

β -FILTRATION OF COHOMOLOGY OF GROUPS

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1. INTRODUCTION

Let G be a compact Lie group, (e.g., finite group). The structure of $H^*(BG; \mathbb{Z}/p)$ seems very complicated, in general. Most cases are difficult to write down it, even if it can be computable. So it may be reasonable to consider an adequate filtration and the associated graded algebra $gr H^*(BG; \mathbb{Z}/p)$.

In this paper, we study natural filtrations such that

$$P^j(F_i) \subset F_i \quad \text{and} \quad \beta(F_i) \subset F_{i-1}$$

where P^j is the reduced power operation and β is the Bockstein operation. (We say such filtrations to be a β -filtration. For the precise definition, see §2 bellow.) The β -filtration seems not unique. However, we see it is determined uniquely for the cases $G = (\mathbb{Z}/p)^n, O_3, PGL_3, p_+^{1+2}$. (However the ring structures of $gr H^*(BG; \mathbb{Z}/p)$ do not become more simple.)

An example of β -filtrations is given in §6 by using the motivic cohomology $H^{*,*'}(BG; \mathbb{Z}/p)$ (over $k = \mathbb{C}$). In fact we define the (motivic) filtration such that $x \in F_i$ if

$$x \in \bigoplus_{i=2*'-*} Im(t_{\mathbb{C}} : H^{*,*'}(BG; \mathbb{Z}/p) \rightarrow H^*(BG; \mathbb{Z}/p))$$

where $t_{\mathbb{C}}$ is the realization map. Moreover, we see that this filtration coincides the coniveau filtration defined by Grothendieck [Bl-Og] for cases $(\mathbb{Z}/p)^n, O_n, PGL_p$.

2. β -FILTRATION

Let p be a prime number. Let G be a compact Lie group (e.g., finite group) and BG be their classifying spaces.

Definition. We say that a filtration

$$(0) = F_{-1} \subset F_0 \subset F_1 \subset \dots \subset F_{\infty} = H^*(BG; \mathbb{Z}/p)$$

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is a β -filtration if it satisfies the following (1) – (5) ;

- (1) It is natural for the induced map and the transfer. That is, for $f : G \rightarrow G'$ and an injection $g : G \rightarrow G'$ of finite cokernel, let F' be the filtration of $H^*(BG'; \mathbb{Z}/p)$. Then

$$f^*(F'_i) \subset F_i \quad \text{and} \quad g_*(F_i) \subset F'_i,$$

- (2) $H^i(BG; \mathbb{Z}/p) \subset F_i$,
 (3) $F_i \cdot F_j \subset F_{i+j}$,
 (4) $Ch(G) \subset F_0$ where $Ch(G)$ is the Chern subring.
 (5) For the reduced and the (usual and higher) Bockstein operations P^j and β ,

$$P^j(F_i) \subset F_i \quad \text{and} \quad \beta(F_i) \subset F_{i-1}.$$

Let us write the associated graded algebra

$$gr H^*(BG; \mathbb{Z}/p) = \oplus_i F_i / F_{i-1} \quad \text{and} \quad gr^i H^*(BG; \mathbb{Z}/p) = F_i / F_{i-1}.$$

For each element $x \in H^*(BG; \mathbb{Z}/p)$, define the (weight) degree

$$w(x) = i \quad \text{when } 0 \neq x \in gr^i H^*(BG; \mathbb{Z}/p).$$

First note that the β -filtration of a finite group is decided from that of its Sylow p -subgroup.

