

# Integral representations of weighted Beppo Levi functions

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## §1. Introduction

It is well known that functions in Sobolev spaces can be represented as Bessel potentials ([17; Chapter V, Theorem 3]). In this paper we shall consider a similar problem for weighted Beppo Levi functions.

Let  $1 < p < \infty$  and let  $w$  be a weight (nonnegative Lebesgue measurable function) satisfying the Muckenhoupt  $A_p$  condition:

$$(A_p) \quad \sup_Q \left( \frac{1}{|Q|} \int_Q w dx \right) \left( \frac{1}{|Q|} \int_Q w^{1/(1-p)} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes  $Q$  with sides parallel to the axes and  $|Q|$  stands for the Lebesgue measure of  $Q$  (see [1]).

By  $A_p$  we denote the class of weights  $w$  satisfying  $(A_p)$ . We write

$$\|f\|_{L^p, w} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}, \quad L^p(\mathbb{R}^n, w) = \{f; \|f\|_{L^p, w} < \infty\}.$$

By  $BL_m(L^p(\mathbb{R}^n, w))$ ,  $m \geq 1$ , we denote the space of distributions whose partial derivatives of  $m$ -th order all belong to  $L^p(\mathbb{R}^n, w)$  (see [2]).

Since  $w^{1/(1-p)}$  is locally integrable by  $(A_p)$ , it follows from

Hölder's inequality and Kryloff's theorem [16; Chapitre VI, Théorème 15] that a distribution in  $BL_m(L^p(\mathbb{R}^n, w))$  is a locally integrable function whenever  $w \in A_p$ . Therefore we call a locally integrable

function in  $BL_m(L^p(\mathbb{R}^n, w))$  a Beppo Levi function of order  $m$  with

weight  $w$ . If  $w = 1$ , then we write simply  $\|f\|_{L^p}$ ,  $L^p(\mathbb{R}^n)$  and  $BL_m(L^p(\mathbb{R}^n))$  for  $\|f\|_{L^p, w}$ ,  $L^p(\mathbb{R}^n, w)$  and  $BL_m(L^p(\mathbb{R}^n, w))$ , respectively. Hereafter we limit ourselves to the case  $1 \leq m \leq n - 1$ .

For a multiindex  $\alpha = (\alpha_1, \dots, \alpha_n)$  we write  $|\alpha| = \alpha_1 + \dots + \alpha_n$ ,  $\alpha! = \alpha_1! \dots \alpha_n!$  and

$$D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}.$$

By  $h_m$  we denote the Riesz kernel  $\gamma(m)^{-1} |x|^{m-n}$  with  $\gamma(m) = \pi^{n/2} \Gamma(\frac{m}{2}) / \Gamma(\frac{n-m}{2})$  (cf. [17; p.117]). If  $|\alpha| = m$ , then  $D^\alpha h_m$  is not a locally integrable function. It will be stated in Lemma 2 in §2 that  $D^\alpha h_m$  is the sum of a principal value distribution  $S_\alpha$  and a multiple  $b_\alpha \delta$  of the Dirac measure  $\delta$  at the origin. It is proved in [1] that this kind of distribution is related to weights  $w$  in  $A_p$  as follows:

$$(1) \quad \|(D^\alpha h_m) * g\|_{L^p, w} \leq \text{const.} \|g\|_{L^p, w} \quad \text{for } g \in L^p(\mathbb{R}^n, w).$$

Let  $\mathcal{D}$  be the space of indefinitely differentiable functions with compact support and  $P_{m-1}$  the space of all polynomials of degree smaller than or equal to  $m - 1$ . Let  $c_\alpha = (-1)^m m! / \alpha!$ . Mizuta proved

Theorem A ([8; Theorem 5.2]). Let  $2m < n$  and let  $f \in BL_m(L^p(\mathbb{R}^n))$ . If

$$(2) \quad \text{there is a sequence } \{\psi_j\}_j \subset \mathcal{D} \text{ such that } D^\alpha \psi_j \rightarrow D^\alpha f \text{ in } L^p(\mathbb{R}^n) \\ \text{for } |\alpha| = m,$$

and  $g = \sum_{|\alpha|=m} c_\alpha (D^\alpha h_m) * D^\alpha f$  satisfies

$$(3) \quad \int_{\mathbb{R}^n} (1 + |x|)^{m-n} |g(x)| dx < \infty,$$

then  $f = h_m * g + P$  a.e. on  $R^n$  with some  $P \in P_{m-1}$ .

We shall show that assumption (2) is superfluous and the theorem extends to the case when  $1 \leq m \leq n - 1$  and  $f \in BL_m(L^p(R^n, w))$  with general  $w \in A_p$ . More precisely, we shall prove

Theorem 1. Let  $w \in A_p$ . Suppose that  $f \in BL_m(L^p(R^n, w))$  and  $g = \sum_{|\alpha|=m} c_\alpha (D^\alpha h_m) * D^\alpha f$ . If  $g$  satisfies (3), then  $f = h_m * g + P$  a.e. on  $R^n$  with some  $P \in P_{m-1}$ . Moreover this representation is unique in the sense that if  $f = h_m * \mu + P'$  a.e. on  $R^n$ , where  $P' \in P_{m-1}$  and  $\mu$  is a signed measure such that

$$(4) \quad \int_{R^n} (1 + |x|)^{m-n} d|\mu|(x) < \infty,$$

then  $\mu$  is absolutely continuous,  $d\mu = gdx$  and  $P' = P$ .

Ohtsuka [13] proved Theorem 1 for  $m = 1$  and  $w = 1$  by using extremal length (see also [12] for the definition and the properties of extremal length). In case  $m > 1$ , however, the theory of extremal length is not applicable to  $BL_m(L^p(R^n, w))$ , so our argument will depend on the general theory of distributions and singular integrals (see [1], [11], [16] and [17]).

Let  $\omega_{n-1} = 2\pi^{n/2}/\Gamma(\frac{n}{2})$  be the surface area of the unit sphere in  $R^n$  and let  $a_\alpha = m/(\alpha! \omega_{n-1})$ . Mizuta proved

Theorem B ([8; Theorem 3.1]). Let  $f \in BL_m(L^p(R^n))$  satisfy (2).

If

$$(5) \quad \int_{R^n} (1 + |x|)^{m-n} |D^\alpha f(x)| dx < \infty \quad \text{for any } \alpha \text{ with } |\alpha| = m,$$

then

$$(6) \quad f(x) = \sum_{|\alpha|=m} a_\alpha \int_{\mathbb{R}^n} \frac{(x-y)^\alpha D^\alpha f(y)}{|x-y|^n} dy + P(x) \quad \text{a.e. on } \mathbb{R}^n,$$

where  $P \in P_{m-1}$ .

In case  $m = 1$  Ohtsuka [13; Theorem 29] proved that (2) can be dropped. We shall extend Ohtsuka's result to higher order Beppo Levi functions with weight  $w$  in  $A_p$ .

