

ON PARTIALLY CONFORMAL QC DEFORMATIONS

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1. Let $M(R)$ be the Banach space of all Beltrami differentials $\mu = \mu(z) \frac{d\bar{z}}{dz}$ on a Riemann surface R with norm $\|\mu\|_\infty := \text{ess sup } |\mu(z)|$. We denote by $M(R)_1$ the open unit ball of $M(R)$. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. For each $\mu \in M(\mathbb{D})_1$, there is a unique normalized quasiconformal self-mapping W^μ of $\mathbb{D}$ whose Beltrami coefficient $\mu(W^\mu) := W^\mu_{\bar{z}}/W^\mu_z$ is μ , that is, $W^\mu: \mathbb{D} \rightarrow \mathbb{D}$ is a homeomorphism whose generalized derivatives satisfy the Beltrami equation $f_{\bar{z}} = \mu f_z$, and its continuous extension to the closed unit disk $\bar{\mathbb{D}}$ fixes 1, i and -1 . Two elements μ and ν in $M(\mathbb{D})_1$ are said to be *equivalent* if W^μ and W^ν have the same boundary values. Let R be a hyperbolic Riemann surface and $\pi: \mathbb{D} \rightarrow R$ be a universal covering mapping. We define $\mu, \nu \in M(R)_1$ are equivalent when so are their pull-backs $\pi^*\mu$ and $\pi^*\nu$, and quasiconformal mappings $f: R \rightarrow f(R)$ and $g: R \rightarrow g(R)$ are equivalent if so are their Beltrami coefficients $\mu(f)$ and $\mu(g)$. It is known that f and g are equivalent if and only if there is a conformal mapping $h: f(R) \rightarrow g(R)$ such that $h \circ f$ is homotopic to g modulo the border of R . The Teichmüller space $T(R)$ of R is the quotient space of $M(R)_1$ with respect to this equivalence relation. We denote by $[\mu]$ the equivalence class containing μ , and identify it with the marked Riemann surface $[f(R), f]$, $\mu(f) = \mu$.

Let V be a measurable subset of R and set

$$M(V)_1 := \{\mu \in M(R)_1 : \mu|_{R \setminus V} = 0\}.$$

A quasiconformal mapping f is ‘conformal’ outside V if $\mu(f) \in M(V)_1$, so we say $[f(R), f]$ is a partially conformal qc deformation of $[R, \text{id}_R]$. A family of partially conformal qc mappings is useful to investigate Teichmüller spaces and extremal problems on them (see for example Krushkal [5], Gardiner [2], [3], Reich [10] and Fehlmann-Sakan [1]).

2. We summarize some known facts. First of all, in general, $[M(V)_1] \neq T(R)$ (cf. Savin [11]). For example, if $R \setminus V$ is an incompressible annular domain, then $[M(V)_1] \neq T(R)$. But if $R \setminus V$ is a topological disk, then $[M(V)_1] = T(R)$.

If R is of finite conformal type, that is, R is a Riemann surface obtained by removing a finite number of punctures from a compact one, then $[M(V)_k]$ is a

neighborhood of the origin $[0]$ of $T(R)$ for any V with positive measure and any $0 < k \leq 1$. This is a classical result. While there are R of infinite conformal type and a subset V of R with positive measure such that $[M(V)_1]$ is not a neighborhood of $[0]$ (Oikawa [9]).

A general necessary condition for V to insure that $[M(V)_1]$ becomes a neighborhood of $[0]$ is

$$(1) \quad r(V) := \inf \left\{ \iint_V |\phi| \, dxdy : \phi \in A_2^1(R), \|\phi\|_1 = 1 \right\} > 0.$$

Moreover, when $R = \mathbb{D}$, the condition (1) is equivalent to a simple geometric one:

$$\inf \{ \text{Area}(V \cap \Delta(z; \rho)) : z \in \mathbb{D} \} > 0 \quad \text{for some } \rho > 0$$

where $\Delta(z; \rho)$ is the hyperbolic disk with center z and radius ρ , and Area means its hyperbolic area (Ohtake [7]).

On the other hand, a known sufficient condition is as follows. Set

$$\omega(z) := \sup \{ \lambda(z)^{-2} |\phi(z)| : \phi \in A_2^1(R), \|\phi\|_1 = 1 \}.$$

It is not difficult to see that the function ω on R is continuous and vanishing at the punctures of R . If V has positive measure and if

$$\iint_V \max\{\omega(z)^2, 1\} \, dxdy < \infty,$$

then $[M(V)_k]$ contains $[0]$ in its interior for any $0 < k \leq 1$ (Ohtake [6]).

3. We give here a quantitative version of the necessary condition (1) above.

Theorem 1. *If $[M(R)_k] \subset [M(V)_{k'}]$, then we have*

$$(2) \quad r(V) \geq \frac{K-1}{K'-1}.$$

where $K := (1-k)/(1+k)$, $K' := (1-k')/(1+k')$.