Lemma 2.1. *Let G be a finite group, S its Sylow p -subgroup and $i : S \subset G$ the inclusion. Let $F_j(G)$ and $F_j(S)$ be its β -filtration of G and S . Then $i_* F_j(S) = F_j(G)$.*

Proof. From the property (1),

$$i_* i^* F_j(G) \subset i_* F_j(S) \subset F_j(G).$$

Let $[G; S] = m$ the number prime to p . Then $i_* i^* F_j(G) = m F_j(G) = F_j(G)$. Hence we see $i_* F_j(S) = F_j(G)$. $\square$

Quillen showed that the following restriction map r is an F-isomorphism

$$r : H^*(BG; \mathbb{Z}/p) \rightarrow \lim_{\leftarrow A} H^*(BA; \mathbb{Z}/p)$$

where A runs over a set of conjugacy classes of elementary abelian p -subgroups of G . Here an F-isomorphism means that its kernel is generated by nilpotent elements and for each $x \in H^*(BG; \mathbb{Z}/p)$ there is $s \geq 0$ such that $x^{p^s} \in \text{Im}(r)$. Let $\bar{Ch}(G)$ be the Mackey closure of $Ch(G)$, which is recursively defined by transfers of Chern classes from subgroups. From (1) and (4), we see $\bar{Ch}(G) \subset F_0$. It is known from Green-Leary [Gr-Le] that the inclusion $\bar{Ch}(G) \subset H^*(BG; \mathbb{Z}/p)$ is F -epic. Hence we see ;

Lemma 2.2. *Each element in $gr^i H^*(BG; \mathbb{Z}/p)$, $i > 0$ is nilpotent in $gr H^*(BG; \mathbb{Z}/p)$.*

Next we recall the Milnor operation Q_i which is inductively defined as $Q_0 = \beta$ and $Q_{i+1} = [Q_i, P^{p^i}]$. It is known that $Q_i Q_j = -Q_j Q_i$ and $Q_i^2 = 0$. Let us write the exterior algebra

$$Q(n) = \Lambda(Q_0, \dots, Q_n).$$

From the property (5), we see $Q_j F_i \subset F_{j-1}$. Moreover $Q_{i_1} \dots Q_{i_s} F_j \subset F_{j-s}$. Thus we have

Lemma 2.3. *Let $H^*(BG; \mathbb{Z}/p) \cong \oplus_i Q(n) G_n$ with $G_n \subset F_{n+1}$ where $Q(n) G_n$ means the free $Q(n)$ -module generated by G_n . Then*

$$gr^i H^*(BG; \mathbb{Z}/p) = \oplus_{i=n-s} Q_{i_1} \dots Q_{i_s} G_n.$$

That means if $0 \neq x \in gr^i H^*(BG; \mathbb{Z}/p)$ if and only if $Q_0 \dots Q_{i-1} x \neq 0$ and $Q_0 \dots Q_i(x) = 0$.

Proof. Let $x' = Q_{i_1} \dots Q_{i_s} x$ for some $0 \neq x \in G_n$. Then $x' \in F_{n-s}$. Moreover

$$0 \neq Q_0 \dots Q_n x = (\pm) Q_{j_1} \dots Q_{j_{n-s}} x'$$

with $(j_1, \dots, j_{n-s}, i_1, \dots, i_s) = (0, \dots, n)$. This implies $x \notin F_{n-s-1}$. Thus $0 \neq x' \in gr^{n-s} H^*(BG; \mathbb{Z}/p)$. $\square$

Remark. If the condition (4) is weakened so that $\beta(F_i) \subset F_i$ for only the usual Bockstein operation, then letting $w(x) = n$ as the largest number n such that $Q_{i_1} \dots Q_{i_s}(x) \neq 0$, gives an example of such filtrations.

3. EXAMPLES

We do not see that β -filtrations are unique. However we give here the cases where the β -filtration (so $gr H^*(BG; \mathbb{Z}/p)$) is determined uniquely.

At first consider the case $G = \mathbb{Z}/2$. Of course $H^*(B\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]$ with $\deg(x) = 1$. Since x^2 is represented as the first Chern class c_1 of the canonical bundle, $x^2 \in F_0$. Let us write x^2 by y . Since $\beta(x) = y$, we show $w(x) \geq 1$. But $x \in F_1$ from (2). So $w(x) = 1$ exactly. Thus we have the isomorphism

$$gr H^*(B\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[y] \otimes \Lambda(x) \quad w(y) = 0, \quad w(x) = 1$$

as the case for odd prime. Indeed, we have

$$gr H^*((B\mathbb{Z}/p)^n; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, \dots, y_n] \otimes \Lambda(x_1, \dots, x_n)$$

with $\beta(x_i) = y_i$ for all primes p . Of course for $i_1 < \dots < i_s$,

$$w(yx_{i_1} \dots x_{i_s}) = s, \quad \text{with } 0 \neq y \in \mathbb{Z}/p[y_1, \dots, y_n].$$

