

Convolution properties for certain subclasses of analytic functions involving Silverman paper

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are univalent in $\mathbb{U}$.

Let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy the following inequality

$$\operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathbb{U}),$$

for some α ($0 \leq \alpha < 1$). A function $f(z) \in \mathcal{S}^*(\alpha)$ is said to be starlike of order α in $\mathbb{U}$. Furthermore, let $\mathcal{K}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy the following inequality

$$\operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \alpha \quad (z \in \mathbb{U}),$$

for some α ($0 \leq \alpha < 1$). A function $f(z) \in \mathcal{K}(\alpha)$ is said to be convex of order α in $\mathbb{U}$. We note that

$$f(z) \in \mathcal{K}(\alpha) \iff z f'(z) \in \mathcal{S}^*(\alpha).$$

In 1975, Silverman [1] gave the following coefficient inequalities for the functions in the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$.

Theorem A. *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality*

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha \quad (0 \leq \alpha < 1),$$

then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha \quad (z \in \mathbb{U}, 0 \leq \alpha < 1),$$

that is, that $f(z) \in \mathcal{S}^*(\alpha)$.

Theorem B. If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq 1 - \alpha \quad (0 \leq \alpha < 1),$$

then

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1 - \alpha \quad (z \in \mathbb{U}, 0 \leq \alpha < 1),$$

that is, that $f(z) \in \mathcal{K}(\alpha)$.

In this paper, we consider a new subclass $\mathcal{M}(\alpha)$ of $\mathcal{A}$ consisting of functions $f(z)$ such that

$$\left| \frac{f(z)}{zf'(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \quad (z \in \mathbb{U}),$$

for some α ($0 < \alpha < 1$). We also introduce and investigate here the subclass $\mathcal{N}(\alpha)$ of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following inclusion relationship

$$zf'(z) \in \mathcal{M}(\alpha).$$

2 Properties of the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$

Theorem 1 If $f(z) \in \mathcal{A}$ satisfies

$$(2.1) \quad \left| \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < 1 - 2\alpha \quad (z \in \mathbb{U})$$

for some α ($\frac{1}{4} \leq \alpha < \frac{1}{2}$), then

$$\left| \frac{f(z)}{zf'(z)} - 1 \right| < \frac{1}{2\alpha} - 1 \quad (z \in \mathbb{U}),$$

therefore, $f(z) \in \mathcal{M}(\alpha)$.

Corollary 1 If $f(z) \in \mathcal{A}$ satisfies

$$(2.6) \quad \left| \frac{zf''(z)}{f'(z)} - \frac{z(2f''(z) + zf'''(z))}{f'(z) + zf''(z)} \right| < 1 - 2\alpha \quad (z \in \mathbb{U})$$

for some α ($\frac{1}{4} \leq \alpha < \frac{1}{2}$), then

$$\left| \frac{f'(z)}{f'(z) + zf''(z)} - 1 \right| < \frac{1}{2\alpha} - 1 \quad (z \in \mathbb{U}),$$

therefore, $f(z) \in \mathcal{N}(\alpha)$.

Theorem 2 If $f(z) \in \mathcal{M}(\alpha)$ $\left(\frac{1}{2} \leq \alpha < 1\right)$, then

$$\left| \left(\frac{z}{f(z)} \right)^\beta - 1 \right| < 1 - \gamma \quad (z \in \mathbb{U}),$$

where, $0 \leq \gamma < 1$ and $0 < \beta \leq 1 - \gamma$.

Corollary 2 If $f(z) \in \mathcal{M}(\alpha)$ $\left(\frac{1}{2} \leq \alpha < 1\right)$, then

$$\left| \left(\frac{z}{f(z)} \right)^\beta - 1 \right| < 1 \quad (z \in \mathbb{U})$$

where $0 < \beta \leq 1$.

Theorem 3 If $f(z) \in \mathcal{N}(\alpha)$ $\left(\frac{1}{2} \leq \alpha < 1\right)$, then

$$\left| \left(\frac{1}{f'(z)} \right)^\beta - 1 \right| < 1 - \gamma \quad (z \in \mathbb{U}),$$

where $0 \leq \gamma < 1$ and $0 < \beta \leq 1 - \gamma$.

Corollary 3 If $f(z) \in \mathcal{N}(\alpha)$ $\left(\frac{1}{2} \leq \alpha < 1\right)$, then

$$\left| \left(\frac{1}{f'(z)} \right)^\beta - 1 \right| < 1 \quad (z \in \mathbb{U}),$$

where $0 < \beta \leq 1$.

3 Coefficient inequalities

Theorem 4. If $f(z) \in \mathcal{A}$ satisfies

$$(3.1) \quad \sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq \frac{1}{2} (1 - |1 - 2\alpha|) = \begin{cases} \alpha & (0 < \alpha \leq \frac{1}{2}) \\ 1 - \alpha & (\frac{1}{2} \leq \alpha < 1) \end{cases}$$

for some α $(0 < \alpha < 1)$, then $f(z) \in \mathcal{M}(\alpha)$.

