# Uniruledness of some unitary Shimura varieties

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## Introduction

Today, I will talk about *uniruledness* of unitary Shimura varieties.

- Give some sufficient conditions for uniruledness of unitary Shimura varieties in terms of Hermitian lattices.
- Construct certain uniruled unitary Shimura varieties for $\mathrm{U}(1,n)$ $(n = 3, 4, 5,\ F = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2}))$.

We use modular forms constructed by Borcherds lifts and Gritsenko lifts.
An irreducible variety $X$ over $\mathbb{C}$ is called uniruled if there exists a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$ where $Y$ is an irreducible variety over $\mathbb{C}$ with $\dim Y = \dim X - 1$.

**Remark**

Uniruled varieties have Kodaira dimension $-\infty$. The converse is conjectured, but it is not known in dimension $> 3$.

## Unitary Shimura varieties

$F := \mathbb{Q}(\sqrt{d})$ $(d < 0)$,
$\langle\ ,\ \rangle \colon L \times L \to F$ : Hermitian lattice of sign $(1, n)$ over $\mathscr{O}_F$ $(n > 0)$.
$\mathrm{U}(L)$ : unitary group of $(L, \langle\ ,\ \rangle)$

$$\begin{aligned} D_L &:= \{w \in \mathbb{P}(L \otimes_{\mathscr{O}_F} \mathbb{C}) \mid \langle w, w\rangle > 0\} \\ &\cong \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 < 1\} \end{aligned}$$

: Hermitian symmetric domain associated with $\mathrm{U}(L)(\mathbb{R}) \cong \mathrm{U}(1, n)$.
For a finite index subgroup $\Gamma \subset \mathrm{U}(L)$, we define

$$\mathscr{F}_L(\Gamma) := \Gamma \backslash D_L \text{ (unitary Shimura variety)}.$$

This is a quasi-projective variety of dimension $n$ over $\mathbb{C}$.



## Unitary Shimura varieties

For a quadratic lattice $M$ over $\mathbb{Z}$, we define

$$\mathscr{D}_M := \{w \in \mathbb{P}(M \otimes_{\mathbb{Z}} \mathbb{C}) \mid (w, w) = 0,\ (w, \overline{w}) > 0\}^+.$$

We say a quadratic lattice $M$ over $\mathbb{Z}$ is 2-elementary if

$$M^\vee / M \cong (\mathbb{Z}/2\mathbb{Z})^{\ell(M)}.$$

$$\delta(M) := \begin{cases} 0 & ((v, v) \in \mathbb{Z} \text{ for any } v \in M^\vee) \\ 1 & ((v, v) \notin \mathbb{Z} \text{ for some } v \in M^\vee). \end{cases}$$

$(L, \langle\ ,\ \rangle)$ : Hermitian lattice of sign $(1, n)$ over $\mathscr{O}_F$
$(L_Q, (\ ,\ ))$ : quadratic lattice of signature $(2, 2n)$ associated with $L$ over $\mathbb{Z}$
Here $L_Q$ is $L$ considered as a free $\mathbb{Z}$-module and $(\ ,\ ) := \mathrm{Tr}_{F/\mathbb{Q}}\langle\ ,\ \rangle$.
We have an embedding

$$\mathrm{U}(L)(\mathbb{R}) \cong \mathrm{U}(1, n) \hookrightarrow \mathrm{O}^+(L_Q)(\mathbb{R}) \cong \mathrm{O}^+(2, 2n)$$
$$\iota \colon D_L \hookrightarrow \mathscr{D}_{L_Q}.$$

## Main results

**Theorem ([M, arXiv:2008.13106])**

Let $(L, \langle\ ,\ \rangle)$ be a Hermitian lattice over $\mathscr{O}_F$ of signature $(1,5)$ and let $(L_Q, (\ ,\ ))$ be the associated quadratic lattice over $\mathbb{Z}$ of signature $(2,10)$. Assume that

1. $L_Q$ is even 2-elementary, $\delta(L_Q) = 0$ and $\ell(L_Q) \leqslant 8$. Moreover, $\ell(L_Q) \leqslant 6$ if $F = \mathbb{Q}(\sqrt{-3})$.
2. $2\langle \ell, r\rangle \in \mathscr{O}_F$ for any $\ell, r \in L$ with $\langle r, r\rangle = -1$.

Then $\mathscr{F}_L(\mathrm{U}(L)(\mathbb{Z}))$ is uniruled.

**Remark**

- To prove this Theorem, we use reflective modular forms constructed by Yoshikawa.
- Using reflective modular forms constructed by Gritsenko-Hulek, we can give 3 more sufficient conditions for uniruledness in terms of Hermitian lattices.



## Main results

**Theorem (Uniruledness [M2, arXiv:2008.13106])**

1. For $F = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-2})$, there exist Hermitian lattices $L$ over $\mathscr{O}_F$ of signature $(1,5)$ such that $\mathscr{F}_L(\mathrm{U}(L)(\mathbb{Z}))$ are uniruled.
2. For $F = \mathbb{Q}(\sqrt{-1})$, there exist Hermitian lattices $L$ over $\mathscr{O}_F$ of signature $(1,4)$ such that $\mathscr{F}_L(\mathrm{U}(L)(\mathbb{Z}))$ are uniruled.
3. For $F = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-2})$, there exist Hermitian lattices $L$ over $\mathscr{O}_F$ of signature $(1,3)$ such that $\mathscr{F}_L(\mathrm{U}(L)(\mathbb{Z}))$ are uniruled.