Note that $Q_i(x) = y^{p^i}$ and each Q_i is a derivation, and hence

$$Q_0 \dots Q_{s-1}(yx_1 \dots x_s) = y \sum \text{sgn}(j_1, \dots, j_n) y_1^{p^{j_1}} y_2^{p^{j_2}} \dots y_s^{p^{j_s}} \neq 0$$

where $(j_1, \dots, j_s)$ are permutations of $(0, \dots, s-1)$.

For the n -th unitary group U_n , it is immediate from (4),

$$F_0 = H^*(BU_n; \mathbb{Z}/p) \cong \mathbb{Z}/p[c_1, \dots, c_n].$$

The mod 2 cohomology of the classifying space BO_n of the n -th orthogonal group is

$$H^*(BO_n; \mathbb{Z}/2) \cong H^*((B\mathbb{Z}/2)^n; \mathbb{Z}/2)^{S_n} \cong \mathbb{Z}/2[w_1, \dots, w_n]$$

where S_n is the n -th symmetry group, w_i is the Stiefel-Whitney class which restricts the elementary symmetric polynomial in $\mathbb{Z}/2[x_1, \dots, x_n]$. Each element w_i^2 is represented by Chern class c_i of the induced representation $O(n) \subset U(n)$. Let us write w_i^2 by c_i .

Since $Q_{i-1} \dots Q_0(w_i) \neq 0$, we see each $w(w_i) = i$. However even the module structure of $gr^* H^*(BO_n; \mathbb{Z}/2)$ seems complicated. W.S.Wilson ([Wi],[Ko-Ya]) found a good $Q(i) = \Lambda(Q_0, \dots, Q_i)$ -module decomposition for $X = BO_n$, namely,

$$H^*(X; \mathbb{Z}/2) = \oplus_{-1} Q(i)G_i \quad \text{with } Q_0 \dots Q_i G_i \in \mathbb{Z}/2[c_1, \dots, c_n].$$

Here G_{k-1} is quite complicated, namely, it is generated by symmetric functions

$$\sum x_1^{2i_1+1} \dots x_k^{2i_k+1} x_{k+1}^{2j_1} \dots x_{k+q}^{2j_q}, \quad k+q \leq n,$$

with $0 \leq i_1 \leq \dots \leq i_k$ and $0 \leq j_1 \leq \dots \leq j_q$; and if the number of j equal to j_u is odd, then there is some $s \leq k$ such that $2i_s + 2^s < 2j_u < 2i_s + 2^{s+1}$.

From Lemma 2.3 in the preceding section, we have

Proposition 3.1. *If $w(G_i) = i+1$, then we have the bidegree module isomorphism*

$$gr H^*(BO_n; \mathbb{Z}/2) \cong (gr H^*((B\mathbb{Z}/2)^n; \mathbb{Z}/2))^{S_n} \cong (\oplus_i Q(i)G_i).$$

We can prove the assumption $w(G_i) = i+1$ for the motivic filtration, which is an example of β -filtrations.

Since the direct decomposition of BO_3 is complicated to write, we only write here that of SO_3 since $O_3 \cong SO_3 \times \mathbb{Z}/2$.

$$\begin{aligned} H^*(BSO_3; \mathbb{Z}/2) &\cong \mathbb{Z}/2[w_1, w_2, w_3]/(w_1) \cong \mathbb{Z}/2[w_2, w_3] \\ &\cong \mathbb{Z}/2[c_2, c_3]\{1, w_2, w_3 = Q_0 w_2, w_2 w_3 = Q_1 w_2\} \end{aligned}$$

$$\begin{aligned} &\cong \mathbb{Z}/2[c_2, c_3]\{w_2, Q_0w_2, Q_1w_2, c_3 = Q_0Q_1w_2\} \oplus \mathbb{Z}/2[c_2] \\ &\cong \mathbb{Z}/2[c_2, c_3]Q(1)\{w_2\} \oplus \mathbb{Z}/2[c_2]. \end{aligned}$$

Of course, this case $w(w_2) = 2$ and the assumption in the above proposition is satisfied.