Theorem 5. If $f(z) \in \mathcal{A}$ satisfies

$$(3.2) \quad \sum_{n=2}^{\infty} n(n - \alpha) |a_n| \leq \frac{1}{2} (1 - |1 - 2\alpha|) = \begin{cases} \alpha & (0 < \alpha \leq \frac{1}{2}) \\ 1 - \alpha & (\frac{1}{2} \leq \alpha < 1) \end{cases}$$

for some α $(0 < \alpha < 1)$, then $f(z) \in \mathcal{N}(\alpha)$.

4 Convolution

Definition If $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$ are given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}.$$

Then the convolution $(f * g)(z)$ of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

In this section, we define

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2, 3, \dots)$$

and

$$g_k(z) = z + \sum_{n=2}^{\infty} b_{n,k} z^n \quad (k = 1, 2, 3, \dots).$$

Theorem 6 If $f_j(z) \in \mathcal{M}^*(\alpha_j)$ with $\frac{1}{2} \leq \alpha_j < 1$ for each $j = 1, 2$, then

$$(f_1 * f_2)(z) \in \mathcal{M}^*(\alpha)$$

where

$$\alpha = 1 - \frac{(1 - \alpha_1)(1 - \alpha_2)}{(2 - \alpha_1)(2 - \alpha_2) - (1 - \alpha_1)(1 - \alpha_2)}.$$

Theorem 7 If $f_j(z) \in \mathcal{M}^*(\alpha_j)$ with $\frac{1}{2} \leq \alpha_j < 1$ for each $j = 1, 2, \dots, m$, then

$$(f_1 * f_2 * \dots * f_m)(z) \in \mathcal{M}^*(\alpha),$$

where

$$\alpha = 1 - \frac{\prod_{j=1}^m (1 - \alpha_j)}{\prod_{j=1}^m (2 - \alpha_j) - \prod_{j=1}^m (1 - \alpha_j)}$$

Theorem 8 If $g_k(z) \in \mathcal{N}^*(\alpha_k)$ with $\frac{1}{2} \leq \alpha_k < 1$ for each $k = 1, 2$, then

$$(g_1 * g_2)(z) \in \mathcal{N}^*(\alpha),$$

where

$$\alpha = 1 - \frac{(1 - \alpha_1)(1 - \alpha_2)}{2(2 - \alpha_1)(2 - \alpha_2) - (1 - \alpha_1)(1 - \alpha_2)}.$$

Theorem 9 If $g_k(z) \in \mathcal{N}^*(\alpha_k)$ with $\frac{1}{2} \leq \alpha_k < 1$ for each $k = 1, 2, 3, \dots, m$, then

$$(g_1 * g_2 * \dots * g_m)(z) \in \mathcal{N}^*(\alpha),$$

where

$$\alpha = 1 - \frac{\prod_{k=1}^m (1 - \alpha_k)}{2^{m-1} \prod_{k=1}^m (2 - \alpha_k) - \prod_{k=1}^m (1 - \alpha_k)}$$

Theorem 10 If $f_j(z) \in \mathcal{M}^*(\alpha_j)$ with $0 < \alpha_j \leq \frac{1}{2}$ for each $j = 1, 2$, then

$$(f_1 * f_2)(z) \in \mathcal{M}^*(\alpha)$$

where

$$\alpha = \frac{2\alpha_1\alpha_2}{\alpha_1\alpha_2 + (2 - \alpha_1)(2 - \alpha_2)}.$$

Theorem 11 If $f_j(z) \in \mathcal{M}^*(\alpha_j)$ with $0 < \alpha_j \leq \frac{1}{2}$ for each $j = 1, 2, 3, \dots, m$, then

$$(f_1 * f_2 * \dots * f_m)(z) \in \mathcal{M}^*(\alpha),$$

where

$$\alpha = \frac{2 \prod_{j=1}^m \alpha_j}{\prod_{j=1}^m \alpha_j + \prod_{j=1}^m (2 - \alpha_j)}$$

Theorem 12 If $g_k(z) \in \mathcal{N}^*(\alpha_k)$ with $0 < \alpha_k \leq \frac{1}{2}$ for each $k = 1, 2$, then

$$(g_1 * g_2)(z) \in \mathcal{N}^*(\alpha)$$

where

$$\alpha = \frac{2\alpha_1\alpha_2}{\alpha_1\alpha_2 + 2(2 - \alpha_1)(2 - \alpha_2)}.$$

Theorem 13 If $g_k(z) \in \mathcal{N}^*(\alpha_k)$ with $0 < \alpha_k \leq \frac{1}{2}$ for each $k = 1, 2, 3, \dots, m$, then

$$(g_1 * g_2 * \dots * g_m)(z) \in \mathcal{N}^*(\alpha),$$

where

$$\alpha = \frac{2 \prod_{k=1}^m \alpha_k}{\prod_{k=1}^m \alpha_k + 2^{m-1} \prod_{k=1}^m (2 - \alpha_k)}$$