Theorem 2. Let  $w \in A_p$  and let  $f \in BL_m(L^p(\mathbb{R}^n, w))$ . If  $f$  satisfies (5), then (6) holds.

It is easy to see that  $w(x) = (1 + |x|)^{rp}$  belongs to  $A_p$  if and only if  $-n < rp < n(p-1)$ . Hence this theorem includes Kurokawa [4; Theorem 2.6].

In case  $g = \sum_{|\alpha|=m} c_\alpha (D^\alpha h_m) * D^\alpha f$  does not satisfy (3), the weighted Beppo Levi function  $f$  cannot be represented as the sum of a Riesz potential and a polynomial. However, a certain modification of the Riesz kernel (cf. [3; Chapter IV]) will enable us to represent  $f$  as the sum of a modified Riesz potential and a polynomial, and to show

Theorem 3 (cf. [14], [4; Theorem 3.2]). Let  $w \in A_p$ . If  $f \in BL_m(L^p(\mathbb{R}^n, w))$ , then there is a sequence  $\{\psi_j\}_j \subset \mathcal{D}$  such that

$$\lim_{j \rightarrow \infty} \sum_{|\alpha|=m} \|D^\alpha f - D^\alpha \psi_j\|_{L^p, w} = 0.$$

In the rest of this section we deal with  $w \in A_p$  for which every

$g \in L^p(\mathbb{R}^n, w)$  satisfies (3). In order to simplify the notation we denote by  $A_{p,m}$  the class of all weights  $w \in A_p$  such that every  $g \in L^p(\mathbb{R}^n, w)$  satisfies (3). We shall show that  $w \in A_{p,m}$  if and only if

$$(7) \quad \int_{\mathbb{R}^n} (1 + |x|)^{(m-n)p/(p-1)} w(x)^{1/(1-p)} dx < \infty.$$

See Theorem 7 in §5. Since  $w(x) = (1 + |x|)^{rp}$  belongs to  $A_{p,m}$  if and only if  $m - n/p < r < n(1 - 1/p)$ , it follows that  $A_{p,m}$  is a proper subclass of  $A_p$ . If  $w \in A_{p,m}$ , then Theorem 1 gives a decomposition

$$BL_m(L^p(\mathbb{R}^n, w)) = I_m(L^p(\mathbb{R}^n, w)) \oplus P_{m-1},$$

where  $I_m(L^p(\mathbb{R}^n, w)) = \{h_m * g; g \in L^p(\mathbb{R}^n, w)\}$ . We shall consider a condition for  $f \in BL_m(L^p(\mathbb{R}^n, w))$  to belong to  $I_m(L^p(\mathbb{R}^n, w))$ . For this purpose we introduce a notion which describes the behavior at  $\infty$  of a weighted Beppo Levi function.

Definition. Let  $f_j$  and  $f \in BL_m(L^p(\mathbb{R}^n, w))$ . We say that  $f_j$  converges to  $f$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense if

$$\lim_{j \rightarrow \infty} \sum_{|\alpha|=m} \|D^\alpha f_j - D^\alpha f\|_{L^p, w} = 0,$$

$$\lim_{j \rightarrow \infty} f_j = f \text{ a.e. on } \mathbb{R}^n.$$

We say that  $f$  vanishes at  $\infty$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense if there is a sequence  $\{\psi_j\}_j \subset \mathcal{D}$  converging to  $f$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense.

We shall show

Theorem 4. Let  $w \in A_{p,m}$ . Then  $f \in BL_m(L^p(\mathbb{R}^n, w))$  belongs to

$I_m(L^p(\mathbb{R}^n, w))$  if and only if  $f$  vanishes at  $\infty$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense.

Corollary 1. Let  $w \in A_{p,m}$ . If  $f \in BL_m(L^p(\mathbb{R}^n, w))$  and  $\lim_{|x| \rightarrow \infty} f(x) = 0$ , then  $f \in I_m(L^p(\mathbb{R}^n, w))$ , and hence  $f$  vanishes at  $\infty$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense.

We shall give a criterion for  $f \in BL_m(L^p(\mathbb{R}^n, w))$  to vanish at  $\infty$  in terms of the integrability of  $f$  in case  $w = v^p$  is a weight introduced by Muckenhoupt and Wheeden [11].

Lemma A ([11; Theorem 4]). Let  $1 < p < n/m$  and  $1/p^* = 1/p - m/n$ . Suppose that  $v \geq 0$  satisfies

$$(8) \quad \sup_Q \left( \frac{1}{|Q|} \int_Q v^{p^*} dx \right)^{1/p^*} \left( \frac{1}{|Q|} \int_Q v^{-p'} dx \right)^{1/p'} < \infty,$$

where  $p' = p/(p-1)$  and the supremum is taken over all cubes  $Q$  with sides parallel to the axes. Then

$$\|(h_m * g)v\|_{L^{p^*}} \leq \text{const.} \|gv\|_{L^p} \quad \text{for } g \in L^p(\mathbb{R}^n, v^p).$$

Obviously, Hölder's inequality yields that if  $v$  satisfies (8), then  $v^p \in A_p$ . Hence we can easily deduce from (1) and this lemma that  $v^p \in A_{p,m}$ . We shall show

Theorem 5. Let  $1 < p < n/m$ ,  $1/p^* = 1/p - m/n$  and  $v$  satisfy (8).

(i) A function  $f$  in  $BL_m(L^p(\mathbb{R}^n, v^p))$  vanishes at  $\infty$  in the

$BL_m(L^p(R^n, V^p))$  sense if and only if  $f \in L^{p^*}(R^n, V^{p^*})$ .

(ii) If  $f \in BL_m(L^p(R^n, V^p))$  satisfies

$$\int_{R^n} |f(x)|^q V(x)^r dx < \infty,$$

for some  $q > 0$  and some  $r$ ,  $0 < r \leq p^*$ , then  $f$  vanishes at  $\infty$  in the  $BL_m(L^p(R^n, V^p))$  sense.

This theorem yields the implication

$$BL_m(L^p(R^n, V^p)) \cap \left( \bigcup_{\substack{q>0 \\ 0<r\leq p^*}} L^q(R^n, V^r) \right)$$

$$\subset BL_m(L^p(R^n, V^p)) \cap L^{p^*}(R^n, V^{p^*}) = I_m(L^p(R^n, V^p)).$$

By virtue of (1), Theorem 1 and Lemma A we readily have an improvement of [11; Theorem 9].

Corollary 2. Let  $m$ ,  $p$ ,  $p^*$  and  $V$  be as in Theorem 5. Then there is a positive constant  $C$  depending only on  $m$ ,  $p$  and  $V$  such that

$$\|fV\|_{L^{p^*}} \leq C \sum_{|\alpha|=m} \|(D^\alpha f)V\|_{L^p}$$

for  $f \in BL_m(L^p(R^n, V^p)) \cap \left( \bigcup_{\substack{q>0 \\ 0<r\leq p^*}} L^q(R^n, V^r) \right)$ .

