

On a universal framework of the homogenization problems for infinite dimensional diffusions

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Abstract

By restricting the universal frame work of the homogenization problem of infinite dimensional diffusions posed in [AY] to the case where the state space of the ergodic process, that corresponds to the original infinite dimensional diffusion for which the homogenization problem is considered, a sufficient condition for the mapping between these processes under which the ergodic process is a unique Markov process that corresponds to a unique Markovian extension of a closable symmetric bilinear form is considered.

1 Introduction

In This note, by restricting the universal frame work of the homogenization problem of infinite dimensional diffusions posed in [AY] to the case where the state space of the ergodic process denoted by $(Y_\theta(t))_{t \geq 0}$, that corresponds to $(X^\theta(t))_{t \geq 0}$, the original infinite dimensional diffusion, for which the homogenization problem is considered, we discuss a sufficient condition for the mapping between these processes (denoted by $T_x(\theta)$) under which the ergodic process is the one that corresponds to a unique Markovian extension of a closable symmetric bilinear form. Since, the present announcement plays a part of introduction of our subsequent researches on this subject, we give here a statement in a rough style without proof. All the exact and new results on this concern will be found in forthcoming papers.

2 Probability space $(\Theta, \bar{\mathcal{B}}, \bar{\mu})$, the ergodic flow and the core

Suppose that we are given the following:

$\{(\Theta_k, \mathcal{B}_k, \lambda_k)\}_{k \in \mathbb{Z}^d}$: a system of complete probability (*resp. measure*) spaces,

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where d is a given natural number. (*resp. for each k , λ_k is a σ -finite measure.*)
 $(\Theta, \bar{\mathcal{B}}, \bar{\lambda})$: the probability (*resp. complete measure*) space that is the completion of $(\prod_k \Theta_k, \bigotimes_k \mathcal{B}_k, \prod_k \lambda_k)$, i.e., the completion of the direct product probability (*resp. complete measure*) space.

$(\Theta, \bar{\mathcal{B}}, \mu)$: a complete probability space (corresponding to a Gibbs state) defined as follows:

for $\forall D \subset \mathbb{Z}^d$ and for any bounded measurable function φ defined on $\prod_{k \in D'} \Theta_k$ with some $\forall D' \subset \mathbb{Z}^d$, μ satisfies

$$(\mathbb{E}^D \varphi, \mu) = (\varphi, \mu), \quad (2.1)$$

where

$$\begin{aligned} (\mathbb{E}^D \varphi)(\theta) &\equiv \int_{\Theta} \varphi(\theta'_D \cdot \theta_{D^c}) \mathbb{E}^D(d\theta' | \theta_{D^c}) \\ &\equiv \int_{\Theta} \varphi(\theta'_D \cdot \theta_{D^c}) m_D(\theta'_D \cdot \theta_{D^c}) \bar{\lambda}(d\theta'), \end{aligned} \quad (2.2)$$

and

$$m_D(\theta'_D \cdot \theta_{D^c}) \equiv \frac{1}{Z_D(\theta_{D^c})} e^{-U_D(\theta'_D \cdot \theta_{D^c})}, \quad U_D \equiv \sum_{k \in D^+} U_k, \quad (2.3)$$

$$\Theta \ni \theta \longmapsto \theta_D \in \prod_{k \in D} \Theta_k$$

is the natural projection,

$\theta'_D \cdot \theta_{D^c}$ is the element $\theta'' \in \Theta$ such that

$$\theta''_D = \theta'_D, \quad \theta''_{D^c} = \theta_{D^c},$$

$$D^+ = \{k' | \text{support of } U_{k'} \cap D \neq \emptyset\},$$

also, for each $k \in \mathbb{Z}^d$, U_k is a given bounded measurable function of which support is in $\prod_{|k'-k| \leq L} \Theta_{k'}$, where the number L (the range of interactions) does not depend on k , and $Z_D(\theta_{D^c})$ is the normalizing constant.