Theorem 14 If $g_k(z) \in \mathcal{N}^*(\alpha_k)$ with $\frac{1}{2} \leq \alpha_k < 1$ for each $k = 1, 2$, then

$$(g_1 * g_2)(z) \in \mathcal{M}^*(\alpha),$$

where

$$\alpha = 1 - \frac{(1 - \alpha_1)(1 - \alpha_2)}{4(2 - \alpha_1)(2 - \alpha_2) - (1 - \alpha_1)(1 - \alpha_2)}.$$

Theorem 15 If $g_k(z) \in \mathcal{N}^*(\alpha_k)$ with $\frac{1}{2} \leq \alpha_k < 1$ for each $k = 1, 2, 3, \dots, m$, then

$$(g_1 * g_2 * \dots * g_m)(z) \in \mathcal{M}^*(\alpha)$$

where

$$\alpha = 1 - \frac{\prod_{k=1}^m (1 - \alpha_k)}{4^{m-1} \prod_{k=1}^m (2 - \alpha_k) - \prod_{k=1}^m (1 - \alpha_k)}.$$

Theorem 16 If $f_1(z) \in \mathcal{M}^*(\alpha_1)$ with $\frac{1}{2} \leq \alpha_1 < 1$, and if $g_1(z) \in \mathcal{N}^*(\beta_1)$ with $\frac{1}{2} \leq \beta_1 < 1$, then

$$(f_1 * g_1)(z) \in \mathcal{M}^*(\alpha)$$

where

$$\alpha = 1 - \frac{(1 - \alpha_1)(1 - \beta_1)}{2(2 - \alpha_1)(2 - \beta_1) - (1 - \alpha_1)(1 - \beta_1)}.$$

Theorem 17 If $f_j(z) \in \mathcal{M}^*(\alpha_j)$ with $\frac{1}{2} \leq \alpha_j < 1$ for each $j = 1, 2, 3, \dots, m$, and if $g_k(z) \in \mathcal{N}^*(\beta_k)$ with $\frac{1}{2} \leq \beta_k < 1$ for each $k = 1, 2, 3, \dots, p$, then

$$(f_1 * \dots * f_m * g_1 * \dots * g_p)(z) \in \mathcal{M}^*(\gamma)$$

where

$$\gamma = 1 - \frac{\prod_{j=1}^m (1 - \alpha_j) \prod_{k=1}^p (1 - \beta_k)}{2^p \prod_{j=1}^m (2 - \alpha_j) \prod_{k=1}^p (2 - \beta_k) - \prod_{j=1}^m (1 - \alpha_j) \prod_{k=1}^p (1 - \beta_k)}$$

Theorem 18 If $g_k(z) \in \mathcal{N}^*(\alpha_k)$ with $0 < \alpha_k \leq \frac{1}{2}$ for each $k = 1, 2$, then

$$(g_1 * g_2)(z) \in \mathcal{M}^*(\alpha),$$

where

$$\alpha = \frac{2\alpha_1\alpha_2}{4(2 - \alpha_1)(2 - \alpha_2) + \alpha_1\alpha_2}.$$

Theorem 19 If $g_k(z) \in \mathcal{N}^*(\alpha_k)$ with $0 < \alpha_k \leq \frac{1}{2}$ for each $k = 1, 2, 3, \dots, m$, then

$$(g_1 * g_2 * \dots * g_m)(z) \in \mathcal{M}^*(\alpha),$$

where

$$\alpha = \frac{2 \prod_{k=1}^m \alpha_k}{\prod_{k=1}^m \alpha_k + 4^{m-1} \prod_{k=1}^m (2 - \alpha_k)}.$$

Theorem 20 If $f_1(z) \in \mathcal{M}^*(\alpha_1)$ with $0 < \alpha_1 \leq \frac{1}{2}$, and if $g_1(z) \in \mathcal{N}^*(\beta_1)$ with $0 < \beta_1 \leq \frac{1}{2}$, then

$$(f_1 * g_1)(z) \in \mathcal{M}^*(\alpha)$$

where

$$\alpha = \frac{2\alpha_1\beta_1}{2(2 - \alpha_1)(2 - \beta_1) + \alpha_1\beta_1}.$$

Theorem 21 If $f_j(z) \in \mathcal{M}^*(\alpha_j)$ with $0 < \alpha_j \leq \frac{1}{2}$, for each $j = 1, 2, 3, \dots, m$, and if $g_k(z) \in \mathcal{N}^*(\beta_k)$ with $0 < \beta_k \leq \frac{1}{2}$ for each $k = 1, 2, 3, \dots, p$, then

$$(f_1 * \dots * f_m * g_1 * \dots * g_p)(z) \in \mathcal{M}^*(\gamma),$$

where

$$\gamma = \frac{2 \prod_{j=1}^m \alpha_j \prod_{k=1}^p \beta_k}{2^p \prod_{j=1}^m (2 - \alpha_j) \prod_{k=1}^p (2 - \beta_k) + \prod_{j=1}^m \alpha_j \prod_{k=1}^p \beta_k}.$$

References

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