## §2. Preliminaries

We collect some basic results on the theory of distributions. We shall mainly use the notation of [16]. We write

$$\langle T, \psi \rangle = T(\psi)$$

for a distribution  $T$  and a test function  $\psi$ . In order to avoid confusion, we write

$$\langle T_x, \psi \rangle$$

if  $\psi$  involves two variables  $x$  and  $y$ , and the distribution  $T$  acts on  $\psi(\cdot, y)$  for each fixed  $y$ . As in [16; Chapitre VII] we define the Fourier transform of  $\psi \in \mathcal{S}$  and that of  $T \in \mathcal{S}'$  by

$$\mathcal{F}\psi(y) = \phi(y) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} \psi(x) dx,$$

$$\langle \mathcal{F}T, \psi \rangle = \langle \hat{T}, \psi \rangle = \langle T, \phi \rangle \quad \text{for } \psi \in \mathcal{S},$$

where  $\mathcal{S}$  is the space of indefinitely differentiable functions decreasing rapidly at  $\infty$  and  $\mathcal{S}'$  is the space of tempered distributions. We note that the Fourier transform defined here corresponds to the inverse Fourier transform in [17]. By  $\mathcal{E}'$  and  $\mathcal{D}'_{L^p}$  we denote the space of distributions of compact support and that of distributions  $T$  of the form

$$T = \sum_{|\alpha| \leq k} D^\alpha f_\alpha, \quad \text{where } k \geq 0 \text{ and } f_\alpha \in L^p(\mathbb{R}^n).$$

Schwartz [16; Chapitres VI and VII] proved

Lemma B. (i) If  $1 \leq p \leq q \leq \infty$ , then

$$\mathcal{E}' \subset \mathcal{D}'_{L^p} \subset \mathcal{D}'_{L^q} \subset \mathcal{S}' \subset \mathcal{D}'.$$

(ii) If  $0 \leq 1/r = 1/p + 1/q - 1 \leq 1$ , then the convolution  $S * T$  exists and belongs to  $\mathcal{D}'_{L^r}$  for  $S \in \mathcal{D}'_{L^p}$  and  $T \in \mathcal{D}'_{L^q}$ .

(iii) If  $S$  and  $T$  belong to  $\mathcal{D}'_{L^2}$ , then  $\mathcal{F}S$  and  $\mathcal{F}T$  belong to  $L^2_{\text{loc}}(\mathbb{R}^n)$  and  $\mathcal{F}(S * T) = \mathcal{F}S \cdot \mathcal{F}T$ .

We can easily give another condition for the convolution of a

function and a measure to be defined.

Lemma 1. (i) Let  $\ell$  be a real number. Suppose that  $f \in L^1_{loc}(\mathbb{R}^n)$  and  $|f(x)| \leq \text{const.} |x|^\ell$  for  $|x| \geq 1$ . If a signed measure  $\mu$  satisfies

$$\int_{\mathbb{R}^n} (1 + |x|)^\ell d|\mu|(x) < \infty,$$

then  $f*\mu$  is well-defined and belongs to  $L^1_{loc}(\mathbb{R}^n)$ ; moreover

$$D^\beta(f*\mu) = (D^\beta f)*\mu = f*(D^\beta \mu) \quad \text{for any multiindex } \beta.$$

(ii) Let  $0 < m < n$ . If a signed measure  $\mu$  satisfies (4), then  $h_m*\mu$  exists and belongs to  $L^1_{loc}(\mathbb{R}^n)$ . If  $\mu$  does not satisfy (4), then  $h_m*|\mu| \equiv \infty$  on  $\mathbb{R}^n$ .

We need several results from the theory of singular integrals. Consider the class consisting of all distributions  $T$  of the form

$$(9) \quad T = c\delta + \text{v.p.} \frac{\Omega(x)}{|x|^n},$$

$$\text{i.e., } \langle T, \psi \rangle = c\psi(0) + \lim_{\epsilon \downarrow 0} \int_{|x| > \epsilon} \frac{\Omega(x)}{|x|^n} \psi(x) dx \quad \text{for } \psi \in \mathcal{D},$$

where  $c$  is a constant;  $\Omega$  is a homogeneous function of degree 0, which is indefinitely differentiable on the unit sphere and

$$\int_{|x|=1} \Omega(x) d\sigma(x) = 0.$$

Lemma C ([17; Chapter III, Theorem 6]). A distribution  $T$  in  $\mathcal{S}'$  is written as (9) if and only if the Fourier transform  $\hat{T}$  is a homogeneous function of degree 0, which is indefinitely differentiable on the unit sphere.

The Muckenhoupt  $A_p$  condition is related to distributions of the form (9) as follows:

Lemma D ([1]). Let  $w \in A_p$  and let  $T$  be a distribution of the form (9). Then

$$(10) \quad \|T * g\|_{L^p, w} \leq \text{const.} \|g\|_{L^p, w} \quad \text{for } g \in L^p(\mathbb{R}^n, w).$$

From Lemmas C and D we can derive a generalization of (1).

Lemma 2 (cf. [8; §3]). Let  $m \geq 1$  and  $\ell \geq 0$ . If  $|\alpha| = m$  and  $|\beta| = \ell$ , then the distribution

$$T = D^\alpha \left( \frac{x^\beta}{|x|^{n-m+\ell}} \right),$$

in particular  $D^\alpha h_m$ , is of the form (9) and satisfies (10).

By  $\mathcal{D}'_{L^p, w}$  we denote the class of distributions of the form

$$\sum_{|\alpha| \leq k} D^\alpha f_\alpha, \quad \text{where } k \geq 0 \text{ and } f_\alpha \in L^p(\mathbb{R}^n, w).$$

We shall have

Lemma 3. Let  $S$  and  $T$  be distributions of the form (9) and  $w \in A_p$ . Then

- (i)  $S$  and  $T$  belong to  $\mathcal{D}'_{L^q}$  for any  $q > 1$ .
- (ii) The convolution  $S * T$  exists and is of the form (9).
- (iii) If  $f \in L^p(\mathbb{R}^n, w)$ , then  $(S * T) * f = S * (T * f) \in L^p(\mathbb{R}^n, w)$ .
- (iv) If  $U \in \mathcal{D}'_{L^p, w}$ , then  $(S * T) * U = S * (T * U) \in \mathcal{D}'_{L^p, w}$ .

It is proved in [1] that every  $w \in A_p$  satisfies the Muckenhoupt  $A_\infty$  condition:

There are positive constants  $C, \delta > 0$  such that given any cube  $Q$  and any measurable subset  $E$  of  $Q$ ,

$$(A_\infty) \quad \frac{w(E)}{w(Q)} \leq C \left( \frac{|E|}{|Q|} \right)^\delta, \text{ where } w(A) = \int_A w dx \text{ for } A \subset \mathbb{R}^n.$$

We shall denote by  $A_\infty$  the class of weights  $w$  satisfying  $(A_\infty)$ . Then

$A_\infty = \bigcup_{p>1}^\infty A_p$  (see [1]). We collect some properties of  $A_p$  and  $A_\infty$  weights.