From here we consider the case $p = \text{odd}$. One of the easiest examples is the case $G = PGL_3$ and $p = 3$. The mod 3 cohomology is given by ([Ko-Ya],[Ve])

$$H^*(BPGL_3; \mathbb{Z}/3) \cong (\mathbb{Z}/3[y_2]\{y_2\} \oplus \mathbb{Z}/3[y_8]\{1, y_2, y_3, y_7\}) \otimes \mathbb{Z}/3[y_{12}]$$

where the suffice i of y_i means its degree. It is known that y_2^2, y_2^3, y_8^2 and y_{12} are represented by Chern classes. The cohomology operations are given

$$y_2 \xrightarrow{\beta} y_3 \xrightarrow{P^1} y_7 \xrightarrow{\beta} y_8$$

Hence $Q_1Q_0(y_2) = y_8$. (The element y_8 is not represented by a Chern class.) Thus we see

$$F_0 = (\mathbb{Z}/3[y_2]\{y_2^2\} \oplus \mathbb{Z}/3[y_8]) \otimes \mathbb{Z}/3[y_{12}].$$

Theorem 3.2. *Let $w(y_2) = 2$. Then the bidegree module $gr H^*(BPGL_3; \mathbb{Z}/3)$ is isomorphic to*

$$(\mathbb{Z}/3[y_2]\{y^2\} \oplus \mathbb{Z}/3\{1\} \oplus \mathbb{Z}/3[y_8] \otimes Q(1)\{y_2\}) \otimes \mathbb{Z}/3[y_{12}].$$

In §10, we will study the motivic filtration of $BPGL_p$ for each odd prime.

Next consider the extraspecial p -group $E = p_+^{1+2}$ for odd prime. (The similar argument also holds for the dihedral group $D_8 = 2_+^{1+2}$.) This group is a noncommutative group of exponent p such that there is the central extension

$$0 \rightarrow \mathbb{Z}/p \rightarrow p_+^{1+2} \xrightarrow{\pi} \mathbb{Z}/p \oplus \mathbb{Z}/p \rightarrow 0.$$

The ordinary cohomology is known by Lewis (see also Leary [Lew], [Le], [Te-Ya]), namely,

$$H^{\text{even}}(BE)/p \cong (\mathbb{Z}/p[y_1, y_2]/(y_1^p y_2 - y_1 y_2^p) \oplus \mathbb{Z}/p\{c_2, \dots, c_{p-1}\}) \otimes \mathbb{Z}/p[c_p].$$

$$H^{\text{odd}}(BE) \cong \mathbb{Z}/p[y_1, y_2, c_p]\{a_1, a_2\}/(y_1 a_2 - y_2 a_1, y_1^p a_2 - y_2^p a_1).$$

Here y_1, y_2 are the 1-st Chern classes of 1-dimensional representation induced from the map π , c_j is the j -th Chern class of the inductive p -dimensional representation from the maximal abelian subgroup ($\cong \mathbb{Z}/p \oplus \mathbb{Z}/p$). The 3-dimensional elements a_i satisfies $Q_1(a_i) = y_i c_p$.

Theorem 3.3. *Let $\partial_p : H^*(BE; \mathbb{Z}/p) \rightarrow H^*(BE)/p$ be the (higher) Bockstein map. Then*

$$gr H^*(BE; \mathbb{Z}/p) \cong \{1, \partial_p^{-1}\}(H^*(BE)/p) - \{\partial_p^{-1}1\}$$

where $w(H^{even}(BE)/p) = 0$, $w(H^{odd}(BE)) = 1$ and ∂_p^{-1} ascends the weight one.