**Remark**

Gritsenko-Hulek (2014) proved certain orthogonal Shimura varieties are uniruled.

## Uniruled unitary Shimura varieties

### Example

$F = \mathbb{Q}(\sqrt{-2})$

$L_{\mathbb{U}\oplus\mathbb{U}(2)}$ : Hermitian lattice of sign (1,1) defined by $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$.

$L_{\mathbb{D}_4}$ : Hermitian lattice of sign (0,2) defined by $\begin{pmatrix} -1 & -\frac{1+\sqrt{-2}}{2} \\ -\frac{1-\sqrt{-2}}{2} & -1 \end{pmatrix}$.

$(L_{\mathbb{U}\oplus\mathbb{U}(2)} \oplus L_{\mathbb{D}_4} \oplus L_{\mathbb{D}_4})_Q \cong \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{D}_4(-1) \oplus \mathbb{D}_4(-1)$.
Then $\mathscr{F}_L(\mathrm{U}(L)(\mathbb{Z}))$ is uniruled for

$$L := L_{\mathbb{U}\oplus\mathbb{U}(2)} \oplus L_{\mathbb{D}_4} \oplus L_{\mathbb{D}_4}.$$



## Uniruled unitary Shimura varieties

A modular form $F_k \in M_k(\Gamma, \chi)$ on $D_L$ is called reflective if $\mathrm{Supp}(\mathrm{div}(F_k))$ is contained in the ramification divisors of $D_L \to \mathscr{F}_L(\Gamma)$. A reflective modular form $F_k$ is called strongly reflective if the multiplicity of each irreducible component of $\mathrm{div}(F_k)$ is 1.
For modular forms on $\mathscr{D}_{L_Q}$, we define the notions similarly.

### Theorem (Uniruledness criterion [GH])

Let $n > 1$. Let $a, k > 0$ be positive integers satisfying $k > an$. If there exists a non-zero reflective modular form $F_{a,k} \in M_k(\Gamma, \chi)$ of weight $k$ for which the multiplicity of every irreducible component of $\mathrm{div}(F_{a,k})$ is less than or equal to $a$, then $\mathscr{F}_L(\Gamma)$ is uniruled.

**Proof.**

Use the numerical criterion of uniruledness due to Miyaoka and Mori. □

## Uniruled unitary Shimura varieties

Reflective modular forms are very rare. In some special cases, we can construct reflective modular forms by Borcherds lifts and Gritsenko lifts.

**Theorem (Yoshikawa (2013))**

Let $M$ be an even 2-elementary quadratic lattice over $\mathbb{Z}$ of signature $(2,10)$ and $\delta(M)=0$. There exists a strongly reflective modular form $\Psi_M$ of weight $2^{(16-\ell(M))/2}-4$ on $\mathscr{D}_M$ for $\mathrm{O}^+(M)$.

**Theorem ([M, arXiv:2008.13106])**

Let $(L,\langle\ ,\ \rangle)$ be a Hermitian lattice over $\mathscr{O}_F$ of signature $(1,5)$ and let $(L_Q,(\ ,\ ))$ be the associated quadratic lattice over $\mathbb{Z}$ of signature $(2,10)$. Assume that

1. $L_Q$ is even 2-elementary, $\delta(L_Q)=0$ and $\ell(L_Q)\leqslant 8$. Moreover, $\ell(L_Q)\leqslant 6$ if $F=\mathbb{Q}(\sqrt{-3})$.
2. $2\langle \ell, r\rangle \in \mathscr{O}_F$ for any $\ell, r\in L$ with $\langle r,r\rangle=-1$.

Then $\mathscr{F}_L(\mathrm{U}(L)(\mathbb{Z}))$ is uniruled.



## Uniruled unitary Shimura varieties

| Quadratic lattices of sign (2,10) | $\ell(L_Q)$ | $\delta(L_Q)$ | $F$ |
|---|---|---|---|
| $\mathbb{U}\oplus\mathbb{U}(2)\oplus\mathbb{E}_8(-2)$ | 10 | 0 | $\mathbb{Q}(\sqrt{-1})$ |
| $\mathbb{U}\oplus\mathbb{U}\oplus\mathbb{E}_8(-2)$ | 8 | 0 | $\mathbb{Q}(\sqrt{-1})$ |
| $\mathbb{U}\oplus\mathbb{U}(2)\oplus\mathbb{D}_4(-1)\oplus\mathbb{D}_4(-1)$ | 6 | 0 | $\mathbb{Q}(\sqrt{-2})$ |
| $\mathbb{U}\oplus\mathbb{U}\oplus\mathbb{D}_4(-1)\oplus\mathbb{D}_4(-1)$ | 4 | 0 | $\mathbb{Q}(\sqrt{-1})$ |
| $\mathbb{U}\oplus\mathbb{U}\oplus\mathbb{D}_8(-1)$ | 2 | 0 | $\mathbb{Q}(\sqrt{-1})$ |
| $\mathbb{U}\oplus\mathbb{U}\oplus\mathbb{E}_8(-1)$ | 0 | 0 | $\mathbb{Q}(\sqrt{-1})$ |

# Summary

- Give some sufficient conditions for uniruledness of unitary Shimura varieties in terms of Hermitian lattices.
- Construct certain uniruled unitary Shimura varieties for $\mathrm{U}(1, n)$ $(n = 3, 4, 5,\ \ F = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2}))$.

To construct reflective modular forms on $D_L$, we need $F = \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$.

**Problem.**

- Unitary Shimura varieties having non-negative Kodaira dimension
- The Kodaira dimension of unitary Shimura varieties over $F \neq \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$