

On the order of strongly starlikeness and order of starlikeness of a certain convex functions

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Let $\mathcal{A}$ denote the set of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in $\mathbb{D} = \{z : |z| < 1\}$. Let $f(z) \in \mathcal{A}$ and suppose that for $0 < \alpha < 1$ and $0 < \beta < 1$,

$$(1) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad \text{in } \mathbb{D},$$

$$(2) \quad 1 + \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) > \alpha \quad \text{in } \mathbb{D},$$

$$(3) \quad \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \beta \quad \text{in } \mathbb{D},$$

$$(4) \quad \left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2} \beta \quad \text{in } \mathbb{D},$$

$$(5) \quad \left| \arg \left(\frac{zf'(z)}{f(z)} - \alpha \right) \right| < \frac{\pi}{2} \beta \quad \text{in } \mathbb{D},$$

$$(6) \quad \left| \arg \left(1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \right| < \frac{\pi}{2} \beta \quad \text{in } \mathbb{D}.$$

Then if $f(z)$ satisfies the above conditions (1), (2), (3), (4), (5) and (6), we call $f(z)$ starlike of order α , convex of order α , strongly starlike of order β , strongly convex of order β , strongly starlike of order β and starlike of order α , and strongly convex of order β and convex of order α respectively and let us denote the class of functions which satisfy the conditions (1), (2), (3), (4), (5) and (6) by $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$, $\mathcal{SS}^*(\beta)$, $\mathcal{SC}(\beta)$, $\mathcal{S}^*(\alpha, \beta)$ and $\mathcal{C}(\alpha, \beta)$ respectively.

Marx [2] and Stroh  cker [5] showed that

$$f(z) \in \mathcal{C}(0) \quad \text{implies} \quad f(z) \in \mathcal{S}^* \left(\frac{1}{2} \right)$$

and MacGregor [1] and Wilken and Feng [6] obtained more general result that

$$f(z) \in \mathcal{C}(\alpha) \text{ implies } f(z) \in \mathcal{S}^*(\beta(\alpha))$$

where $0 \leq \alpha < 1$ and

$$(7) \quad \beta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}[1-2^{2\alpha-1}]} & \text{if } \alpha \neq \frac{1}{2} \\ \frac{1}{2 \log 2} & \text{if } \alpha = \frac{1}{2}. \end{cases}$$

Mocanu [3] showed that

$$f(z) \in \mathcal{SC}(\gamma) \text{ implies } f(z) \in \mathcal{SS}^*(\beta)$$

where

$$\tan \frac{\pi\gamma}{2} = \tan \frac{\pi\beta}{2} + \frac{\beta}{(1-\beta) \cos \frac{\pi\beta}{2}} \left(\frac{1-\beta}{1+\beta} \right)^{\frac{1+\beta}{2}}$$

and $0 < \beta < 1$.

On the other hand, Nunokawa [4] obtained that

$$f(z) \in \mathcal{SC}(\alpha(\beta)) \text{ implies } f(z) \in \mathcal{SS}^*(\beta)$$

where

$$\alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \frac{\beta q(\beta) \sin \frac{\pi}{2}(1-\beta)}{p(\beta) + \beta q(\beta) \cos \frac{\pi}{2}(1-\beta)}$$

$$p(\beta) = (1+\beta)^{\frac{1+\beta}{2}}, \quad q(\beta) = (1-\beta)^{\frac{\beta-1}{2}}$$

and $0 < \beta < 1$.

In this paper, we need the following lemma due to Nunokawa [4].

Lemma 1 *Let $P(z)$ be analytic in $\mathbb{D}$, $P(0) = 1$, $P(z) \neq 0$ in $\mathbb{D}$ and suppose that there exists a point $z_0 \in \mathbb{D}$ such that*

$$|\arg(P(z))| < \frac{\pi}{2}\delta \quad \text{for } |z| < |z_0|$$

and

$$|\arg(P(z_0))| = \frac{\pi}{2}\delta$$

where $0 < \delta$. Then we have

$$\frac{z_0 P'(z_0)}{P(z_0)} = ik\delta$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when } \arg(P(z_0)) = \frac{\pi}{2} \delta$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when } \arg(P(z_0)) = -\frac{\pi}{2} \delta$$

where

$$P(z_0)^{\frac{1}{\delta}} = \pm ia \quad \text{and} \quad 0 < a.$$

Theorem 1 Let $f(z) \in \mathcal{A}$ and suppose that $\frac{zf'(z)}{f(z)} \neq \beta(\alpha)$ in $\mathbb{D}$ and

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \right| < \frac{\pi}{2} \gamma \quad \text{in } \mathbb{D}$$

where $0 \leq \alpha < 1$ and $0 < \gamma < 1$. Then we have

$$\left| \arg \left(\frac{zf'(z)}{f(z)} - \beta(\alpha) \right) \right| < \frac{\pi}{2} \delta \quad \text{in } \mathbb{D}$$

where $\beta(\alpha)$ is defined by (7), $0 < \delta < 1$,

$$\gamma = \frac{2}{\pi} \tan^{-1} \delta (1 - \beta(\alpha)) \left(\frac{a_0^{\delta+1} + a_0^{\delta-1}}{\beta(\alpha) + (1 - \beta(\alpha))a_0^\delta} \right)$$

and a_0 is the positive root of the equation

$$(1 - \beta(\alpha))x^\delta(x^2 - 1) = \beta(\alpha) \{ (1 - \delta) - (1 + \delta)x^2 \}.$$

Proof. Let us put

$$p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1 \quad \text{and} \quad p(z) \neq \beta(\alpha) \quad \text{in } \mathbb{D}.$$