On $(\Theta, \bar{\mathcal{B}}, \bar{\lambda})$ we are given a *measure preserving map* T_x (which is also a map on $(\Theta, \bar{\mathcal{B}}, \mu)$, but is not a measure preserving map on it *an ergodic flow*) as follows:

Suppose that

$$\exists M_1 < \infty \quad \text{and} \quad \forall k \in \mathbb{Z}^d \quad \text{there exists a } d_k \text{ such that } d_k \leq M_1. \quad (2.4)$$

For each $\mathbf{x} \in \prod_{\mathbf{k}} \mathbb{R}^{d_{\mathbf{k}}}$ such that $\mathbf{x} = (\mathbf{x}^{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ with $\mathbf{x}^{\mathbf{k}} = (x_1^{\mathbf{k}}, \dots, x_{d_{\mathbf{k}}}^{\mathbf{k}})$ the map $T_{\mathbf{x}}$ on $(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$ is defined by

i)

$$T_{\mathbf{x}} : \Theta \longrightarrow \Theta$$

that is a measure preserving transformation with respect to the measure $\overline{\lambda}$;

ii)

$$T_0 = \text{the identity,}$$

$$\text{for } \mathbf{x}, \mathbf{x}' \in \prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}} \quad T_{\mathbf{x}+\mathbf{x}'} = T_{\mathbf{x}} \circ T_{\mathbf{x}'},$$

where

$$\mathbf{x} + \mathbf{x}' \equiv (\mathbf{x}^{\mathbf{k}} + \mathbf{x}'^{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d},$$

with

$$\mathbf{x}^{\mathbf{k}} + \mathbf{x}'^{\mathbf{k}} = (x_1^{\mathbf{k}} + x_1'^{\mathbf{k}}, \dots, x_{d_{\mathbf{k}}}^{\mathbf{k}} + x_{d_{\mathbf{k}}}'^{\mathbf{k}}),$$

for

$$\begin{aligned} \mathbf{x} &= (\mathbf{x}^{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}, & \mathbf{x}^{\mathbf{k}} &= (x_1^{\mathbf{k}}, \dots, x_{d_{\mathbf{k}}}^{\mathbf{k}}), \\ \mathbf{x}' &= (\mathbf{x}'^{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}, & \mathbf{x}'^{\mathbf{k}} &= (x_1'^{\mathbf{k}}, \dots, x_{d_{\mathbf{k}}}'^{\mathbf{k}}), \end{aligned}$$

and

$$\mathbf{0} \equiv (\mathbf{0}^{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}, \quad \mathbf{0}^{\mathbf{k}} = (0, \dots, 0) \in \mathbb{R}^{d_{\mathbf{k}}};$$

iii)

$$(\mathbf{x}, \theta) \in \left(\prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}} \right) \times \Theta \longrightarrow T_{\mathbf{x}}(\theta) \in \Theta$$

is $\mathcal{B}(\prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}) \times \overline{\mathcal{B}}/\overline{\mathcal{B}}$ -measurable, where $\prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}$ is assumed to be the topological space with the direct product topology;

iv) A function which is $T_{\mathbf{x}}$ invariant for all $\mathbf{x} \in \prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}$ is a constant function on $(\Theta, \overline{\mathcal{B}}, \mu)$;

v) For $D \subset \mathbb{Z}^d$, let

$$\prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}} \ni \mathbf{x} \longmapsto \mathbf{x}_D \in \prod_{\mathbf{k} \in D} \mathbb{R}^{d_{\mathbf{k}}}$$

be the natural *projection*. If $\mathbf{x}_{D^c} = \mathbf{0}_{D^c}$, then

$$(T_{\mathbf{x}}(\theta))_{D^c} = \theta_{D^c}, \quad \forall \theta \in \Theta, \quad \forall D \subset \subset \mathbb{Z}^d.$$

□

We assume that an existence of a *core* $\mathcal{D}^\Theta$. Namely, there exists $\mathcal{D}^\Theta$ which is a dense subset of both $L^2(\mu)$ and $L^1(\mu)$, and $\forall \varphi \in \mathcal{D}^\Theta$ satisfies

