

DEFINABLE G -FIBER BUNDLES AND DEFINABLE $C^r G$ -FIBER BUNDLES

TOMOHIRO KAWAKAMI

川上智博 (和歌山大学)

ABSTRACT. Let G be a compact definable group and $f, h : X \rightarrow Y$ definable G -maps between definable G -sets. We prove that if X is compact, η is a definable G -fiber bundle over Y and f and h are G -homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably G -isomorphic.

Let G be a compact subgroup of $GL_n(\mathbb{R})$ and $f, h : X \rightarrow Y$ definable $C^r G$ maps between definable $C^r G$ manifolds. We show that if X is compact and affine, η is a definable $C^r G$ -fiber bundle over Y and f and h are definably $C^r G$ -homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably $C^r G$ -isomorphic.

1. INTRODUCTION

Let $\mathcal{M}$ denote an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. The term “definable” means “definable with parameters in $\mathcal{M}$ ”. In this paper, we are concerned with homotopy property of definable G -fiber bundles and definable $C^r G$ -fiber bundles when $1 \leq r < \infty$. General references on o-minimal structures are [6], [8], see also [18]. Further properties and constructions of them are studied in [7], [9], [17]. Every definable category is a generalization of the semialgebraic category and the definable category on $\mathcal{R}$ coincides the semialgebraic one.

A group G is a *definable group* if G is a definable set and the group operations $G \times G \rightarrow G$ and $G \rightarrow G$ are definable. A *definable G -set* means a G -invariant definable subset of some representation of G . We use a definable space as in the sense of [6], and every definable set is a definable space in this sense. Throughout this paper, definable maps between definable spaces are assumed to be continuous.

Theorem 1.1. *Let G be a compact definable group. Suppose that $\eta = (E, p, Y, F, K)$ is a definable G -fiber bundle over a definable G set Y and $f, h : X \rightarrow Y$ are definable G -maps between definable G -sets. If X is compact and f and h are G -homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably G -isomorphic.*

Two definable G -maps $f, h : X \rightarrow Y$ between definable G -sets are *definably G -homotopic* if there exists a definable G -map $H : X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in X$, where the action on $[0, 1]$ is trivial. By 1.2 [11], two definable G -maps in Theorem 1.1 are definably G -homotopic.

2000 *Mathematics Subject Classification.* 14P10, 14P20, 57R22, 57R35, 57S10, 57S15, 58A05, 58A07, 03C64.

Keywords and Phrases. Definable G -sets, definable G -fiber bundles, definable G -vector bundles, o-minimal, compact definable groups, definable $C^r G$ -manifolds, definable $C^r G$ -fiber bundles, definable $C^r G$ -vector bundles.

In the rest of this paper except section 2, G and K denote compact subgroups of $GL_n(\mathbb{R})$. It is known that they are compact algebraic subgroups of $GL_n(\mathbb{R})$ (e.g. 2.2 [16]).

Let Ω be a representation of G and $k \in \mathbb{N}$. Then we can consider the universal G -vector bundle $\gamma(\Omega, k)$ associated with Ω and k (see Definition 3.1). A definable G -vector bundle $\eta = (E, p, X)$ over a definable G -set X is called *strongly definable* if there exist a representation Ω of G and a definable G -map $f : X \rightarrow G(\Omega, k)$ such that η is definably G -isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η . The following result is a definable version of 1.1 [3].

Theorem 1.2. *Every definable G -vector bundle over a definable G -set is strongly definable.*

Let X be a definable G -set. Let $\text{Vect}_{\text{def}}^G(X)$ (respectively $\text{Vect}^G(X)$) denote the set of definable G -isomorphism (respectively G -isomorphism) classes of definable G -vector bundles (respectively G -vector bundles) over X . Then there is a canonical map $\kappa : \text{Vect}_{\text{def}}^G(X) \rightarrow \text{Vect}^G(X)$ which sends the definable G -isomorphism class $[\eta]_{\text{def}}^G$ of a definable G -vector bundle η over X to the G -isomorphism class $[\eta]^G$ of η .

Theorem 1.3. *Let X be a definable G -set. Then the map $\kappa : \text{Vect}_{\text{def}}^G(X) \rightarrow \text{Vect}^G(X)$ defined by $\kappa([\eta]_{\text{def}}^G) = [\eta]^G$ is bijective.*

As a corollary of Theorem 1.3, we have the following.

Corollary 1.4. *Let $\eta = (E, p, Y)$ be a definable G -vector bundle over a definable G -set Y and $f, h : X \rightarrow Y$ definable G -maps between definable G -sets. If f and h are G -homotopic, then $f^*(\eta)$ and $h^*(\eta)$ are definably G -isomorphic.*

Let $1 \leq r \leq \omega$. A definable $C^r G$ -manifold is a pair (X, θ) consisting of a definable C^r -manifold X and a group action $\theta : G \times X \rightarrow X$ which is a definable C^r -map. We simply write X for (X, θ) . A definable $C^r G$ -manifold is *affine* if it is definably $C^r G$ -diffeomorphic to a G -invariant definable C^r -submanifold of some representation of G .

Two definable $C^r G$ -maps $f, h : X \rightarrow Y$ between definable $C^r G$ -manifolds are *definably $C^r G$ -homotopic* if there exists a definable $C^r G$ -map $H : X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in X$, where G acts on $[0, 1]$ trivially.

The following result is a definable $C^r G$ -version of Theorem 1.1.

Theorem 1.5. *Suppose that $\eta = (E, p, Y, F, K)$ is a definable $C^r G$ -fiber bundle over a definable $C^r G$ -manifold Y and $1 \leq r < \infty$. Let f, h be definable $C^r G$ -maps from a compact affine definable $C^r G$ -manifold X to Y . If f and h are definably $C^r G$ -homotopic and F is affine, then $f^*(\eta)$ and $h^*(\eta)$ are definably $C^r G$ -isomorphic.*

Corollary 1.6. *Let $f, h : X \rightarrow Y$ be definable $C^r G$ -maps between definable $C^r G$ -manifolds and $1 \leq r < \infty$. If X is compact and affine, η is a definable $C^r G$ -vector bundle over Y and f is definably $C^r G$ -homotopic to h , then $f^*(\eta)$ and $h^*(\eta)$ are definably $C^r G$ -isomorphic.*

Let $1 \leq r \leq \omega$. A definable $C^r G$ -vector bundle $\eta = (E, p, X)$ over an affine definable $C^r G$ -manifold X is called *strongly definable* if then there exist a representation Ω of G and a definable $C^r G$ -map $f : X \rightarrow G(\Omega, k)$ such that η is definably $C^r G$ -isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η .