Proof. Since all elements in $H^{even}(BE)$ are generated by Chern classes, we have the isomorphism $F_0 = H^{even}(BE)/p$. We know $H^{odd}(BE)/p$ is generated as an $(H^{even}(BE)/p)$ -module by two elements a_1, a_2 such that $Q_1(a_i) = y_i c_p$ [Te-Ya].

The mod p -cohomology is written additively

$$H^*(BE; \mathbb{Z}/p) \cong \{1, \partial_p^{-1}\}H^*(BE)/p.$$

All elements in $H^{odd}(BE)$ are just p -torsion and we can take $a'_i \in H^2(BE; \mathbb{Z}/p)$ such that $\beta(a'_i) = a_i$. Since $|a'_i| = 2$, $a'_i \in F_2$ from (2). Hence we take $w(a'_i) = 2$ and so $w(a_i) = 1$.

Next consider elements $x = \partial_p^{-1}(y)$, $y \in H^{even}(BE)/p$. If $y \in (Ideal(y_1, y_2))$, then $\partial_p^{-1}(y) = \sum x_i b_i$ for $b_i \in H^{even}(BE)/p$, and hence we can take $w(\partial_p^{-1}(y)) = 1$. For other elements $y = c_i c$ with $c \in \mathbb{Z}/p[c_p]$, we can prove ([Ly]) that those elements are represented by transfer from a subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. Therefore we can also prove that $w(\partial_p^{-1}(y)) = 1$. Thus we complete the proof. $\square$

4. MOTIVIC FILTRATION

The motivic cohomology of the classifying space is defined as follows. Let G be a linear algebraic group over k . Let V be a representation of G such that G acts freely on $V - S$ for some closed subset S . Then $(V - S)/G$ exists as a quasi-projective variety over k . According to Totaro ([To1]) and V.Voevodsky, define

$$H^{*,*}(BG; \mathbb{Z}/p) = \lim_{\dim(V), \text{codim}(S) \rightarrow \infty} H^{*,*}((V - S)/G; \mathbb{Z}/p).$$

The topological space $BG(\mathbb{C}) = \lim((V - S)/G)(\mathbb{C})$ is homotopic to the usual classifying space BG . Hence we write the $\mathbb{C}$ -value points $BG(\mathbb{C})$ simply by BG .

We still know the motivic cohomologies of BG_m and $B\mathbb{Z}/p$. Since BGL_n is cellular, we have (Hu-Kriz [Hu-Kr])

$$H^{*,*'}(BGL_n; \mathbb{Z}/p) \cong \mathbb{Z}/p[c_1, \dots, c_n] \otimes H^{*,*'}(pt; \mathbb{Z}/p)$$

where the Chern class c_i with $\deg(c_i) = (2i, i)$ are identified with the elementary symmetric polynomial in $H^{*,*}((\mathbb{P}^\infty)^n; \mathbb{Z}/p)$. So we can define the Chern class $\rho^*(c_i) \in H^{2*,*}(BG; \mathbb{Z}/p)$ for each representation $\rho : G \rightarrow GL_n$.

Hereafter we assume $k = \mathbb{C}$.

Definition. We define the motivic filtration of $H^*(BG; \mathbb{Z}/p)$ by,

$$F_i = \text{Im}(t_{\mathbb{C}}^{2*-i,*}) = \oplus_n t_{\mathbb{C}}(H^{2n-i,n}(BG; \mathbb{Z}/p)).$$

Note $t_{\mathbb{C}} = cl : H^{*,*'}(X; \mathbb{Z}/p) \rightarrow H_{et}^*(X; \mathbb{Z}/p)$ identifying $H_{et}^*(X; \mathbb{Z}/p) \cong H^*(X(\mathbb{C}); \mathbb{Z}/p)$ when $k = \mathbb{C}$. Moreover the cycle map is identified with

$$\times \tau^{*-*'} : H^{*,*'}(X; \mathbb{Z}/p) \rightarrow H^{*,*}(X; \mathbb{Z}/p),$$

from the $BL(*, p)$ condition. So we know $F_i \subset F_{i+1}$.