($\mathcal{D}$ -1) φ is a bounded measurable function having only a finite number of variables θ_D for some $D \subset \mathbb{Z}^d$,

($\mathcal{D}$ -2)

$$\varphi(T_{\mathbf{x}_D}(\theta)) \in C^\infty(\prod_{\mathbf{k} \in D} \mathbb{R}^{d_{\mathbf{k}}} \rightarrow \mathbb{R}), \quad \forall \theta \in \Theta,$$

(cf. v) in the previous section) where we identify $\mathbf{x}_D \in \prod_{\mathbf{k} \in D} \mathbb{R}^{d_{\mathbf{k}}}$ with an $\mathbf{x} \in (\prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}})$ of which projection to $\prod_{\mathbf{k} \in D} \mathbb{R}^{d_{\mathbf{k}}}$ is $\mathbf{x}_D$,

($\mathcal{D}$ -3) in ($\mathcal{D}$ -2) for each $\theta \in \Theta$, all the partial derivatives of all orders of the function $\varphi(T(\theta))$ (with the variables $\mathbf{x}_D$) are bounded and

$$\forall \varphi \in \mathcal{D}, \exists M < \infty; \quad |\nabla_{\mathbf{k}} \varphi(T_{\mathbf{x}}(\theta))| < M, \quad \forall \theta \in \Theta, \forall \mathbf{x}, \forall \mathbf{k} \in \mathbb{Z}^d, \quad (2.5)$$

where

$$\nabla_{\mathbf{k}} = \left(\frac{\partial}{x_1^{\mathbf{k}}}, \dots, \frac{\partial}{x_{d_{\mathbf{k}}}^{\mathbf{k}}} \right).$$

□

3 Probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ and the processes

Suppose that we are given a system of family of functions $a_{ij}^{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^d$, $1 \leq i, j \leq d_{\mathbf{k}}$ on $(\Theta, \bar{\mathcal{B}}, \bar{\mu})$ such that for each $\mathbf{k} \in \mathbb{Z}^d$ and each $1 \leq i, j \leq d_{\mathbf{k}}$, $a_{ij}^{\mathbf{k}}$ is a measurable function on $\Theta_{\mathbf{k}}$ and there exists $M_2 \in (0, \infty)$ and

$$M_2^{-1} \|\mathbf{x}\|^2 \leq \sum_{1 \leq i, j \leq d_{\mathbf{k}}} a_{ij}^{\mathbf{k}}(\theta_{\mathbf{k}}) x_i x_j \leq M_2 \|\mathbf{x}\|^2, \quad \forall \mathbf{k} \in \mathbb{Z}^d, \forall \theta_{\mathbf{k}} \in \Theta_{\mathbf{k}},$$

$$\forall \mathbf{x} = (x_1, \dots, x_{d_{\mathbf{k}}}) \in \mathbb{R}^{d_{\mathbf{k}}}, \quad (3.1)$$

also

$$a_{ij}^{\mathbf{k}}(\cdot) = a_{ji}^{\mathbf{k}}(\cdot).$$

We assume that

$$U_{\mathbf{k}}, a_{ij}^{\mathbf{k}} \in \mathcal{D}^\Theta, \quad \mathbf{k} \in \mathbb{Z}^d, \quad 1 \leq i, j \leq d_{\mathbf{k}}.$$

Also, we assume that there exists a common $M < \infty$ by which the evaluation (2.5) holds for all $a_{i,j}^{\mathbf{k}}$ and $U_{\mathbf{k}}$.