Theorem 1.7. *Let η be a definable $C^r G$ -vector bundle over an affine definable $C^r G$ -manifold X . If X is compact and $1 \leq r < \infty$, then η is strongly definable. Moreover if $r = \infty$ or ω , then η is strongly definable if and only if the total space of η is affine.*

This paper is organized as follows. In section 2, we give a definition of definable G fiber bundles and prove Theorem 1.1. We prove Theorem 1.2, 1.3 and Corollary 1.4 in section 3 and Theorem 1.5 and 1.7 in section 4.

2. DEFINABLE G -FIBER BUNDLES

A group homomorphism between definable groups is a *definable group homomorphism* if it is a definable map. An *n -dimensional representation* of a definable group G means $\mathbb{R}^n$ with the linear action induced by a definable group homomorphism from G to $O_n(\mathbb{R})$. A subgroup of a definable group G is a *definable subgroup* of G if it is a definable subset of G . A definable map (respectively A definable homeomorphism) between definable G -sets is a *definable G -map* (respectively a *definable G -homeomorphism*) if it is a G -map.

Let G be a definable group. A *definable set with a definable G -action* is a pair (X, θ) consisting of a definable set X and a group action $\theta : G \times X \rightarrow X$ such that θ is a definable map. We simply write X instead of (X, θ) . This action is not necessarily linear (orthogonal). *Definable G -maps* and *definable G -homeomorphisms* between definable sets with definable G -actions are defined similarly.

A *definable space* is an object obtained by pasting finitely many definable sets together along open definable subsets, and definable maps between definable spaces are defined similarly (see Chapter 10 [6]). Definable spaces are generalizations of semialgebraic spaces in the sense of [4].

Definition 2.1. Let G be a definable group.

- (1) A *definable G -space* is a pair (X, θ) consisting of a definable space X and a group action $\theta : G \times X \rightarrow X$ which is definable. For simplicity of notation, we write X for (X, θ) .
- (2) Let X and Y be definable G -spaces. A definable map $f : X \rightarrow Y$ is called a *definable G -map* if it is a G -map. We say that X and Y are *definably G -homeomorphic* if there exist definable G -maps $h : X \rightarrow Y$ and $k : Y \rightarrow X$ such that $h \circ k = id$ and $k \circ h = id$.

Note that clearly an implication “a definable G -set” $\Rightarrow$ “a definable set with a definable G -action” $\Rightarrow$ “a definable G -space” holds.

Definition 2.2. (1) A topological fiber bundle $\eta = (E, p, X, F, K)$ is called a *definable fiber bundle* over X with fiber F and structure group K if the following two conditions are satisfied:

- (a) The total space E is a definable space, the base space X is a definable set, the structure group K is a definable group, the fiber F is a definable set with an effective definable K action, and the projection $p : E \rightarrow X$ is a definable map.
- (b) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \rightarrow U_i \times F\}_i$ of η such that each U_i is a definable open subset of X , $\{U_i\}_i$ is a finite open covering of X . For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \rightarrow F$, $\phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where

π_i stands for the projection $U_i \times F \rightarrow F$. For any i and j with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \rightarrow K$ is a definable map. We call these trivializations *definable*.

Definable fiber bundles with compatible definable local trivializations are identified.

- (2) Let $\eta = (E, p, X, F, K)$ and $\zeta = (E', p', X', F, K)$ be definable fiber bundles whose definable local trivializations are $\{U_i, \phi_i\}_i$ and $\{V_j, \psi_j\}_j$, respectively. A definable map $\bar{f} : E \rightarrow E'$ is said to be a *definable morphism* if the following two conditions are satisfied:

- (a) The map $\bar{f}$ covers a definable map, namely there exists a definable map $f : X \rightarrow X'$ such that $f \circ p = p' \circ \bar{f}$.
- (b) For any i, j such that $U_i \cap f^{-1}(V_j) \neq \emptyset$ and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \bar{f} \circ \phi_{i,x}^{-1} : F \rightarrow F$ lies in K , and $f_{ij} : U_i \cap f^{-1}(V_j) \rightarrow K$ is a definable map.

We say that a bijective definable morphism $\bar{f} : E \rightarrow E'$ is a *definable equivalence* if it covers a definable homeomorphism $f : X \rightarrow X'$ and $(\bar{f})^{-1} : E' \rightarrow E$ is a definable morphism covering $f^{-1} : X' \rightarrow X$. A definable equivalence $\bar{f} : E \rightarrow E'$ is called a *definable isomorphism* if $X = X'$ and $f = id_X$.

- (3) A continuous section $s : X \rightarrow E$ of a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable section* if for any i , the map $\phi_i \circ s|_{U_i} : U_i \rightarrow U_i \times F$ is a definable map.
- (4) We say that a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *principal definable fiber bundle* if $F = K$ and the K -action on F is defined by the multiplication of K . We write (E, p, X, K) for (E, p, X, F, K) .

Definition 2.3. Let G be a definable group.

- (1) A definable fiber bundle (E, p, X, F, K) (respectively A principal definable fiber bundle (E, p, X, K)) is called a *definable G -fiber bundle* (respectively a *principal definable G -fiber bundle*) if the total space E is a definable G -space such that G acts on E through definable equivalences, the base space X is a definable set with a definable G -action and the projection p is a definable G -map.
- (2) A definable morphism (respectively A definable equivalence, A definable isomorphism) between definable G -fiber bundles is a *definable G -morphism* (respectively a *definable G -equivalence*, a *definable G -isomorphism*) if it is a G -map.
- (3) A *definable G -section* of a definable G -fiber bundle means a definable section which is a G -map.

Let $f : X \rightarrow Y$ be a definable map between definable sets. We say that f is *proper* if for any compact subset C of Y , $f^{-1}(C)$ is compact.

Let E be an equivalence relation on a definable set X . We call E *proper* if E is a definable subset of $X \times X$ and the projection $E \rightarrow X$ defined by $(x, y) \mapsto x$ is proper.

Theorem 2.4 (Definable quotients (e.g. 10.2.15 [6])). *Let E be a proper equivalence relation on a definable set X . Then X/E exists a proper quotient, namely X/E is a definable subset of some $\mathbb{R}^n$ and the projection $X \rightarrow X/E$ is a surjective proper definable map.*

In the remainder of this section, G and K denote compact definable groups. The following is a corollary of Theorem 2.4.

Corollary 2.5 (e.g. 10.2.18 [6]). *Let X be a definable set with a definable G -action. Then X/G is a definable subset of some $\mathbb{R}^n$ and the orbit map $p : X \rightarrow X/G$ is a surjective proper definable map.*

By similar proofs of 2.10 [14] and 2.11 [14], the standard construction of the associated principal bundle from a fiber bundle and by Theorem 2.4, we have the following.