Proof. Take arbitrary $0 < t < k$ and $\phi \in A_2^1(R)$ with $\|\phi\|_1 = 1$. Let $f_0: R \rightarrow R_0$ be a quasiconformal mapping whose Beltrami coefficient is $t\bar{\phi}/|\phi|$ and $\psi \in A_2^1(R_0)$ be the terminal differential of the Teichmüller mapping f_0 (cf. Lehto [4]). Then $f_0^{-1}: R_0 \rightarrow R$ is a Teichmüller mapping with $\mu(f_0^{-1}) = -k\bar{\psi}/|\psi|$. By assumption, there is a quasiconformal mapping $f: R \rightarrow R_0$ which is equivalent to f_0 and whose Beltrami coefficient $\mu(f)$ is in $M(V)_{k'}$. Applying Reich-Strebel inequality (Strebel [12], [13]) to $-\psi$ and $f \circ f_0^{-1}$ equivalent to the identity mapping of R_0 , we have

$$\|\psi\|_1 \leq \iint_{R_0} |\psi| \frac{|1 + \mu(f \circ f_0^{-1})\psi/|\psi||^2}{1 - |\mu(f \circ f_0^{-1})|^2} \, dudv.$$

Since

$$\begin{aligned} K(f_0)|\phi(z)| dx dy &= |\psi(w)| du dv, \quad w = f_0(z) \\ \frac{\bar{\psi}(w)}{|\psi(w)|} &= \frac{p(z)}{\bar{p}(z)} \cdot \frac{\bar{\phi}(z)}{|\phi(z)|}, \quad p = (f_0)_{\bar{z}} \\ \mu(f \circ f_0^{-1})(w) &= \frac{\mu(f)(z) - \mu(f_0)(z)}{1 - \bar{\mu}(f_0)(z)\mu(f)(z)} \cdot \frac{p(z)}{\bar{p}(z)}, \end{aligned}$$

change of variable gives us

$$\begin{aligned} K(f_0) &\leq K(f_0) \iint_R \frac{\left|1 - \mu(f_0) \frac{\phi}{|\phi|}\right|^2 \left|1 + \mu(f) \frac{\phi}{|\phi|} \cdot \frac{1 - \bar{\mu}(f_0) \bar{\phi}/|\phi|}{1 - \mu(f_0) \phi/|\phi|}\right|^2}{(1 - |\mu(f_0)|^2)(1 - |\mu(f)|^2)} |\phi| dx dy \\ &= \iint_R \frac{\left|1 + \mu(f) \phi/|\phi|\right|^2}{1 - |\mu(f)|^2} |\phi| dx dy \\ &\leq K' \iint_V |\phi| dx dy + \iint_{R \setminus V} |\phi| dx dy \\ &= (K' - 1) \iint_V |\phi| dx dy + 1. \end{aligned}$$

Letting $t \rightarrow k$, we have a desired inequality (2). $\square$

We can show a partial converse of Theorem 1. Its proof and the details are omitted and will appear elsewhere.

Theorem 2. For $A > 0$ and $l > 0$, there are positive constants C and $t_0 \leq 1$ such that if a Riemann surface R has hyperbolic area less than A and if the length of each closed geodesic of R is not shorter than l , then

$$[M(R)_t] \subset [M(V)_{Ct/r(V)^2}] \quad \text{for any } 0 \leq t \leq t_0.$$

where the constants C and t_0 depend only on A , l and $r(V)$ but not on R nor V .

REFERENCES

1. R. Fehlmann and K. Sakan, *On extremal quasiconformal mappings with varying dilatation bounds*, Osaka J. Math. **23** (1986), 751–764.
2. F. P. Gardiner, *On partially Teichmüller Beltrami differentials*, Michigan Math. J. **29** (1982), 237–242.
3. F. P. Gardiner, *Schiffer's interior variation and quasiconformal mapping*, Duke Math. J. **42** (1975), 371–380.
4. O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, 1987.
5. S. L. Krushkal, *Quasiconformal Mappings and Riemann Surfaces*, Winston and Sons, 1979.
6. H. Ohtake, *On the deformation of Fuchsian groups by quasiconformal mappings with partially vanishing Beltrami coefficients*, J. Math. Kyoto Univ. **29** (1989), 69–90.

7. H. Ohtake, *Partially conformal qc mappings and the universal Teichmüller space*, J. Math. Kyoto Univ. **31** (1991), 171–180.
8. H. Ohtake, *Extremal Quasiconformal Mappings and Teichmüller Spaces (Japanese)*, informal notes, 1992.
9. K. Oikawa, *A problem on the deformation and extension of Riemann surfaces*, Hokkaido Math., Special issue **10** (1981), 546–554.
10. E. Reich, *Quasiconformal mappings with prescribed boundary values and dilatation bound*, Arch. Rat. Mech. Anal. **68** (1978), 99–112.
11. V. V. Savin, *On moduli of Riemann surfaces*, Soviet Math. Dokl. **12** (1971), 267–270.
12. K. Strebel, *On the trajectory structure of quadratic differentials*, Ann. of Math. Studies **79** (1974), 419–438.
13. K. Strebel, *On quasiconformal mappings of Riemann surfaces*, Comment. Math. Helv. **53** (1978), 301–321.

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