Lemma 4. Let  $w_1$  and  $w_2$  belong to  $A_p$ . Then the weights  $\max\{w_1, w_2\}$ ,  $\min\{w_1, w_2\}$  and  $w_1 + w_2$  belong to  $A_p$ .

Lemma 5. Let  $w \in A_\infty$ . If  $L$  is a cone with vertex at the origin, then  $w(L) = \infty$ . Moreover, there are no nonnegative functions  $u$  and  $v$  such that

$$(11) \quad \begin{aligned} & w \leq u + v \text{ on } L, \\ & \int_L u^q dx + \int_L v^r dx < \infty \text{ for some } q, r, 0 < q \leq r \leq 1. \end{aligned}$$

If a polynomial  $P$  belongs to  $L^s(\mathbb{R}^n, w)$  for some  $s > 0$ , then  $P \equiv 0$ .

Lemma 6. Let  $w \in A_p$ . Then

$$(i) \quad \int_{\mathbb{R}^n} (1 + |x|)^{-n} |g(x)| dx < \infty \text{ for } g \in L^p(\mathbb{R}^n, w).$$

$$(ii) \quad \int_{\mathbb{R}^n} (1 + |x|)^{-np} w(x) dx < \infty.$$

## §3. Proof of Theorems 1 and 2

Proof of Theorem 1. In this proof we let  $\alpha, \beta$  and  $\gamma$  be multiindices of length  $m$ . Take  $\psi \in \mathcal{D}$  such that  $\psi = 1$  on a neighborhood of the origin and let  $h'_m = \psi h_m$  and  $h''_m = (1-\psi)h_m$ . Since  $h''_m \in L^q(\mathbb{R}^n)$  for any  $q > n/(n-m)$  and  $h'_m \in \mathcal{E}'$ , it follows from Lemma B (i) that  $h_m \in \mathcal{D}'_{L^q}$  for any  $q > n/(n-m)$ . Since  $D^\alpha h_m$  belongs to  $\mathcal{D}'_{L^q}$  for any  $q > 1$  by Lemmas 2 and 3, we have from Lemma B (ii) that the convolution  $h_m * D^\alpha h_m$  is well-defined and belongs to  $\mathcal{D}'_{L^q}$  for any  $q > n/(n-m)$ . We observe that

$$D^\beta(h_m * D^\alpha h_m) = D^\alpha h_m * D^\beta h_m = h_m * D^\alpha D^\beta h_m.$$

Noting that  $D^\alpha h_m \in \mathcal{D}'_{L^2}$ , we obtain from Lemma B (iii) that

$$\begin{aligned} \mathcal{F}(h_m * \sum_{|\alpha|=m} c_\alpha D^{2\alpha} h_m) &= \mathcal{F}(\sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha h_m) \\ &= \sum_{|\alpha|=m} c_\alpha \{(2\pi i x)^\alpha (2\pi |x|)^{-m}\}^2 = 1, \end{aligned}$$

because

$$(2\pi |x|)^{2m} = \mathcal{F}((-\Delta)^m) = \mathcal{F}(\sum_{|\alpha|=m} c_\alpha D^{2\alpha}) = \sum_{|\alpha|=m} c_\alpha (2\pi i x)^{2\alpha}.$$

Accordingly

$$(12) \quad h_m * \sum_{|\alpha|=m} c_\alpha D^{2\alpha} h_m = \sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha h_m = \delta.$$

By (3) and Lemma 1 we obtain that the convolution  $h_m * g$  is well-defined and belongs to  $L^1_{loc}(\mathbb{R}^n)$ . We infer from Lemmas 2, 3 and (12) that

$$\begin{aligned} D^\beta D^\gamma (h_m * g) &= D^\beta h_m * D^\gamma g = D^\beta h_m * ((\sum_{|\alpha|=m} c_\alpha D^\alpha h_m) * D^\gamma D^\alpha f) \\ &= (D^\beta h_m * \sum_{|\alpha|=m} c_\alpha D^{2\alpha} h_m) * D^\gamma f = D^\beta \delta * D^\gamma f = D^\beta D^\gamma f. \end{aligned}$$

Since  $\beta$  is arbitrary, it follows that

$$D^\gamma f = D^\gamma(h_m * g) + P_\gamma \quad \text{for any } \gamma \text{ with } |\gamma| = m,$$

where  $P_\gamma \in P_{m-1}$ . However  $D^\gamma f \in L^p(\mathbb{R}^n, w)$  and

$$D^\gamma(h_m * g) = \sum_{|\alpha|=m} c_\alpha D^\gamma h_m * (D^\alpha h_m * D^\alpha f) \in L^p(\mathbb{R}^n, w)$$

by Lemma 3, and hence  $P_\gamma$  must be identically 0 by Lemma 5. Since

$D^\gamma f = D^\gamma(h_m * g)$  for any  $\gamma$  with  $|\gamma| = m$ , it follows that

$$f = h_m * g + P,$$

where  $P \in P_{m-1}$ .

The uniqueness of the representation readily follows from the following proposition, which may be of some independent interest.

**Proposition 1.** Let  $0 < m < n$  and let  $\mu$  be a signed measure satisfying (4). If  $h_m * \mu$  coincides with some polynomial  $P$ , then  $\mu = 0$  and hence  $P$  must be 0.

**Proof.** We define a sequence of signed measures  $\mu_j$  of compact support by  $\mu_j(E) = \mu(\{x \in E; |x| \leq j\})$ . Since

$$\left| \int_{|x| > j} \psi d\mu(x) \right| \leq \text{const.} \int_{|x| > j} (1 + |x|)^{m-n} d|\mu|(x) \quad \text{for } \psi \in \mathcal{S},$$

it follows that  $\mu_j \rightarrow \mu$  in  $\mathcal{S}'$ . We claim that  $h_m * \mu_j \rightarrow h_m * \mu$  in  $\mathcal{S}'$ .

Let  $\psi \in \mathcal{S}$ . Take  $\chi \in \mathcal{D}$  such that  $\chi(x) = 1$  for  $|x| \leq 1$  and write

$$\psi = \chi\psi + (1-\chi)\psi = \psi_1 + \psi_2.$$

It is easy to see that  $h_m * |\psi_1|(x) = O(|x|^{m-n})$  as  $|x| \rightarrow \infty$ . Since  $\psi$

decreases rapidly, we have  $|\psi_2(y)| \leq \text{const.} |y|^{-n-1}$ . Let  $|x| > 2$ .

Then

$$\begin{aligned}
h_m^*|\psi_2|(x) &\leq \text{const.} \int_{|y|>1} |x-y|^{m-n}|y|^{-n-1} dy \\
&= \text{const.} |x|^{m-n-1} \left\{ \int_{|x|^{-1} < |z| < 2^{-1}} + \int_{|z|>2^{-1}} \left| \frac{x}{|x|} - z \right|^{m-n} |z|^{-n-1} dz \right\}.
\end{aligned}$$

We see that the first integral is not greater than

$$\text{const.} \int_{|x|^{-1}}^{2^{-1}} t^{-2} dt \leq \text{const.} |x|,$$

and that the second integral is a finite value independent of  $x$ .