Proposition 2.6. (1) *Let (E, p, X, K) be a principal definable G -fiber bundle and F a definable set with an effective definable K -action. Then $(E \times_K F, p', X, F, K)$ is a definable G -fiber bundle, where $p' : E \times_K F \rightarrow X$ denotes the projection defined by $p'([z, k]) = p(z)$.*
 (2) *The associated principal G -fiber bundle of a definable G -fiber bundle is definable.*
 (3) *Two definable G -fiber bundles having the same base space, fiber and structure group are definably G -isomorphic if and only if their associated principal definable G -fiber bundles are definably G -isomorphic.*

Let X be a definable set with a definable G -action and $x \in X$. A G_x -invariant definable subset S of X is a *definable slice* at x in X if GS is a G -invariant definable open neighborhood of the orbit $G(x)$ of x in X , $G \times_{G_x} S$ is a definable set with the standard definable G -action $G \times (G \times_{G_x} S) \rightarrow G \times_{G_x} S, (g, [g', s]) \mapsto [gg', s]$, and the map $G \times_{G_x} S \rightarrow GS \subset X$ defined by $[g, s] \mapsto gs$ is a definable G -homeomorphism.

Theorem 2.7 (Definable slices). *Let X be a definable G -set and $x \in X$. Then there exists a definable slice S at x in X .*

Let Y be a G -invariant definable subset of a definable G -set X . A *definable G -retraction from X to Y* means a definable G -map $R : X \rightarrow Y$ with $R|_Y = \text{id}_Y$.

For the proof of Theorem 2.7, we recall the following result.

Theorem 2.8 (3.4 [11]). *Let Y be a G -invariant definable closed subset of a definable G -set X . Then there exist a G -invariant definable open neighborhood U of Y in X and a definable G -retraction from U to Y .*

Proof of Theorem 2.7. Since $G(x)$ is a G -invariant definable closed subset of X and by Theorem 2.8, we have a G -invariant definable open neighborhood U of $G(x)$ in X and a definable G -retraction q from U to $G(x)$. Let $S := q^{-1}(x)$. Then S is a definable G_x -set and $U = GS$. By II.4.2 [2], the map $f : G \times_{G_x} S \rightarrow GS (\subset X)$ defined by $f([g, s]) = gs$ is a G -homeomorphism. On the other hand, the map $k : G \times S \rightarrow GS$ defined by $k(g, s) = gs$ and the projection $\pi : G \times S \rightarrow G \times_{G_x} S$ are definable maps. Since the graph of f is the image of that of k by $\pi \times \text{id}_{GS}$, f is a definable G -homeomorphism. $\square$

Definition 2.9. A definable G -fiber bundle $\eta = (E, p, X, F, K)$ satisfies the *definable Bierstone condition* if for any $x \in X$, there exist a G_x -invariant definable open neighborhood U_x of x in X and a definable group homomorphism $\rho_x : G_x \rightarrow K$ such that $\eta|_{U_x}$ is definably G_x -isomorphic to $U_x \times F$ with the definable G_x -action defined by $G_x \times (U_x \times F) \rightarrow U_x \times F, (h, u, y) \mapsto (hu, \rho_x(h)y)$.

Note that a definable G -fiber bundle over a definable G -set satisfies the definable Bierstone condition if and only if the associated principal definable G -fiber bundle satisfies it.

Using Theorem 2.7, similar proofs of 1.4 [15] and 1.5 [15] prove the following proposition.

Proposition 2.10. *Every definable G -fiber bundle over a definable G -set satisfies the definable Bierstone condition.*

A finite definable open covering $\{U_i\}_i$ of a definable G -set is called a *finite definable open G -covering* if each U_i is G -invariant. A finite definable G -open covering is *numerable* if there exists a definable partition of unity $\{\lambda_i\}_i$ subordinate to $\{U_i\}_i$ such that each λ_i is G -invariant.

The following proposition shows existence of (non-equivariant) definable partition of unity.

Proposition 2.11 (e.g. 6.3.7 [6]). *Let X be a definable set in $\mathbb{R}^n$ and $\{U_i\}_{i=1}^n$ a finite definable open covering of X . Then there exists a definable partition of unity subordinate to $\{U_i\}_{i=1}^n$, namely there exist definable functions $\lambda_1, \dots, \lambda_n : X \rightarrow \mathbb{R}$ such that $0 \leq \lambda_i \leq 1$, $\text{supp } \lambda_i \subset U_i$ and $\sum_{i=1}^n \lambda_i = 1$.*

The following is an equivariant version of Proposition 2.11.

Proposition 2.12 (Equivariant definable partition of unity). *Every finite definable open G -covering of a definable G -set X is numerable.*

Proof. Let $\{U_i\}_{i=1}^n$ be a finite definable open G -covering of a definable G -set X . By Corollary 2.5, the orbit map $p : X \rightarrow X/G$ is a surjective proper definable map. Since $p : X \rightarrow X/G$ is open, $\{p(U_i)\}_{i=1}^n$ is a finite definable open covering of X/G . By Proposition 2.11, one can find a definable partition of unity $\{\bar{\lambda}_i\}_{i=1}^n$ subordinate to $\{p(U_i)\}_{i=1}^n$. Hence $\lambda_1 := \bar{\lambda}_1 \circ p, \dots, \lambda_n := \bar{\lambda}_n \circ p$ are G -invariant and subordinate to $\{U_i\}_{i=1}^n$. $\square$

Note that in Proposition 2.11 and 2.12, we can replace $\sum_{i=1}^n \lambda_i = 1$ by $\max_{1 \leq i \leq n} \lambda_i = 1$.

Theorem 1.1 follows from Theorem 2.13 below.

Theorem 2.13. *If X is a compact definable G -set, then every definable G -fiber bundle $\eta = (E, p, X \times [0, 1], F, K)$ is definably G -isomorphic to $(p^{-1}(X \times \{0\}) \times [0, 1], p', X \times [0, 1], F, K)$, where G acts on $[0, 1]$ trivially, $X \times \{0\}$ is identified with X and $p' = p|_{p^{-1}(X \times \{0\})} \times \text{id}_{[0, 1]}$.*

To prove Theorem 2.13, we need the following three results.

Lemma 2.14. *Let A be a definable G -set, $X_1 = A \times [a, b]$, $X_2 = A \times [b, c]$, and $\eta = (E, p, X, F, K)$ a definable G -fiber bundle over $X = X_1 \cup X_2$, where G acts trivially on $[a, b]$ and $[b, c]$. If $\eta|_{X_1}$ and $\eta|_{X_2}$ are definably G -isomorphic to $X_1 \times F$ and $X_2 \times F$, respectively, then so is η , where the action on F is induced by a definable group homomorphism from G to K .*

Proof. Let $u_i : X_i \times F \rightarrow p^{-1}(X_i)$, ($i = 1, 2$), be definable G -isomorphisms and $w_i := u_i|(X_1 \cap X_2) \times F$, ($i = 1, 2$). Then $h := w_2^{-1} \circ w_1 : (X_1 \cap X_2) \times F \rightarrow (X_1 \cap X_2) \times F$

is a definable G -isomorphism. Hence there exists a definable map $l : X_1 \cap X_2 \rightarrow K$ such that $h(x, y) = (x, l(x)y)$, where $(x, y) \in (X_1 \cap X_2) \times F$. Let $i_A : A \rightarrow K, i_A(a) = l(a, b)$. Then we can extend h to a definable G -isomorphism

$$\tilde{h} : X_2 \times F \rightarrow X_2 \times F, \tilde{h}(x_1, x_2, y) = (x_1, x_2, i_A(x_1)y).$$