Finally, suppose that we are given a complete probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, $(t \in \mathbb{R}_+)$ with a filtration $\mathcal{F}_t$. On $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ suppose that there exists a system of independent 1-dimensional $\mathcal{F}_t$ -adapted Brownian motion processes

$$\{(B^{\mathbf{k},i}(t))_{t \geq 0}\}_{\mathbf{k} \in \mathbb{Z}^d, 1 \leq i \leq d_{\mathbf{k}}}.$$

Now, for each $\theta \in \Theta$, let

$$X^\theta \equiv \{(X^{\theta,\mathbf{k},i}(t))_{t \geq 0}\}_{\mathbf{k} \in \mathbb{Z}^d, 1 \leq i \leq d_{\mathbf{k}}}.$$

be the unique solution of

$$\begin{aligned} X^{\theta,\mathbf{k},i}(t) = & X^{\theta,\mathbf{k},i}(0) + \int_0^t \sum_{1 \leq j \leq d_{\mathbf{k}}} \left\{ \frac{\partial}{\partial x_j^{\mathbf{k}}} a_{ij}^{\mathbf{k}}(T_{X^{\theta,\mathbf{k}}(s)}(\theta)) \right. \\ & \left. - a_{ij}^{\mathbf{k}}(T_{X^{\theta,\mathbf{k}}(s)}(\theta)) \left(\frac{\partial}{\partial x_j^{\mathbf{k}}} \left(\sum_{\mathbf{k}' \in \{\mathbf{k}\}^+} U_{\mathbf{k}'}(T_{X^{\theta}(s)}(\theta)) \right) \right) \right\} ds \\ & + \int_0^t \sum_{1 \leq j \leq d_{\mathbf{k}}} \sigma_{ij}^{\mathbf{k}}(T_{X^{\theta,\mathbf{k}}(s)}(\theta)) dB^{\mathbf{k},j}(s), \quad t \geq 0, \end{aligned} \quad (3.2)$$

where, as the matrix sense,

$$(\sigma_{ij}^{\mathbf{k}}) = (2a_{ij}^{\mathbf{k}})^{\frac{1}{2}},$$

and

$$X^{\theta,\mathbf{k}}(t) = (X^{\theta,\mathbf{k},1}(t), \dots, X^{\theta,\mathbf{k},d_{\mathbf{k}}}(t)), \quad \{\mathbf{k}\}^+ = \{\mathbf{k}' \mid \text{support of } U_{\mathbf{k}'} \cap \{\mathbf{k}\} \neq \emptyset\},$$

also, by $X^\theta(t)$ we denote the vector

$$(X^{\theta,\mathbf{k}}(t))_{\mathbf{k} \in \mathbb{Z}^d} \in \prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}.$$

To get the unique solution for (3.2) we assume the following:

Assumption 1. *All the coefficients appeared in (3.2) are uniformly bounded and equi-continuous for all $1 \leq i, j \leq d_{\mathbf{k}}$ and $\mathbf{k} \in \mathbb{Z}^d$.*

□

Proposition 3.1 *Under Assumption 1, for each $\theta \in \Theta$ the SDE (3.2) has a unique solution, and the random variable X^θ on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ is the one taking values in*

$$C([0, \infty) \rightarrow \prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}).$$

□

Definition 3.1 *For $\theta \in \Theta$, let $(X_0^\theta(t))_{t \geq 0}$ be the stochastic process defined by (3.2) with the initial condition $X_0^\theta(0) = \mathbf{0}$. By using $(X_0^\theta(t))_{t \geq 0}$ and the map $T_{\mathbf{x}}(\cdot)$ we define a Θ -valued process $(Y_\theta(t))_{t \geq 0}$ on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ as follows:*

$$(Y_\theta(t))_{t \geq 0} = (X_0^\theta(t))_{t \geq 0}.$$

□

4 A homeomorphism

The problem of homogenization of the process $(X_0^\theta(t))_{t \geq 0}$ is described as follows:

Problem. For each $\theta \in \Theta$, $\mu - a.s.$, we are concerning the scaling limit of $(X_0^\theta(t))_{t \geq 0}$ such that

$$\lim_{\epsilon \downarrow 0} \{\epsilon X_0^\theta(\frac{t}{\epsilon^2})\}_{t \geq 0} \quad (4.1)$$

More precisely, we consider the weak convergence of (4.1), where the sequence of the processes $\{\epsilon X_0^\theta(\frac{t}{\epsilon^2})\}_{t \geq 0}$ is understood as the sequence of random variables on $(\Omega \times \Theta, \mathcal{F} \times \overline{\mathcal{B}}, P \times \overline{\mu}; \mathcal{F}_t \times \{\Theta, \emptyset\})$ taking values in the direct product space $\prod_{\mathbf{k} \in \mathbb{Z}^d} C([0, \infty) \rightarrow \mathbb{R}^{d_{\mathbf{k}}})$ equipped with the direct product topology.