Therefore  $h_m^*|\psi_2|(x) = O(|x|^{m-n})$  as  $|x| \rightarrow \infty$ . Accordingly

$$\left| \int \psi(h_m^*(\mu - \mu_j)) dx \right| \leq \text{const.} \int_{|x|>j} |x|^{m-n} d|\mu|(x) \rightarrow 0$$

by (4) and Fubini's theorem. Thus  $h_m^*\mu_j \rightarrow h_m^*\mu$  in  $\mathcal{S}'$ .

Now we see that  $\mathcal{F}(h_m^*\mu_j) = (2\pi|x|)^{-m} \hat{\mu}_j \in L_{loc}^1(\mathbb{R}^n)$ . In fact, since  $\mu_j \in \mathcal{E}' \subset \mathcal{D}'_{L^q}$  for any  $q > 1$ , it follows that  $h_{m/2}$  and  $h_{m/2}^*\mu_j$  belong to  $\mathcal{D}'_{L^2}$ , and from Riesz's composition formula that  $h_m^*\mu_j = h_{m/2}^*(h_{m/2}^*\mu_j)$ . We infer from Lemma B (iii) that

$$\mathcal{F}(h_m^*\mu_j) = \mathcal{F}(h_{m/2}^*(h_{m/2}^*\mu_j)) = \hat{h}_{m/2} \cdot \hat{h}_{m/2} \cdot \hat{\mu}_j = (2\pi|x|)^{-m} \hat{\mu}_j.$$

Since the total variation of  $\mu_j$  is finite, it follows that  $\hat{\mu}_j$  is a bounded function, so that  $(2\pi|x|)^{-m} \hat{\mu}_j \in L_{loc}^1(\mathbb{R}^n)$ .

Noting that  $\mu_j \rightarrow \mu$  and  $h_m^*\mu_j \rightarrow h_m^*\mu = P$  in  $\mathcal{S}'$ , we obtain that

$$\hat{\mu}_j \rightarrow \hat{\mu} \quad \text{and} \quad (2\pi|x|)^{-m} \hat{\mu}_j \rightarrow \mathcal{F}(P) = P\left(\frac{-1}{2\pi i} \frac{\partial}{\partial x}\right) \delta \quad \text{in } \mathcal{S}'.$$

For any  $\psi \in \mathcal{S}$  vanishing on a neighborhood of the origin we have  $\psi = (2\pi|x|)^m \psi(x) \in \mathcal{S}$  and

$$\begin{aligned}
\langle \hat{\mu}, \psi \rangle &= \lim_{j \rightarrow \infty} \langle \hat{\mu}_j, \psi \rangle = \lim_{j \rightarrow \infty} \int \hat{\mu}_j(x) \psi(x) dx \\
&= \lim_{j \rightarrow \infty} \int (2\pi|x|)^{-m} \hat{\mu}_j(x) \psi(x) dx = \langle P\left(\frac{-1}{2\pi i} \frac{\partial}{\partial x}\right) \delta, \psi \rangle = 0.
\end{aligned}$$

This implies that  $\hat{\mu}$  is supported on  $\{0\}$ . Hence we can write

$$\hat{\mu} = P' \left( \frac{-1}{2\pi i} \frac{\partial}{\partial x} \right) \delta$$

with some polynomial  $P'$ . By the inverse Fourier transform we have  $\mu = P'$ , i.e.,  $\mu$  is absolutely continuous and  $d\mu = P'dx$ . Since  $\mu$  satisfies (4), it follows that  $(1 + |x|)^{m-n}P'(x)$  is integrable, so that  $P'$  must be identically zero. Hence  $\mu = 0$  and  $P = 0$ .

Remark 1. The above proof works even if  $m$  is not an integer. In case  $m$  is an integer,  $d\mu = gdx$ ,  $g \in L^p(\mathbb{R}^n, w)$  and  $P \in P_{m-1}$ , it is possible to give a simple proof. In fact by (12) and Lemma 3

$$g = \sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha (h_m * g) = \sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha P = 0.$$

Proof of Theorem 2. By using polar coordinates and integration by parts, we can prove

$$(13) \quad \sum_{|\alpha|=m} a_\alpha D^\alpha \left( \frac{x^\alpha}{|x|^n} \right) = \delta$$

(see [15; Lemma 6.2]). Let  $|\beta| = |\gamma| = m$ . Applying Lemma 1 to  $\ell = m - n$ ,  $f = x^\alpha / |x|^n$  and  $d\mu = D^\gamma f dx$ , we obtain that

$$\left( \frac{x^\alpha}{|x|^n} \right) * D^\gamma f \in L^1_{loc}(\mathbb{R}^n) \text{ for each } \alpha \text{ and } \gamma.$$

We infer from Lemma 2 and (13) that

$$\begin{aligned} D^\beta D^\gamma \left( \sum_{|\alpha|=m} a_\alpha \left( \frac{x^\alpha}{|x|^n} \right) * D^\alpha f \right) &= \sum_{|\alpha|=m} a_\alpha D^\beta \left( \frac{x^\alpha}{|x|^n} \right) * D^\alpha D^\gamma f \\ &= \sum_{|\alpha|=m} a_\alpha D^\beta D^\alpha \left( \frac{x^\alpha}{|x|^n} \right) * D^\gamma f = D^\beta \left( \sum_{|\alpha|=m} a_\alpha D^\alpha \left( \frac{x^\alpha}{|x|^n} \right) \right) * D^\gamma f \\ &= D^\beta \delta * D^\gamma f = D^\beta D^\gamma f. \end{aligned}$$

Now the same argument as in the proof of Theorem 1 completes the proof.

## §4. Proof of Theorem 3

Let us begin with modifying the Riesz kernel. The following technique is found in [3; Chapter IV] and [9, 10]. Observe that if  $y \neq 0$ , then  $h_m(x - y)$  has a multiple power series expansion in  $x_1, x_2, \dots, x_n$ , convergent in a neighborhood of the origin. We write

$$h_m(x - y) = \sum_{v=0}^{\infty} a_v(x, y),$$

where, for fixed  $v$  and  $y \neq 0$ ,  $a_v(x, y)$  is a homogeneous polynomial in  $x_1$  to  $x_n$  of degree  $v$  and continuous in  $x, y$  jointly for  $y \neq 0$  (cf. [3; Lemma 4.1]). We now set

$$k_m(x, y) = \begin{cases} h_m(x - y) & \text{if } |y| \leq 1 \\ h_m(x - y) - \sum_{v=0}^{m-1} a_v(x, y) & \text{if } |y| > 1. \end{cases}$$

Obviously  $D_x^\alpha k_m(x, y) = D_x^\alpha h_m(x - y)$  for  $|\alpha| \geq m$ . Since

$$|k_m(x, y)| \leq \text{const.} |x|^m |y|^{-n} \quad \text{if } 2|x| \leq |y|$$

(cf. [3; Lemma 4.2]), we can easily prove from Lemma 6 (i)

Lemma 7. Let  $w \in A_p$ . If  $g \in L^p(\mathbb{R}^n, w)$ , then

$$\int_{\mathbb{R}^n} k_m(x, y) g(y) dy \in L^1_{\text{loc}}(\mathbb{R}^n),$$

$$D^\alpha \left( \int_{\mathbb{R}^n} k_m(x, y) g(y) dy \right) = (D^\alpha h_m) * g \quad \text{for } |\alpha| \geq m.$$

This lemma and the same argument as in Theorem 1 yield

Theorem 6. Let  $w \in A_p$ . If  $f \in BL_m(L^p(\mathbb{R}^n, w))$ , then

$$f = \int_{R^n} k_m(x, y) g(y) dy + P, \quad g = \sum_{|\alpha|=m} c_\alpha (D^\alpha h_m)^* D^\alpha f,$$

where  $P \in P_{m-1}$ .