Since two definable G -isomorphisms $u_1 : X_1 \times F \rightarrow p^{-1}(X_1)$ and $u_2 \circ \tilde{h} : X_2 \times F \rightarrow p^{-1}(X_2)$ coincide on $(X_1 \cap X_2) \times F$ and $X_1 \times F$ and $X_2 \times F$ are closed in $(X_1 \cup X_2) \times F = X \times F$, the gluing map provides the required definable G -isomorphism. $\square$

Let H be a definable subgroup of G , $\rho : H \rightarrow K$ a definable group homomorphism between definable groups, and F a definable set with an effective definable K -action. For any definable H -set S , we define a definable G -fiber bundle $\epsilon^\rho(S)$ by $(G \times_H (S \times F), p, G \times_H S, F, K)$, where $p : G \times_H (S \times F) \rightarrow G \times_H S, p([g, (s, y)]) = [g, s]$ and H acts on F via ρ .

Lemma 2.15. *Let X be a compact definable G -set and $\eta = (E, p, X \times [0, 1], F, K)$ a definable G -fiber bundle over $X \times [0, 1]$. Then there exist finitely many points $x_1, \dots, x_n$ with definable slices $S_{x_1}, \dots, S_{x_n}$ and definable group homomorphisms $\{\rho_i : G_{x_i} \rightarrow K\}_{i=1}^n$ such that $\{GS_{x_i}\}_{i=1}^n$ is a finite definable open G -covering of X and each $\eta|(GS_{x_i} \times [0, 1])$ is definably G -equivalent to $\epsilon^{\rho_i}(S_{x_i}) \times [0, 1]$.*

Proof. By Proposition 2.10, for any $(x, t) \in X \times [0, 1]$, there exist a G_x -invariant definable open neighborhood U_x of x in X and $\delta > 0$ such that $\eta|(U_x \times [t - \delta, t + \delta])$ is definably G_x -isomorphic to $(U_x \times [t - \delta, t + \delta]) \times F$, where the action on F is induced by a definable group homomorphism $\rho_x : G_x \rightarrow K$. Since $[0, 1]$ is compact and by Lemma 2.14, we have a G_x -invariant definable open neighborhood V_x of x in X such that $\eta|V_x \times [0, 1]$ is definably G_x -isomorphic to $(V_x \times [0, 1]) \times F$. By Theorem 2.7, we have a definable slice S_x at x with $S_x \subset V_x$. Hence there exists a definable G_x -isomorphism $l_x : S_x \times [0, 1] \times F \rightarrow \eta|S_x \times [0, 1]$. Thus $h_x : G \times_{G_x} (S_x \times [0, 1] \times F) = \epsilon^{\rho_x}(S_x) \times [0, 1] \rightarrow \eta|GS_x \times [0, 1]$ defined by $h_x([g, (s, t, f)]) = gl_x(s, t, f)$ is a definable G -equivalence. Since X is compact, there exist finitely many points $x_1, \dots, x_n$ of X such that $\{GS_{x_i}\}_{i=1}^n$ is a finite definable open G -covering of X . $\square$

Theorem 2.16. *Let X be a compact definable G -set, $r : X \times [0, 1] \rightarrow X \times [0, 1], r(x, t) = (x, 1)$ and $\eta = (E, p, X \times [0, 1], F, K)$ a definable G -fiber bundle over $X \times [0, 1]$. Then there exists a definable G -morphism $\phi : E \rightarrow E$ covering r .*

Proof. By Lemma 2.15, we can find finitely many points $x_1, \dots, x_n$ with definable slices $S_{x_1}, \dots, S_{x_n}$ and definable group homomorphisms $\{\rho_i : G_{x_i} \rightarrow K\}_{i=1}^n$ such that $\{GS_{x_i}\}_{i=1}^n$ is a finite definable open G -covering of X and each $\eta|(GS_{x_i} \times [0, 1])$ is definably G -equivalent to $\epsilon^{\rho_i}(S_{x_i}) \times [0, 1]$. By Proposition 2.12, there exist G -invariant definable functions $l_1, \dots, l_n : X \rightarrow [0, 1]$ such that:

- (a) The support of each l_i is contained in GS_{x_i} .
- (b) $\max_{1 \leq i \leq n} l_i(x) = 1$ for all $x \in X$.

Let $h_{x_i} : (G \times_{G_{x_i}} (S_{x_i} \times F)) \times [0, 1] \rightarrow p^{-1}(GS_{x_i} \times [0, 1])$ be a definable G -equivalence covering a definable G -homeomorphism $f_{x_i} \times id_{[0, 1]} : (G \times_{G_{x_i}} S_{x_i}) \times [0, 1] \rightarrow GS_{x_i} \times [0, 1]$.

Define

$$\begin{aligned}
 (u_i, r_i) &: (E, X \times [0, 1]) \rightarrow (E, X \times [0, 1]), 1 \leq i \leq n, \\
 r_i(x, t) &= \begin{cases} (x, \max(l_i(f_{x_i}([g, s])), t)), & ([g, s], t) \in (G \times_{G_{x_i}} S_{x_i}) \times [0, 1] \\ (x, t), & \text{otherwise} \end{cases}, \\
 u_i(h_{x_i}([g, (s, f)], t) &= h_{x_i}([g, (s, f)], \max(l_i(f_{x_i}([g, s])), t)), \\
 &\quad \text{for any } ([g, (s, f)], t) \in (G \times_{G_{x_i}} (S_{x_i} \times F)) \times [0, 1], \\
 u_i &\text{ is the identity outside } p^{-1}(GS_{x_i} \times [0, 1]).
 \end{aligned}$$

Then $r = r_n \circ \cdots \circ r_1$. Therefore $\phi = u_n \circ \cdots \circ u_1 : E \rightarrow E$ is the required definable G -morphism. $\square$

Theorem 2.13 follows from Theorem 2.16.

3. DEFINABLE G -VECTOR BUNDLES AND PROOF OF THEOREM 1.2, 1.3 AND COROLLARY 1.4

We recall that G and K denote compact subgroups of $GL_n(\mathbb{R})$ except section 2. Then remember that G is a compact algebraic subgroup of $GL_n(\mathbb{R})$ and any closed subgroup of G is a compact algebraic subgroup of G .

Note that a definable group homomorphism from G to $O_n(\mathbb{R})$ is a definable C^∞ -map because it is a continuous group homomorphism between Lie groups.

Recall universal G -vector bundles (e.g. [12]).