□

In order to prove the weak convergence of (4.1), the ergodicity of the process $(Y_\theta(t))_{t \geq 0}$ plays a crucial role (cf. [ABRY 1,2,3] and [AY]). Hence, for a concrete analysis on this problem, in any case, we have to characterize both the probabilistic and analytic properties of $(Y_\theta(t))_{t \geq 0}$. In this report, assuming in particular that $\Theta_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d$, are topological spaces, and then we consider a sufficient condition under which $(Y_\theta(t))_{t \geq 0}$ is a process corresponding to a unique Markovian extension of a symmetric quadratic form.

Definition 4.1 For each $\mathbf{k} \in \mathbb{Z}^d$ and $i = 1, \dots, d_{\mathbf{k}}$, define an operator $D^{\mathbf{k},i} : \mathcal{D}^\Theta \rightarrow \mathcal{D}^\Theta$ such that

$$(D^{\mathbf{k},i}\varphi)(\theta) \equiv \frac{\partial}{\partial x_i^{\mathbf{k}}} \varphi(T_{\mathbf{x}}(\theta))|_{\mathbf{x}=0}, \quad \varphi \in \mathcal{D}^\Theta, \quad \theta \in \Theta.$$

Also, define a quadratic form $\mathcal{E}$ on $L^2(\mu)$ such that

$$\mathcal{E}(\varphi, \psi) \equiv \sum_{\mathbf{k} \in \mathbb{Z}^d} \sum_{1 \leq i, j \leq d_{\mathbf{k}}} \int_{\Theta} (D^{\mathbf{k},i}\varphi)(\theta) a_{i,j}^{\mathbf{k}}(\theta) (D^{\mathbf{k},j}\psi)(\theta) \mu(d\theta), \quad \varphi, \psi \in \mathcal{D}^\Theta.$$

□

Theorem 4.1 Let $\prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}$ be the topological space with the direct product topology, and for each $M > 0$ let $C^{X,M}$ be the space of continuous functions with the uniform convergence topology such that

$$C^{X,M} \equiv \{\mathbf{x}(\cdot) \mid \mathbf{x}(\cdot) \in C([0, M] \rightarrow \prod_{\mathbf{k} \in \mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}) \text{ with } \mathbf{x}(0) = \mathbf{0}\}.$$

Suppose that for each $\mathbf{k} \in \mathbb{Z}^d$, $\Theta_{\mathbf{k}}$ is a topological space and let $\mathcal{B}_{\mathbf{k}}$ be its Borel σ -field, also $\Theta = \prod_{\mathbf{k}} \Theta_{\mathbf{k}}$ be the direct product space with the direct product topology. for each $\theta \in \Theta$ and $M > 0$ let $C^{\theta,Y,M}$ be the space of continuous functions with the uniform convergence topology such that

$$C^{\theta,Y,M} \equiv \{\mathbf{y}(\cdot) \mid \mathbf{y}(\cdot) \in C([0, M] \rightarrow \Theta) \text{ with } \mathbf{y}(0) = \theta\}.$$

For any $\theta \in \Theta$ and $M > 0$ if the map f defined by

$$f : C^{X,M} \ni \mathbf{x}(\cdot) \mapsto T_{\mathbf{x}(\cdot)}(\theta) \in C^{\theta,Y,M}$$

is a continuous onto one to one map of which inverse map f^{-1} is also continuous (i.e. $C^{X,M}$ and $C^{\theta,Y,M}$ are homeomorphic), then the probability law of the process $(Y_\theta(t))_{t \geq 0}$ is identical with the probability law of the Markov process which corresponds to a unique Markovian extension of the quadratic form $\mathcal{E}(\varphi, \psi)$ defined by Definition 4.1.

□

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