Let  $\mathcal{E}$  be the space of all indefinitely differentiable functions on  $R^n$ . We show

Lemma 8. Let  $f \in I_m(L^p(R^n, w)) \cap \mathcal{E}$ . Then for  $\varepsilon > 0$  and  $r > 0$  there is a function  $\psi \in \mathcal{D}$  such that

$$(14) \quad \sum_{|\alpha|=m} \|D^\alpha \psi - D^\alpha f\|_{L^p, w} < \varepsilon \quad \text{and} \quad \sup_{|x| < r} |\psi(x) - f(x)| < \varepsilon.$$

Proof. First we treat the case when  $f = h_m^* g$  with  $g \in \mathcal{D}$ . Let  $R > r$  and  $\text{supp } g \in \{y; |y| < R\}$ . Take  $\psi \in \mathcal{D}$  such that  $0 \leq \psi \leq 1$  and  $\psi(x) = 1$  for  $|x| < 3R$  and put  $\psi_j(x) = \psi(x/j)$ . We observe that

$$(15) \quad 0 \leq \psi_j \leq 1, \quad \psi_j(x) = 1 \quad \text{for } |x| < 3Rj,$$

$$\sum_{k=0}^m \sum_{|\alpha|=k} \sup(|x|^k |D^\alpha \psi_j(x)|) = \sum_{k=0}^m \sum_{|\alpha|=k} \sup(|x|^k |D^\alpha \psi(x)|) < \infty.$$

Let  $h_{m,j}(x) = \psi_j(x) h_m(x)$ . Then  $h_{m,j}^* g \in \mathcal{D}$  and

$$\begin{aligned} h_{m,j}^* g(x) &= \int_{|y| < R} \psi_j(x-y) h_m(x-y) g(y) dy \\ &= \int_{|y| < R} h_m(x-y) g(y) dy = h_m^* g(x) \quad \text{for } |x| < 2Rj \end{aligned}$$

by (15). Let  $\alpha$  be a multiindex of length  $m$ . We have

$$D^\alpha h_{m,j}^* g(x) = D^\alpha h_m^* g(x) \quad \text{for } |x| < 2Rj,$$

and hence

$$D^\alpha h_{m,j}^* g \rightarrow D^\alpha h_m^* g \quad \text{on } R^n.$$

In view of (15) and Leibniz's formula we have

$$|D^{\alpha}h_{m,j}(x-y)| \leq \text{const.}|x|^{-n} \quad \text{for } |x| > 2R \text{ and } |y| < R,$$

and hence

$$|D^{\alpha}h_{m,j}*g(x) - D^{\alpha}h_m*g(x)| \leq \text{const.}|x|^{-n} \quad \text{for } |x| > 2R.$$

Now it follows from Lemma 6 (ii) and the dominated convergence theorem that

$$\begin{aligned} & \int_{\mathbb{R}^n} |D^{\alpha}h_{m,j}*g(x) - D^{\alpha}h_m*g(x)|^{p_w(x)} dx \\ &= \int_{|x| > 2R} |D^{\alpha}h_{m,j}*g(x) - D^{\alpha}h_m*g(x)|^{p_w(x)} dx \rightarrow 0, \end{aligned}$$

so that  $D^{\alpha}(h_{m,j}*g) \rightarrow D^{\alpha}(h_m*g)$  in  $L^p(\mathbb{R}^n, w)$ . Therefore  $\psi = h_{m,j}*g$  satisfies (14) if  $j$  is sufficiently large.

Next we consider the general case. From the uniqueness in Theorem 1  $f$  is written as  $f = h_m*g$  with  $g = \sum_{|\alpha|=m} c_{\alpha}(D^{\alpha}h_m)*D^{\alpha}f \in L^p(\mathbb{R}^n, w) \cap \mathcal{E}$ . It is easy to find  $\psi \in \mathcal{D}$  such that  $0 \leq \psi \leq 1$ ,

$$\sum_{|\alpha|=m} \|D^{\alpha}h_m*(\psi g) - D^{\alpha}f\|_{L^p, w} \leq \text{const.}\|\psi g - g\|_{L^p, w} < \varepsilon/2,$$

and

$$\sup_{|x| < R} |h_m*(\psi g)(x) - h_m*g(x)| < \varepsilon/2.$$

From the first part there is a function  $\psi \in \mathcal{D}$  such that

$$\begin{aligned} & \sum_{|\alpha|=m} \|D^{\alpha}\psi - D^{\alpha}h_m*(\psi g)\|_{L^p, w} < \varepsilon/2, \\ & \sup_{|x| < R} |\psi(x) - h_m*(\psi g)(x)| < \varepsilon/2. \end{aligned}$$

This  $\psi$  satisfies (14).

**Proof of Theorem 3.** Let  $g$  be as in Theorem 6. It is easy to find a sequence  $\{g_j\}_j \subset \mathcal{D}$  such that  $\|g_j - g\|_{L^p, w} \rightarrow 0$ . Since  $g_j$  has compact support,  $h_m*g_j$  exists and by Lemma 7

$$\int_{\mathbb{R}^n} k_m(x, y) g_j(y) dy = h_m * g_j + P_j$$

with some  $P_j \in P_{m-1}$ . Now Lemma 8 gives a sequence  $\{\psi_j\}_j \subset \mathcal{D}$  such that

$$\sum_{|\alpha|=m} \|D^\alpha h_m * g_j - D^\alpha \psi_j\|_{L^p, w} < 1/j.$$

We infer from Theorem 6, Lemmas 2 and 7 that

$$\begin{aligned} & \sum_{|\alpha|=m} \|D^\alpha f - D^\alpha \psi_j\|_{L^p, w} \\ &= \sum_{|\alpha|=m} \|D^\alpha \left( \int_{\mathbb{R}^n} k_m(x, y) g(y) dy \right) - D^\alpha \psi_j\|_{L^p, w} \\ &= \sum_{|\alpha|=m} \|(D^\alpha h_m) * g - D^\alpha \psi_j\|_{L^p, w} \\ &\leq \sum_{|\alpha|=m} \|(D^\alpha h_m) * (g - g_j)\|_{L^p, w} + 1/j \rightarrow 0. \end{aligned}$$

The theorem is proved.