Definition 3.1. Let Ω be an n -dimensional representation of G induced by a definable group homomorphism $B : G \rightarrow O_n(\mathbb{R})$ of Ω . Suppose that $M(\Omega)$ denotes the vector space of $n \times n$ -matrices with the action $(g, A) \in G \times M(\Omega) \rightarrow B(g)AB(g)^{-1} \in M(\Omega)$. For any positive integer k , we define the vector bundle $\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))$ as follows:

$$\begin{aligned}
 G(\Omega, k) &= \{A \in M(\Omega) \mid A^2 = A, A = A', \text{Tr } A = k\}, \\
 E(\Omega, k) &= \{(A, v) \in G(\Omega, k) \times \Omega \mid Av = v\}, \\
 u : E(\Omega, k) &\rightarrow G(\Omega, k), u((A, v)) = A,
 \end{aligned}$$

where A' denotes the transposed matrix of A and $\text{Tr } A$ stands for the trace of A . Then $\gamma(\Omega, k)$ is an algebraic vector bundle. Since the action on $\gamma(\Omega, k)$ is algebraic, it is an algebraic G -vector bundle. We call it the *universal G -vector bundle associated with Ω and k* . Remark that $G(\Omega, k) \subset M(\Omega)$ and $E(\Omega, k) \subset M(\Omega) \times \Omega$ are nonsingular algebraic G -sets.

Definition 3.2. (1) A *definable G -vector bundle of rank k* is a definable G -fiber bundle with fiber $\mathbb{R}^k$ and structure group $GL_k(\mathbb{R})$. We usually write (E, p, X) instead of $(E, p, X, \mathbb{R}^k, GL_k(\mathbb{R}))$.
 (2) Let $\eta = (E, p, X)$ and $\eta' = (E', p', X)$ be definable G -vector bundles. A definable G -map $f : E \rightarrow E'$ is called a *definable G -morphism* if $p = p' \circ f$ and f is linear on each fiber. A definable G -morphism $h : E \rightarrow E'$ is said to be a *definable G -isomorphism* if there exists a definable G -morphism $h' : E' \rightarrow E$ such that $h \circ h' = id$ and $h' \circ h = id$.

- (3) A *definable G -section* of a definable G -vector bundle means a definable G -section as a definable G -fiber bundle.

By a way similar to 3.1 [10], we have the following proposition.

Proposition 3.3. *If η and η' are two definable G -vector bundles over a definable G -set X , then $\eta \oplus \eta'$, $\eta \otimes \eta'$, $\text{Hom}(\eta, \eta')$ and the dual bundle $\eta^\vee$ of η are definable G -vector bundles over X .*

The next result states equivalent properties of strong definability of definable G vector bundles, which is obtained in a way similar to the proof of 3.6 [3].

Theorem 3.4. *Let $\eta = (E, p, X)$ be a definable G -vector bundle of rank k over a definable G -set X . Then the following five properties are equivalent.*

- (1) *The bundle η is strongly definable.*
- (2) *There exists a surjective definable G -morphism from a trivial G -vector bundle $X \times \Omega$ onto η for some representation Ω of G .*
- (3) *There exists an injective definable G -morphism from η to a trivial G -vector bundle $X \times \Omega$ for some representation Ω of G .*
- (4) *There exists a definable G -vector bundle η' over X such that $\eta \oplus \eta'$ is definably G -isomorphic to a trivial G -vector bundle.*
- (5) *There exist non-equivariant definable sections $s_1, \dots, s_n : X \rightarrow E$ of η such that:*
 - (a) *For any $x \in X$, the vectors $s_1(x), \dots, s_n(x)$ generate the fiber $p^{-1}(x)$ over x .*
 - (b) *The sections $s_1, \dots, s_n$ generate a finite dimensional G -invariant vector subspace of $\Gamma(\eta)$, where $\Gamma(\eta)$ denotes the set of all continuous sections of η with the natural G -action, namely $(g \cdot s)(x) = g(s(g^{-1}x))$ for all $g \in G$ and $x \in X$.*

Theorem 1.2 follows from Theorem 3.4 and Theorem 3.5 below.

Theorem 3.5. *Every definable G vector bundle over a definable G set satisfies Condition (5) in Theorem 3.4.*

By a way similar to the proof of 3.9 [3], we have the following proposition.

Proposition 3.6. *Let $\eta = (E, p, X)$ be a definable G -vector bundle over a definable set X with the trivial G -action and A a closed definable subset of X such that $\eta|_A$ is strongly definable. If A admits a definable retraction from X to A , then there exists some open definable neighborhood V of A in X such that $\eta|_V$ is strongly definable.*

The following is the equivariant definable version of Urysohn's lemma, and its semialgebraic version is proved in 1.6 [5]. We use only a non-equivariant version of it to prove Theorem 3.5.

Lemma 3.7. *Let X be a definable set with a definable G -action and A and B disjoint closed definable G -subsets of X . Then there exists a G -invariant definable function $f : X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.*

Proof. By Corollary 2.5, X/G is a definable subset of some $\mathbb{R}^n$ and the orbit map $p : X \rightarrow X/G$ is a surjective proper definable map. Hence $\pi(A)$ and $\pi(B)$ are closed definable

subsets of X/G . Then the function $h : X/G \rightarrow [0, 1]$ defined by $h(x) = \frac{d(x, \pi(A))}{d(x, \pi(A)) + d(x, \pi(B))}$ is a definable function such that $h^{-1}(0) = \pi(A)$ and $h^{-1}(1) = \pi(B)$, where $d(x, \pi(A))$ (respectively $d(x, \pi(B))$) denotes the distance between x and $\pi(A)$ (respectively x and $\pi(B)$). Therefore $f := h \circ \pi : X \rightarrow [0, 1]$ is the required G -invariant definable function. $\square$

Proposition 3.8. *Let H be a closed subgroup of G , D the closed unit ball of a representation Ω of H . Then $G \times_H D$ is a compact affine definable $C^\infty G$ manifold with boundary. In particular, $G \times_H D$ is definably G -imbeddable into some representation of G .*

Proof. Note that G and Ω are affine definable $C^\infty H$ -manifolds. Thus by 4.4 [13] and 4.5 [13], $G \times_H \Omega$ is a definable $C^\infty G$ -manifold whose underlying manifold is a definable C^∞ -submanifold of some $\mathbb{R}^k$. Since $G \times_H D$ is compact, there exists a $C^\infty G$ -imbedding i from $G \times_H D$ to some representation Ξ of G . Applying the polynomial approximation theorem to i and averaging it, we have a definable $C^\infty G$ -imbedding from $G \times_H D$ to Ξ . $\square$

A *definable G -CW-complex* is a finite G -CW-complex such that the characteristic map of each G -cell is a definable G -map (see [11]).