Theorem 4.1. *The motivic filtration is a β -filtration.*

Proof. By the dimensional condition, we know

$$H^{2*-i,*}(BG; \mathbb{Z}/p) = 0 \quad \text{for } i < 0.$$

This implies $F_{-1} = 0$. By the $BL(i, p)$ condition, we see $F_i = H^i(BG; \mathbb{Z}/p)$, which is the condition (2) of the β -filtration. Hence of course $F_\infty = H^*(BG; \mathbb{Z}/p)$. Thus the motivic filtration is indeed filtration of $H^*(BG; \mathbb{Z}/p)$.

The condition (1) is satisfied, indeed $H^{*,*'}(X; \mathbb{Z}/p)$ has the Gysin exact sequence and transfers. Of course the sum of the bidegrees are

$$(2* - n, *) + (2*' - n', *') = (2(* + *') - (n + n'), * + *'),$$

which shows the condition (3). Since we can also define the Chern class $c_i \in H^{2i,i}(X; \mathbb{Z}/p)$ from the above argument, we get (4), namely $Ch(G) \subset F_0$. The existence of cohomology operations implies the condition (5). $\square$

Remark. The motivic filtration can be extend for all smooth X and $k \subset \mathbb{C}$ but not only BG and $k = \mathbb{C}$ with changing the realization map to the cycle map, and $H^*(X(\mathbb{C}); \mathbb{Z}/p)$ to $H_{et}^*(X; \mathbb{Z}/p)$. Moreover the condition (1) should be extended for each projective map g and the condition (3) be changed $cl(CH^*(X)) = F_0$.

From the above theorem, we can know some information of the motivic cohomology without using any theory of algebraic geometry (just arguments in §2 or §3). In [Ya2], we define

$$h^{*,*'}(X; \mathbb{Z}/p) = \oplus_{m,n} H^{m,n}(X; \mathbb{Z}/p) / (Ker(t_{\mathbb{C}}^{m,n})),$$

and compute them, for example, the cases $X = BG$ stated in §3. In fact, it is immediate that

$$h^{*,*'}(X; \mathbb{Z}/p) \cong gr^{2*' - *} H^*(X(\mathbb{C}); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau].$$

5. CONIVEAU FILTRATION

The motivic cohomology is known to be a cohomology of a complex of some Zarisky sheaves. Recall that $\mathbb{Z}/p(n)$ ([Vo1],[Vo4]) is the complex of sheaves in Zarisky topology such that $H^{m,n}(X; \mathbb{Z}/p) \cong H_{Zar}^m(X; \mathbb{Z}/p(n))$. Let α be the obvious map of sites from Zarisky topology to etale topology so that

$$H_{et}^m(X; \mathbb{Z}/p) \cong H_{et}^m(X; \mu_p^{\otimes n}) \cong H_{Zar}^m(X, R\alpha_* \alpha^* \mathbb{Z}/p(n)).$$

For $k \leq n$, let $\tau_{\leq k+1} R\alpha_* \alpha^* \mathbb{Z}/p(n)$ be the canonical truncation of $R\alpha_* \alpha^* \mathbb{Z}/p(n)$ of level $k+1$. Then we have the short exact sequence of sheaves

$$0 \rightarrow \tau_{\leq n} R\alpha_* \alpha^* \mathbb{Z}/p(n) \rightarrow \tau_{\leq n+1} R\alpha_* \alpha^* \mathbb{Z}/p(n) \rightarrow H_{\mathbb{Z}/p}^n \rightarrow 0$$

where $H_{\mathbb{Z}/p}^n$ the Zarisky sheaf induced from the presheaf $H_{et}^n(V; \mathbb{Z}/p)$ for open subset V of X . The Beilinson and Lichtenbaum conjecture (hence $BL(n, p)$ condition) (see [Vo4],[Vo5]) implies

$$\mathbb{Z}/p(k) \cong \tau_{\leq k+1} R\alpha_* \alpha^* \mathbb{Z}/p(n) \quad \text{quasi equivalence.}$$