#### §5. Proof of Theorem 4

Lemma 9. If  $\psi \in \mathcal{D}$ , then  $\psi = h_m * g \in I_m(L^p(\mathbb{R}^n, w))$ , where  $g = \sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha \psi \in L^p(\mathbb{R}^n, w) \cap \mathcal{E}$ .

Proof. Since  $D^\alpha h_m$  is of the form (8) and  $D^\alpha \psi$  has compact support, it follows that  $g(x) = O(|x|^{-n})$  as  $|x| \rightarrow \infty$ , so that  $g$  satisfies (3) and  $h_m * g = o(1)$  as  $|x| \rightarrow \infty$ . By Theorem 1 we have  $\psi = h_m * g + P$  with some  $P \in P_{m-1}$ . However  $P$  must be equal to zero, for  $P(x) = \psi(x) - h_m * g(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

Lemma 10. A function  $f = h_m * g + P$  in  $BL_m(L^p(\mathbb{R}^n, w))$  vanishes at  $\infty$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense if and only if there is a sequence

$\{g_j\}_j \subset L^p(\mathbb{R}^n, w) \cap \mathcal{E}$  such that

$$(16) \quad \|g_j - g\|_{L^p, w} \rightarrow 0 \quad \text{and} \quad h_m * g_j \rightarrow f \text{ a.e. on } \mathbb{R}^n.$$

Proof. First suppose that  $\{g_j\}_j \subset L^p(\mathbb{R}^n, w) \cap \mathcal{E}$  satisfies (16). Then by Lemma 8 there is a sequence  $\{\psi_j\}_j \subset \mathcal{D}$  such that

$$\sum_{|\alpha|=m} \|D^\alpha h_m * g_j - D^\alpha \psi_j\|_{L^p, w} < 1/j,$$

$$\sup_{|x| < j} |h_m * g_j(x) - \psi_j(x)| < 1/j.$$

We easily see that  $\psi_j$  converges to  $f$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense.

Conversely suppose that  $\{\psi_j\}_j \subset \mathcal{D}$  converges to  $f$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense. We infer from Lemmas 2 and 9 that  $\psi_j = h_m * g_j$ ,  $g_j \in L^p(\mathbb{R}^n, w) \cap \mathcal{E}$  and

$$\|g_j - g\|_{L^p, w} = \left\| \sum_{|\alpha|=m} c_\alpha (D^\alpha h_m)^* (D^\alpha \psi_j - D^\alpha f) \right\|_{L^p, w}$$

converges to zero. Thus  $\{g_j\}_j$  satisfies (16).

For  $E \subset \mathbb{R}^n$  we define a capacity  $R_{m,p,w}(E)$  by

$$R_{m,p,w}(E) = \inf \{ \|g\|_{L^p, w}^p ; g \geq 0, h_m * g \geq 1 \text{ on } E \}.$$

The next theorem combines conditions (3) and (7), the capacity  $R_{m,p,w}$  and the vanishing property of Beppo Levi functions.

Theorem 7. The following statements on  $w \in A_p$  are equivalent:

(a)  $w \in A_{p,m}$ .

(b) For every  $g \in L^p(\mathbb{R}^n, w)$  the convolution  $h_m * g$  exists and

belongs to  $L^1_{loc}(R^n)$ .

(c) If  $\|g_j\|_{L^p, w} \rightarrow 0$ , then  $h_m * g_j \rightarrow 0$  in measure on any ball.

(c') If  $\|g_j\|_{L^p, w} \rightarrow 0$ , then  $h_m * g_j \rightarrow 0$  in measure on some ball.

(d) The constant function 1 does not vanish at  $\infty$  in the  $BL_m(L^p(R^n, w))$  sense.

(e) There is a set of positive  $R_{m,p,w}$  capacity.

(f) If  $|E| > 0$ , then  $R_{m,p,w}(E) > 0$ .

(g)  $w$  satisfies (7).

Proof. The equivalence between (a) and (b) readily follows from Lemma 1. The implications (c)  $\Rightarrow$  (c') and (f)  $\Rightarrow$  (e) are obvious. We have (f)  $\Rightarrow$  (c) from [6; Theorem 4] and (g)  $\Rightarrow$  (a) from Hölder's inequality. We shall complete the proof by showing (b)  $\Rightarrow$  (f)  $\Rightarrow$  (g), (e)  $\Rightarrow$  (b) and (c')  $\Rightarrow$  (d)  $\Rightarrow$  (a).

(b)  $\Rightarrow$  (f): Suppose that there is a measurable set  $E$  such that  $|E| > 0$  but  $R_{m,p,w}(E) = 0$ . By [6; Theorem 3] we find a nonnegative function  $g$  in  $L^p(R^n, w)$  such that  $h_m * g = \infty$  on  $E$ . Since  $|E| > 0$ , it follows that  $h_m * g$  is not locally integrable, so that (b) does not hold.

(f)  $\Rightarrow$  (g): Since the unit ball  $B$  has positive capacity, it follows from [6; Theorem 14] that there exists a measure  $\mu$  concentrated on  $B$  such that  $\mu(B) > 0$  and  $h_m * \mu \in L^{p'}(R^n, w^{1/(1-p)})$ . Noting that  $h_m * \mu(x) \geq \text{const.} \mu(B) h_m(x)$  for  $|x| > 1$ , we obtain

$$\int_{|x|>1} |x|^{(m-n)p'} w(x)^{1/(1-p)} dx < \infty,$$

which is equivalent to (7).

(e)  $\Rightarrow$  (b): If (b) does not hold, then there is a nonnegative

function  $g$  in  $L^p(\mathbb{R}^n, w)$  such that  $h_m * g \equiv \infty$  on  $\mathbb{R}^n$ . By definition

$$0 \leq R_{m,p,w}(E) \leq R_{m,p,w}(\mathbb{R}^n) \leq \inf_{t>0} \|tg\|_{L^p,w}^p = 0.$$

Thus (e) does not hold.

(c')  $\Rightarrow$  (d): If 1 vanishes at  $\infty$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense, then there is a sequence  $\{g_j\}_j \subset L^p(\mathbb{R}^n, w)$  such that

$$\|g_j\|_{L^p,w} \rightarrow 0 \quad \text{and} \quad h_m * g_j \rightarrow 1 \text{ a.e. on } \mathbb{R}^n$$

by Lemma 10. This contradicts (c').

