(\nu)) & \xrightarrow{H^0(\text{Tr}(\text{res}_\nu^* \otimes H^0(\nu)))} & \mathfrak{k} \\ \uparrow H^0(\text{res}_\nu^* \otimes H^0(\nu)) & & \uparrow \pi_G \\ H^0(2(p^s - 1)\rho - \nu) \otimes H^0(\nu) & \longrightarrow & H^0(2(p^s - 1)\rho) \end{array}$$

It follows that $H^0(\text{Tr} \circ (\text{res}_\nu^* \otimes H^0(\nu))) \neq 0$, as desired.

References

- [BB] Beilinson, A. and Bernstein, J., *Localisation de $\mathfrak{g}$ -modules*, C. R. Acad. Sci. Paris 292 (1981), 15–18
- [H87] Haastert, B., *Über Differentialoperatoren und $\mathbb{D}$ -Moduln in positiver Charakteristik*, Manuscr. Math. 58 (1987), 385–415
- [J] Jantzen, J.C., “Representations of Algebraic Groups”, Academic Press 1987
- [K98a] Kaneda M., *Some generalities on $\mathcal{D}$ -modules in positive characteristic*, PJM 183 (1998), 103–141
- [K98b] Kaneda M., *Based modules and good filtrations in algebraic groups*, Hiroshima Math. J. 28 (1998), 337–344
- [Ka] Kashiwara M., *Representation theory and $\mathcal{D}$ -modules on flag varieties*, Astérisque 173-174 (1989), 55–109,