Theorem 3.9 (1.1 [11]). *Let X be a definable G -set and Y a closed definable G -subset of X . Then there exist a definable G -CW-complex Z in a representation Ω of G , a G -CW-subcomplex W of Z , and a definable G -map $f : X \rightarrow Z$ such that:*

- (1) *The map f takes X and Y definably G -homeomorphically onto G -invariant definable subsets Z_1 and W_1 of Z and W obtained by removing some open G -cells from Z and W , respectively.*
- (2) *The orbit map $\pi : Z \rightarrow Z/G$ is a definable cellular map.*
- (3) *The orbit space Z/G is a finite simplicial complex compatible with $\pi(Z_1)$ and $\pi(W_1)$.*
- (4) *For each open G -cell c of Z , $\pi|\bar{c} : \bar{c} \rightarrow \pi(\bar{c})$ has a definable section $s : \pi(\bar{c}) \rightarrow \bar{c}$, where $\bar{c}$ denotes the closure of c in Z .*

Furthermore, if X is compact, then $Z = f(X)$ and $W = f(Y)$.

Using Proposition 3.6, Lemma 3.7, Proposition 3.8, Theorem 3.9, a similar proof of 3.5 [3] proves Theorem 3.5.

By Theorem 1.2 and by the proof of 4.7 [11], we have the following.

Proposition 3.10. *Let η a definable G -vector bundle over a compact definable G -set X . Then every continuous G -section of η can be approximated by definable G -sections.*

We obtain the following theorem using Proposition 3.3 and Proposition 3.10.

Theorem 3.11. *Let η and ζ be definable G -vector bundles over a compact definable G -set. If η is G -isomorphic to ζ , then they are definably G -isomorphic.*

Proposition 3.12 (2.11 [15]). *Let X, Y be definable G -sets. If η is G -vector bundle over Y and $f, h : X \rightarrow Y$ are G -homotopic continuous G -maps, then $f^*(\eta)$ is G -isomorphic to*

Proposition 3.13 ([1], [20]). *Let X be a compact G -set. If η is a G -vector bundle over X , then there exist a representation Ω of G and a continuous G -map $f : X \rightarrow G(\Omega, k)$ such that η is G -isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η .*

Theorem 3.14. *If X is a compact definable G -set, $\kappa : \text{Vect}_G^{\text{def}}(X) \rightarrow \text{Vect}_G(X)$ is bijective.*

Proof. Injectivity follows from Theorem 3.11.

Let η be a G -vector bundle over X . Then by Proposition 3.13, there exist a representation Ω of G and a continuous G -map $f : X \rightarrow G(\Omega, k)$ such that η is G -isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η . By 3.5 [11], f is G -homotopic to a definable G -map $h : X \rightarrow G(\Omega, k)$. Hence by Proposition 3.12, $f^*(\gamma(\Omega, k))$ is G -isomorphic to $h^*(\gamma(\Omega, k))$. Therefore η is G -isomorphic to a definable G -vector bundle $h^*(\gamma(\Omega, k))$. $\square$

A G -set X is G -contractible if there exist a fixed point $x_0 \in X$ and a continuous G -map $F : X \times [0, 1] \rightarrow X$ such that $F(x, 0) = x$ and $F(x, 1) = x_0$ for all $x \in X$, where G acts on $[0, 1]$ trivially. We have the following as a corollary of Theorem 1.1.

Corollary 3.15. *Let X be a compact G -contractible definable G -set. Then every definable G -vector bundle over X is definably G -isomorphic to a trivial G -bundle.*

Theorem 3.16 (3.3 [11]). *Let X be a definable G -set. Then there exists a definable G -deformation retraction R from X to a compact definable G -subset Y of X .*

By a way similar to the proof of 4.10 [11], we have the following proposition.

Proposition 3.17. *The map $R^* : \text{Vect}_G^{\text{def}}(Y) \rightarrow \text{Vect}_G^{\text{def}}(X)$ defined by $\eta \mapsto R^*(\eta)$ is bijective.*

Theorem 1.3 follows from Theorem 3.14 and Proposition 3.17. Corollary 1.4 follows from Theorem 1.3 and Proposition 3.12.

4. DEFINABLE $C^r G$ -FIBER BUNDLES AND DEFINABLE $C^r G$ -VECTOR BUNDLES

Definition 4.1 ([12]). Let $1 \leq r \leq \omega$.

- (1) A definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable C^r -fiber bundle* if the total space E and the base space X are definable C^r -manifolds, the structure group K is a definable C^r -group, the fiber F is a definable $C^r K$ -manifold with an effective action, the projection p is a definable C^r -map and all transition functions of η are definable C^r -maps. A *principal definable C^r -fiber bundle* is defined similarly.
- (2) *Definable C^r -morphisms, definable C^r -equivalences, definable C^r -isomorphisms* between definable C^r -fiber bundles and *definable C^r -sections* of a definable C^r fiber bundle are defined similarly.
- (3) A definable C^r -fiber bundle $\eta = (E, p, X, F, K)$ is a *definable $C^r G$ -fiber bundle* if the total space E and the base space X are definable $C^r G$ -manifolds, the projection p is a definable $C^r G$ -map and G acts on E through definable C^r -equivalences. A *principal definable $C^r G$ -fiber bundle* is defined similarly.

- (4) A definable C^r -morphism (resp. a definable C^r -equivalence, a definable C^r -isomorphism, a definable C^r -section) is a *definable $C^r G$ -morphism* (resp. a *definable $C^r G$ -equivalence*, a *definable $C^r G$ -isomorphism*, a *definable $C^r G$ -section*) if it is a G -map.

The following is a definable $C^r G$ -version of Proposition 2.6, which is obtained similarly.

Proposition 4.2. *Suppose that $1 \leq r \leq \omega$.*

- (1) *Let (E, p, X, K) be a principal definable $C^r G$ -fiber bundle and F an affine definable $C^r K$ -manifold with an effective action. Then $(E \times_K F, p', X, F, K)$ is a definable $C^r G$ -fiber bundle, where $p' : E \times_K F \rightarrow X$ denotes the projection defined by $p'([z, k]) = p(z)$.*
- (2) *The associated principal G -fiber bundle of a definable $C^r G$ -fiber bundle is a principal definable $C^r G$ -fiber bundle.*
- (3) *Two definable $C^r G$ -fiber bundles having the same base space, fiber and structure group are definably $C^r G$ -isomorphic if and only if their associated principal definable $C^r G$ -fiber bundles are definably $C^r G$ -isomorphic.*

Proposition 4.3. *Let X be a definable $C^r G$ -submanifold of a representation Ω of G and $1 \leq r < \infty$. Then for any $x \in X$, there exists a linear definable C^r -slice at x in X , namely there exists a definable $C^r G_x$ -imbedding i from a representation Ξ of G_x into X such that $i(0) = x$, $G \times_{G_x} \Xi$ is a definable $C^r G$ -manifold with the standard action $(g, [g', x]) \mapsto [gg', x]$ and the map $\mu : G \times_{G_x} \Xi \rightarrow X$ defined by $[g, x] \mapsto gi(x)$ is a definable $C^r G$ -diffeomorphism onto some G -invariant definable open neighborhood of $G(x)$ in X .*