Thus we get the long exact sequence

$$\begin{aligned} & \rightarrow H^{m,n-1}(X; \mathbb{Z}/p) \xrightarrow{\times \tau} H^{m,n}(X; \mathbb{Z}/p) \\ & \rightarrow H_{Zar}^{m-n}(X; H_{\mathbb{Z}/p}^n) \rightarrow H^{m+1,n-1}(X; \mathbb{Z}/p) \xrightarrow{\times \tau}. \end{aligned}$$

Hence we have the isomorphism ;

Lemma 5.1. ([Or-Vi-Vo], [Ya4])

$$H_{Zar}^{*-*'}(X; H_{\mathbb{Z}/p}^{*'}) \cong H^{*,*'}(X; \mathbb{Z}/p)/(\tau) \oplus \text{Ker}(\tau|H^{*+1,*'-1}(X; \mathbb{Z}/p)).$$

The filtration *coniveau* is given by

$$N^c H_{et}^m(X; \mathbb{Z}/p) = \cup_Z \text{Ker}\{H_{et}^m(X; \mathbb{Z}/p) \rightarrow H_{et}^m(X - Z; \mathbb{Z}/p)\}$$

where Z runs in the set of closed subschemes of X of *codim* = c .

Grothendieck wrote down the E_1 -term of the spectral sequence induced from the above *coniveau* filtration.

$$E(c)_1^{c,m-c} \cong \Pi_{x \in X^{(c)}} H_{et}^{m-c}(k(x); \mathbb{Z}/p) \Longrightarrow gr H_{et}^m(X; \mathbb{Z}/p)$$

where $X^{(c)}$ is the set of primes of codimension c and $k(x)$ is the function field of x . We can regard $i_x H_{et}^{m-c}(k(x); \mathbb{Z}/p)$ as a constant sheaf $H_{et}^{m-c}(k(x); \mathbb{Z}/p)$ on $\{\bar{x}\}$ and extend it by zero to X . Then the differentials of the spectral sequence give us a complex on sheaves on X

$$\begin{aligned} 0 \rightarrow H_{\mathbb{Z}/p}^q & \rightarrow \Pi_{x \in X^{(0)}} i_x H_{et}^q(k(x); \mathbb{Z}/p) \rightarrow \Pi_{x \in X^{(1)}} i_x H_{et}^{q-1}(k(x); \mathbb{Z}/p) \\ & \rightarrow \dots \rightarrow \Pi_{x \in X^{(q)}} i_x H_{et}^0(k(x); \mathbb{Z}/p) \rightarrow 0. \end{aligned}$$

Bloch-Ogus [Bl-Og] proved that the above sequence of sheaves is exact and the E_2 -term is given by

$$E(c)_2^{c,m-c} \cong H_{Zar}^c(X, H_{\mathbb{Z}/p}^{m-c}).$$

Theorem 5.2. ([Ya4]) *For each bidegree $(*, *')$, let $H^{*,*'}(X; \mathbb{Z}/p)$ be a finite group and the cycle map $cl^{*,*}'$ injective. Then the coniveau spectral sequence collapses from the E_2 -term, namely,*

$$E_{\infty}^{*-*,*'} \cong E_2^{*-*,*'} \cong H^{*,*'}(X; \mathbb{Z}/p)/(\tau).$$

Thus the motivic filtration is the coniveau filtration (changing degree)

$$F_i(H_{et}^*(X; \mathbb{Z}/p)) = \oplus_{*-2c=i} N^c H_{et}^*(X; \mathbb{Z}/p).$$

Proof. Let $cl^{*,*}'$ be injective for each $(*, *')$. Then $Ker(\tau)|H^{*,*'}(X; \mathbb{Z}/p) = 0$. From the preceding lemma, we have the isomorphism for $E_2^{*,*}'$ -term in the theorem.