Then it follows that

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}.$$

If there exists a point $z_0 \in \mathbb{D}$ such that

$$|\arg(P(z))| = |\arg(p(z) - \beta(\alpha))| < \frac{\pi}{2} \delta$$

for $|z| < |z_0|$ and

$$|\arg(P(z_0))| = |\arg(p(z_0) - \beta(\alpha))| = \frac{\pi}{2} \delta$$

where $P(z) = \frac{p(z) - \beta(\alpha)}{1 - \beta(\alpha)}$ and $P(0) = 1$, then from Lemma 1, we have

$$\frac{z_0 P'(z_0)}{P(z_0)} = \frac{z_0 p'(z_0)}{p(z_0) - \beta(\alpha)} = i\delta k.$$

For the case $\arg(P(z_0)) = \arg(p(z_0) - \beta(\alpha)) = \frac{\pi}{2}\delta$, it follows that

$$\begin{aligned} (8) \quad & \arg \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha \right) \\ &= \arg \left\{ (p(z_0) - \beta(\alpha)) \left(1 + \frac{z_0 p'(z_0)}{p(z_0) - \beta(\alpha)} \cdot \frac{1}{p(z_0)} + \frac{\beta(\alpha) - \alpha}{p(z_0) - \beta(\alpha)} \right) \right\} \\ &= \frac{\pi\delta}{2} + \arg \left\{ 1 + \frac{i\delta k}{\beta(\alpha) + (1 - \beta(\alpha))(ia)^\delta} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))(ia)^\delta} \right\} \\ &> \frac{\pi\delta}{2} + \arg \left\{ 1 + \frac{i\delta k}{(\beta(\alpha) + (1 - \beta(\alpha))a^\delta)e^{i\frac{\pi}{2}\delta}} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^\delta e^{i\frac{\pi}{2}\delta}} \right\} \\ &= \frac{\pi\delta}{2} + \arg \left\{ e^{-i\frac{\pi}{2}\delta} \left(e^{i\frac{\pi}{2}\delta} + \frac{i\delta k}{\beta(\alpha) + (1 - \beta(\alpha))a^\delta} + \frac{\beta(\alpha) - \alpha}{(1 - \beta(\alpha))a^\delta} \right) \right\} \\ &\geq \arg \left\{ e^{i\frac{\pi}{2}\delta} + \frac{1}{2} \left(\frac{i\delta(a + a^{-1})}{\beta(\alpha) + (1 - \beta(\alpha))a^\delta} + \frac{1}{(1 - \beta(\alpha))a^\delta} \right) \right\} \end{aligned}$$

since we have $0 < \beta(\alpha) - \alpha \leq \frac{1}{2}$ and Lemma 1. Let us put

$$(9) \quad \varphi(x) = \frac{x^\delta(x + x^{-1})}{\beta(\alpha) + (1 - \beta(\alpha))x^\delta} = \frac{x + x^{-1}}{1 - \beta(\alpha) + \beta(\alpha)x^{-\delta}}$$

for $0 < x$. Then it follows that

$$\varphi'(x) = \frac{1}{x^2(1 - \beta(\alpha) + \beta(\alpha)x^{-\delta})^2} [(1 - \beta(\alpha))(x^2 - 1) + \beta(\alpha)x^{-\delta} \{(1 + \delta)x^2 - (1 - \delta)\}].$$

Putting a_0 be the positive root of the equation $\varphi'(x) = 0$ or

$$x^\delta(x^2 - 1) = \beta(\alpha) \{(1 - \delta) - (1 + \delta)x^2\},$$

then $\varphi(x)$ takes its minimum value at $x = a_0$. Therefore, from (8) and (9), we have

$$\begin{aligned} \arg \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha \right) &> \arg \left\{ e^{i\frac{\pi}{2}\delta} + \frac{1}{2} \left(\frac{1}{(1 - \beta(\alpha))a_0^\delta} + \frac{i\delta(a_0 + a_0^{-1})}{\beta(\alpha) + (1 - \beta(\alpha))a_0^\delta} \right) \right\} \\ &\geq \arg \left(\frac{1}{(1 - \beta(\alpha))a_0^\delta} + i \frac{\delta(a_0 + a_0^{-1})}{\beta(\alpha) + (1 - \beta(\alpha))a_0^\delta} \right) \\ &= \tan^{-1} \delta(1 - \beta(\alpha)) \left(\frac{a_0^{\delta+1} + a_0^{\delta-1}}{\beta(\alpha) + (1 - \beta(\alpha))a_0^\delta} \right). \end{aligned}$$

This contradicts hypothesis of Theorem 1.

For the case $\arg(P(z_0)) = \arg(p(z_0) - \beta(\alpha)) = -\frac{\pi}{2}\delta$, applying the same method as the above and Lemma 1, we have

$$\arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha\right) < -\tan^{-1} \delta(1 - \beta(\alpha)) \left(\frac{a_0^{\delta+1} + a_0^{\delta-1}}{\beta(\alpha) + (1 - \beta(\alpha))a_0^\delta}\right).$$

This is also a contradiction and therefore it completes the proof of Theorem 1. $\square$

Remark Theorem 1 shows that

$$f(z) \in \mathcal{SC}(\alpha, \gamma) \text{ implies } f(z) \in \mathcal{SS}^*(\beta(\alpha), \delta)$$

where

$$\gamma = \frac{2}{\pi} \tan^{-1} \delta(1 - \beta(\alpha)) \left(\frac{a_0^{\delta+1} + a_0^{\delta-1}}{\beta(\alpha) + (1 - \beta(\alpha))a_0^\delta}\right)$$

but Theorem 1 is not a sharp result and so, the authors expect that Theorem 1 will be improved by someone in future.

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