Proof. Since G is a compact algebraic subgroup of $GL_n(\mathbb{R})$ and by 4.1 [13], for any $x \in X$, there exists a linear definable C^∞ slice at x in Ω , namely we have a representation Ξ' of G_x and a definable $C^\infty G_x$ imbedding $j : \Xi' \rightarrow \Omega$ such that $j(0) = x$, $G \times_{G_x} \Xi'$ is a definable $C^\infty G$ manifold and the map $\mu' : G \times_{G_x} \Xi' \rightarrow \Omega$ defined by $\mu'([g, x]) = gj(x)$ is a definable $C^\infty G$ diffeomorphism onto a G invariant definable open neighborhood $Gj(\Xi')$ of $G(x)$ in Ω . Then $j^{-1}(X)$ is a definable $C^r G_x$ submanifold of Ξ' and $j|_{j^{-1}(X)} : j^{-1}(X) \rightarrow X$ is a definable $C^r G_x$ imbedding. Hence there exists a sufficiently small G_x invariant definable open neighborhood U of 0 in $j^{-1}(X)$ such that U is definably $C^r G_x$ diffeomorphic to a representation Ξ of G_x . Take a definable $C^r G_x$ diffeomorphism $l : \Xi \rightarrow U$ with $l(0) = 0$ and let $i = j \circ l$. Then i is a definable $C^r G_x$ imbedding from Ξ to X and the map $\mu : G \times_{G_x} \Xi \rightarrow X$ defined by $\mu([g, x]) = gi(x)$ is a definable $C^r G$ diffeomorphism onto a G invariant definable open neighborhood $Gi(\Xi) = Gj(U)$ of $G(x)$ in X . $\square$

Note that if $r = \infty$ or ω , then Proposition 4.3 is proved in 4.1 [13].

We can consider the *definably C^r -Bierstone condition* as a definable $C^r G$ -version of Definition 2.9. Using Proposition 4.2 and 4.3, we have the following definable C^r -version of Proposition 2.10.

Proposition 4.4. *Let $1 \leq r \leq \omega$. Then every definable $C^r G$ -fiber bundle over an affine definable $C^r G$ -manifold satisfies the definable C^r -Bierstone condition.*

The proof of 4.8 [12] proves the following.

Proposition 4.5 (4.8 [12]). (*Definable C^r partition of unity*). Let X be a definable closed subset of $\mathbb{R}^n$, $\{U_i\}_{i=1}^l$ a finite definable open covering of X and $0 \leq r < \infty$. Then there exist definable C^r functions $\lambda_1, \dots, \lambda_l : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $0 \leq \lambda_i \leq 1$, $\text{supp } \lambda_i \subset U_i$ and $\sum_{i=1}^l \lambda_i(x) = 1$ for any $x \in X$.

The following is a definable C^r -version of Proposition 2.12.

Proposition 4.6 (Equivariant definable C^r -partition of unity). Let X be a definable $C^r G$ -submanifold closed in a representation Ω of G and $\{U_i\}_{i=1}^n$ a finite definable open G -covering of X and $0 \leq r < \infty$. Then $\{U_i\}_{i=1}^n$ is numerable, namely there exist G -invariant definable C^r -functions $\lambda_1, \dots, \lambda_n : X \rightarrow \mathbb{R}$ such that $0 \leq \lambda_i \leq 1$, $\text{supp } \lambda_i \subset U_i$ and $\sum_{i=1}^n \lambda_i(x) = 1$ for any $x \in X$.

Proof. First of all, we recall the structure of the orbit space Ω/G . The algebra $\mathbb{R}[\Omega]^G$ of G invariant polynomials on Ω is finitely generated [21]. Let $p_1, \dots, p_n : \Omega \rightarrow \mathbb{R}$ be G invariant polynomials generating $\mathbb{R}[\Omega]^G$, and put $p : \Omega \rightarrow \mathbb{R}^n$, $p = (p_1, \dots, p_n)$. Then p is a proper polynomial map, and it induces a closed imbedding $j : \Omega/G \rightarrow \mathbb{R}^n$ such that $p = j \circ \pi$, where $\pi : \Omega \rightarrow \Omega/G$ denotes the orbit map. Hence we can identify Ω/G (resp. X/G , π) with $j(\Omega/G)$ (resp. $j(X/G)$, p). Thus $\{p(U_i)\}_{i=1}^l$ is a finite definable open covering of X/G because $p|_X : X \rightarrow X/G$ is open. Note that $p(X)$ is closed in $\mathbb{R}^n$ because X is closed in Ω . By Proposition 4.5, one can find a definable partition of unity $\{\bar{\lambda}_i\}_{i=1}^l$ subordinate to $\{p(U_i)\}_{i=1}^l$. Hence $\lambda_1 := \bar{\lambda}_1 \circ p, \dots, \lambda_l := \bar{\lambda}_l \circ p$ are the required G invariant definable C^r functions. $\square$

We can replace $\sum_{i=1}^n \lambda_i = 1$ by $\max_{1 \leq i \leq n} \lambda_i = 1$ in Proposition 4.5 and 4.6.

By the proof of 2.10 [12], we may assume that an affine definable $C^r G$ -manifold is a definable $C^r G$ -submanifold closed in some representation Ω of G . Thus similar proofs of Lemma 2.14, 2.15 and Theorem 2.16 prove the following.

Theorem 4.7. *If X is a compact affine definable $C^r G$ -manifold and $1 \leq r < \infty$, then every definable $C^r G$ -fiber bundle $\eta = (E, p, X \times [0, 1], F, K)$ is definably $C^r G$ -isomorphic to $(p^{-1}(X \times \{0\}) \times [0, 1], p', X \times [0, 1], F, K)$, where G acts on $[0, 1]$ trivially, $X \times \{0\}$ is identified with X and $p' = p|_{p^{-1}(X \times \{0\})} \times \text{id}_{[0, 1]}$.*

Theorem 1.5 follows from Theorem 4.7.

The following result is a definable $C^r G$ -version of Theorem 3.4, which is obtained similarly.