(d)  $\Rightarrow$  (a): Suppose that there is a nonnegative function  $g$  in  $L^p(\mathbb{R}^n, w)$  such that (3) does not hold. Mollifying  $g$ , we may assume that  $g \in L^p(\mathbb{R}^n, w) \cap \mathcal{E}$ . We shall prove that 1 vanishes at  $\infty$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense. By Lemma 10 it is sufficient to show that if  $\varepsilon > 0$  and  $R > 0$ , then there is  $g_1 \in L^p(\mathbb{R}^n, w) \cap \mathcal{E}$  such that

$$\begin{aligned} |h_m * g_1(x) - 1| &< \varepsilon \text{ for } |x| < R, \\ \|g_1\|_{L^p,w} &< \varepsilon. \end{aligned}$$

Take  $R_1 > R$  such that

$$1 - \varepsilon < h_m(x-y)/h_m(y) < 1 + \varepsilon \quad \text{for } |x| < R \text{ and } |y| > R_1.$$

Since (3) does not hold and  $g \in L^p(\mathbb{R}^n, w)$ , we find a function  $\psi \in \mathcal{D}$  such that  $0 \leq \psi \leq 1$ ,  $\text{supp } \psi \subset \{y; |y| > R_1\}$  and  $g_1 = \psi g$  satisfies  $h_m * g_1(0) = 1$  and  $\|g_1\|_{L^p,w} < \varepsilon$ . We observe that

$$1 - \varepsilon < h_m * g_1(x)/h_m * g_1(0) = h_m * g_1(x) < 1 + \varepsilon \quad \text{for } |x| < R.$$

Hence  $g_1$  has the desired property. Thus the theorem is completely proved.

Proof of Theorem 4. Suppose that  $f = h_m * g \in I_m(L^p(\mathbb{R}^n, w))$ .

Take a nonnegative function  $\psi$  in  $\mathcal{D}$  such that  $\int \psi dx = 1$ . Letting  $\psi_j(x) = j^n \psi(jx)$ , we observe that  $g_j = g * \psi_j \in L^p(\mathbb{R}^n, w) \cap \mathcal{E}$  satisfies (16). Hence  $f$  vanishes at  $\infty$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense by Lemma 10.

Conversely suppose that  $f = h_m * g + P \in BL_m(L^p(\mathbb{R}^n, w))$  vanishes at  $\infty$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense. Since  $h_m * g$  vanishes at  $\infty$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense from the only if part of the theorem, it follows that  $P = f - h_m * g$  vanishes at  $\infty$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense. Hence there is a sequence  $\{g_j\}_j \subset L^p(\mathbb{R}^n, w)$  such that

$$\|g_j\|_{L^p, w} \rightarrow 0 \quad \text{and} \quad h_m * g_j \rightarrow P \text{ a.e. on } \mathbb{R}^n$$

by Lemma 10. On account of (c) of Theorem 7 we have  $P = 0$ . The proof is complete.

For the proof of Corollary 1 we prepare

Lemma 11. Let  $L$  be a cone with vertex at the origin. Then  $R_{m,p,w}(L)$  is equal to 0 or  $\infty$ ;  $R_{m,p,w}(L) = 0$  if and only if  $R_{m,p,w}(\mathbb{R}^n) = 0$ . The constant 1 vanishes at  $\infty$  in the  $BL_m(L^p(\mathbb{R}^n, w))$  sense if and only if  $R_{m,p,w}(\mathbb{R}^n) = 0$ .

Proof. If  $0 < R_{m,p,w}(L) < \infty$ , then there would exist a nonnegative function  $g$  in  $L^p(\mathbb{R}^n, w)$  satisfying (3) and  $h_m * g \geq 1$  on  $L$  by Theorem 7. Since  $L$  is not  $m$ -thin at  $\infty$  in the notation of [5], this contradicts

$$\liminf_{|x| \rightarrow \infty, x \in L} h_m * g(x) = 0$$

([5; Theorem 3.3]). By Theorem 7 we can easily prove the remainder.

Proof of Corollary 1. Suppose that  $f = h_m * g + P$  and  $P \neq 0$ . Then we would find  $\varepsilon > 0$ ,  $R > 0$  and a cone  $L$  with vertex at the origin such that

$$h_m * |g|(x) \geq |f(x) - P(x)| \geq \varepsilon \quad \text{if } |x| \geq R \text{ and } x \in L.$$

By definition

$$R_{m,p,w}(L) \leq R_{m,p,w}(\{x; |x| < R\}) + R_{m,p,w}(\{x \in L; |x| \geq R\}) < \infty$$

and hence by Lemma 11  $R_{m,p,w}(L) = 0$ . This contradicts (f) of Theorem 7.

#### §6. Proof of Theorem 5

Proof of Theorem 5. First suppose that  $f$  vanishes at  $\infty$  in the  $BL_m(L^p(\mathbb{R}^n, V^p))$  sense. Thus  $f$  is written as  $h_m * g$  with  $g \in L^p(\mathbb{R}^n, V^p)$ . On account of Lemma 10 there is a sequence  $\{g_j\}_j \subset L^p(\mathbb{R}^n, V^p)$  satisfying (16) with  $w = V^p$ . Since

$$\|(h_m * g_j)V\|_{L^{p*}} \leq \text{const.} \|g_j V\|_{L^p} \leq \text{const.}$$

by Lemma A, Fatou's lemma leads to

$$\|fV\|_{L^{p*}} \leq \liminf_{j \rightarrow \infty} \|(h_m * g_j)V\|_{L^{p*}} < \infty.$$

The if part of (i) is included in (ii). Now we shall prove (ii) by contradiction. Suppose that  $f = h_m * g + P$ ,  $g \in L^p(\mathbb{R}^n, V^p)$ ,  $P \in P_{m-1}$  and  $P \neq 0$ . Then we would find  $\varepsilon > 0$ ,  $R > 0$  and a cone  $L$  with vertex at the origin such that

$$|P(x)| \geq 2\varepsilon \quad \text{for } x \in \{x \in L; |x| \geq R\}.$$

We observe that  $v^{p^*} \leq u + v$  on  $L$ , where

$$u(x) = \begin{cases} v(x)^{p^*} & \text{if } |f(x)| \geq \varepsilon \\ 0 & \text{otherwise} \end{cases}, \quad v(x) = \begin{cases} v(x)^{p^*} & \text{if } |h_m * g(x)| \geq \varepsilon \text{ or } |x| \leq R \\ 0 & \text{otherwise} \end{cases}.$$

Since  $v^{p^*} \in A_\infty$  and  $0 < r/p^* \leq 1$ ,

$$\int_L u^{r/p^*} dx = \int_{\{x \in L; |f(x)| \geq \varepsilon\}} v^r dx \leq \varepsilon^{-q} \int_{\mathbb{R}^n} |f|^q v^r dx < \infty,$$

$$\begin{aligned} \int_L v dx &= \int_{\{x \in L; |h_m * g(x)| \geq \varepsilon\}} v^{p^*} dx + \int_{|x| \leq R} v^{p^*} dx \\ &\leq \varepsilon^{-p^*} \int_{\mathbb{R}^n} |h_m * g|^{p^*} v^{p^*} dx + \int_{|x| \leq R} v^{p^*} dx < \infty, \end{aligned}$$

we have a contradiction by Lemma 5. The theorem is proved.

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