From the injectivity of cl also, we see

$$\oplus_s^{s=*} H^{*,s}(X; \mathbb{Z}/p)/(\tau) \cong H_{et}^*(X; \mathbb{Z}/p).$$

Suppose $d_r(x) \neq 0$ for some $x \in E_2^{*,*}'$ and $r \geq 2$. Then

$$\oplus_s rank_p E_{\infty}^{*-s,s} < \oplus_s rank_p E_2^{*-s,s} = rank_p H_{et}^*(X; \mathbb{Z}/p).$$

(Here note that ranks are finite since all cohomology are finite.)

On the other hand, $E_{\infty}^{*,*}' \cong \tilde{g}r H_{et}(x; \mathbb{Z}/p)$ where $\tilde{g}r$ means the graded algebra associated with the coniveau filtration. This is a contradiction. Thus $d_r(x) = 0$ for all x and $r \geq 2$. $\square$

For example, when $X = (B\mathbb{Z}/p)^n$, the motivic filtration is the coniveau one. We give here some geometric explanation as following.

Let $x \in N^c H_{et}^m(X; \mathbb{Z}/p)$. Take a closed subscheme K of X with $codim = c$ such that $i^* x = 0$ for $i^* : H_{et}^*(X; \mathbb{Z}/p) \rightarrow H_{et}^*(X - K; \mathbb{Z}/p)$. Consider the exact sequence

$$(*) \quad \rightarrow H_{et}^*(X/X - K; \mathbb{Z}/p) \xrightarrow{q^*} H_{et}^*(X; \mathbb{Z}/p) \xrightarrow{i^*} H_{et}^*(X - K; \mathbb{Z}/p) \rightarrow .$$

Here we assume that the embedding $i : K \subset X$ is regular. Since there is the Thom isomorphism

$$Th : H_{et}^{*-2c}(K; \mathbb{Z}/p) \cong H_{et}^*(X/X - K; \mathbb{Z}/p),$$

we can take $x' \in H_{et}^{*-2c}(K; \mathbb{Z}/p)$ such that $q^*(Th(x')) = x$. By $BL(n, p)$ -condition, we have $x'' \in H^{*-2c, *-2c}(K; \mathbb{Z}/p)$ in the motivic cohomology with $cl^{*,*'}(x'') = x'$. We consider the Thom map in the motivic cohomology

$$H^{*-2c, *-2c}(K; \mathbb{Z}/p) \xrightarrow{Th} H^{*, *-c}(Th(K); \mathbb{Z}/p) \cong H^{*, *-c}(X/X - K; \mathbb{Z}/p).$$

So we get the element $\tilde{x} = q^*Th(x'') \in H^{*,*-c}(X; \mathbb{Z}/p)$ with $cl^{*,*-c}(\tilde{x}) = x$. Our case $* = m$ and this means $x \in F_{m-2c}$.

Let $L_p^N \cong (\mathbb{C}^N - \{0\})/\mathbb{Z}/p$ the $2N$ -dimensional lens space so that $H^*(L_p^N; \mathbb{Z}/p) \cong (\mathbb{Z}/p[y] \otimes \Lambda(x))/(y^{N+1}, y^N x)$. Let us write by j the (regular) embedding

$$j : Z = L_p^{N-i_1} \times \dots \times L_l^{N-i_n} \subset X = (L_p^N)^{\times n}.$$

Recall the Gysin map j_* is defined by q^*Th in the exact sequence $(*)$. Since $j_*(1) = y_1^{i_1} \dots y_n^{i_n}$, we see that

$$\begin{aligned} x &= y_1^{i_1} \dots y_n^{i_n} x_{j_1} \dots x_{j_s} \in Im(j_*) \\ &\subset Ker(i^*) \subset N^{i_1+\dots+i_n} H^*((L_p^N)^{\times n}; \mathbb{Z}/p). \end{aligned}$$

This means that $x \in F_s^m$ implies $x \in N^{(m-s)/2}$. When $i_1 + \dots + i_n = c$, the above elements x make a $\mathbb{Z}/p$ -basis of N^c and the embeddings are regular. Thus we have $F_{m-2c} = N^c$ as desired.

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