Theorem 4.8. *Let $\eta = (E, p, X)$ be a definable $C^r G$ -vector bundle of rank k over an affine definable $C^r G$ -manifold X and $1 \leq r < \infty$. Then the following five properties are equivalent.*

- (1) *The bundle η is strongly definable.*
- (2) *There exists a surjective definable $C^r G$ -morphism from a trivial G -vector bundle $X \times \Omega$ onto η for some representation Ω of G .*
- (3) *There exists an injective definable $C^r G$ -morphism from η to a trivial G -vector bundle $X \times \Omega$ for some representation Ω of G .*
- (4) *There exists a definable $C^r G$ -vector bundle η' over X such that $\eta \oplus \eta'$ is definably $C^r G$ -isomorphic to a trivial G -vector bundle.*

- (5) *There exist non-equivariant definable C^r -sections $s_1, \dots, s_n : X \rightarrow E$ of η such that:*
- (a) *For any $x \in X$, the vectors $s_1(x), \dots, s_n(x)$ generate the fiber $p^{-1}(x)$ over x .*
 - (b) *The sections $s_1, \dots, s_n$ generate a finite dimensional G -invariant vector subspace of $\Gamma(\eta)$.*

Proof of Theorem 1.7. Since X is compact, a similar proof of Lemma 2.15 proves that there exist finitely many points $x_1, \dots, x_n \in X$ with definable C^r -slices $S_{x_1}, \dots, S_{x_n}$ and α -dimensional representations $\Omega_{x_1}, \dots, \Omega_{x_n}$ of $G_{x_1}, \dots, G_{x_n}$, respectively, such that $\{GS_{x_i}\}_{i=1}^n$ is a finite definable open G -covering of X and each $\eta|_{GS_{x_i}}$ is definably $C^r G$ -equivalent to $\epsilon(S_{x_i})$, where $\epsilon(S_{x_i}) = (G \times_{G_{x_i}} (S_{x_i} \times \Omega_{x_i}), p, G \times_{G_{x_i}} S_{x_i}), p : G \times_{G_{x_i}} (S_{x_i} \times \Omega_{x_i}) \rightarrow G \times_{G_{x_i}} S_{x_i}, p([g, x, y]) = [g, x]$ and α denotes the rank of η . Clearly each $\epsilon(S_{x_i})$ admits finitely many definable C^r -sections satisfying Condition (5) in Theorem 4.8. Thus every $\eta|_{GS_{x_i}}$ admits definable C^r -sections $s_{i1}, \dots, s_{it_i}$ satisfying the same condition.

By Proposition 4.6, we have an equivariant definable C^r -partition of unity $\{\lambda_i\}_{i=1}^n$ subordinate to $\{GS_{x_i}\}_{i=1}^n$. Let $\overline{s_{iq}} := \lambda_i s_{iq}$. Then for any $g \in G$, $g \cdot \overline{s_{iq}} = \lambda_i(g \cdot s_{iq})$. Therefore a finite family of definable C^r -sections $\overline{s_{11}}, \dots, \overline{s_{1t_1}}, \dots, \overline{s_{n1}}, \dots, \overline{s_{nt_n}}$ satisfies the required conditions.

Now we prove the second part of the theorem. If η is strongly definable, then there exist a representation Ω of G and a definable $C^r G$ -map f from X to $G(\Omega, \alpha)$ such that η is definably $C^r G$ -isomorphic to $f^*(\gamma(\Omega, \alpha))$. Since the total space of $f^*(\gamma(\Omega, \alpha))$ is affine, E is affine.

Conversely, we assume that E is a definable $C^r G$ -submanifold of a representation Ξ of G .

Let

$$F_1 : X \rightarrow M(\Xi), F_1(x) = \text{the matrix projecting } T_x \Xi \text{ onto } T_x E,$$

$$F_2 : X \rightarrow M(\Xi), F_2(x) = \text{the matrix projecting } T_x \Xi \text{ onto } T_x X.$$

Then by a way similar to the proof of I.3.3 [19], F_1 and F_2 are definable maps. Thus they are definable C^r -maps. By the definition of G -action, they are G -maps. Hence they are definable $C^r G$ -maps. Let

$$F : X \rightarrow G(\Xi, \alpha), F = (id - F_2)F_1.$$

Then F is a definable $C^r G$ -map and η is definably $C^r G$ -isomorphic to $F^*(\gamma(\Xi, \alpha))$. Therefore η is strongly definable. $\square$

REFERENCES

- [1] M. F. Atiyah, *K-theory*, Benjamin, 1967.
- [2] G.E. Bredon, *Introduction to compact transformation groups*, Academic Press, 1972.
- [3] M. J. Choi, T. Kawakami, and D.H. Park, *Equivariant semialgebraic vector bundles*, Topology and its appl. **123** (2002), 383-400.
- [4] H. Delfs and M. Knebusch, *Semialgebraic topology over a real closed field II: Basic theory of semialgebraic spaces*, Math. Z. **178** (1981), 175-213.
- [5] H. Delfs and M. Knebusch, *Separation, retraction and homotopy extension in semialgebraic spaces*, Pacific J. Math. **114**(1) (1984), 47-71.
- [6] L. van den Dries, *Tame topology and o-minimal structures*, Lecture notes series **248**, London Math. Soc. Cambridge Univ. Press (1998).

- [7] L. van den Dries, A. Macintyre, and D. Marker, *The elementary theory of restricted analytic field with exponentiation*, Ann. Math. **140** (1994), 183–205.
- [8] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497–540.
- [9] L. van den Dries and P. Speissegger, *The real field with convergent generalized power series*, Trans. Amer. Math. Soc. **350**, (1998), 4377–4421.
- [10] T. Kawakami, *Algebraic G vector bundles and Nash G vector bundles*, Chinese J. Math. **22(3)** (1994), 275–289.
- [11] T. Kawakami, *Definable G CW complex structures of definable G sets and their applications*, preprint.
- [12] T. Kawakami, *Equivariant differential topology in an o-minimal expansion of the field of real numbers*, Topology and its appl. **123** (2002), 323–349.
- [13] T. Kawakami, *Imbedding of manifolds defined on an o-minimal structures on $(\mathbb{R}, +, \cdot, <)$* , Bull. Korean Math. Soc. **36** (1999), 183–201.
- [14] K. Kawakubo, *The theory of transformation groups*, Oxford Univ. Press, 1991.
- [15] R. K. Lashof, *Equivariant Bundles*, Illinois J. Math. **26(2)** (1982), 257–271.
- [16] D.H. Park and D.Y. Suh, *Linear embeddings of semialgebraic G -spaces*, Math. Z. **242**, (2002), 725–742.
- [17] Y. Peterzil, A. Pillay and S. Starchenko, *Definably simple groups in o-minimal structures*, Trans. Amer. Math. Soc. **352** (2000), 4397–4419.
- [18] M. Shiota, *Geometry of subanalytic and semialgebraic sets*, Progress in Math. **150** (1997), Birkhäuser.
- [19] M. Shiota, *Nash manifolds*, Lecture Note in Math. **1269**, Springer-Verlag (1987).
- [20] G. Segal, *Equivariant K -theory*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 129–151.
- [21] H. Weyl, *The classical groups (2nd ed.)*, Princeton Univ. Press, Princeton, N.J., (1946).

DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, WAKAYAMA UNIVERSITY, SAKAEDANI
WAKAYAMA 640-8510, JAPAN

E-mail address: kawa@center.wakayama-u.ac.jp