

On Floating Body Problem

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This paper consists of two parts. In Sections 1, 2 and 3, I give my recent work [8] of 2-dimensional floating bodies. In Section 4, I give two important formulas of 3-dimensional floating bodies. These formulas are already known, maybe, but I do not know where the statements and the proofs are given. In Section 5, we apply the result of Section 4 to Ulam's problem.

1 Ulam's Floating Body Problem of Two Dimension

S. M. Ulam posed a problem: *If a body of uniform density floats in water in equilibrium in every direction, must it be a sphere?* See [3] or [9] for detail. The problem is still open. However, in two dimensional case of the problem, Auerbach [1] gives a counter-example.

Theorem 1. ([1]) *There is a non-circular figure $D \subset \mathbb{R}^2$ of density $\rho = 1/2$ which floats in equilibrium in every direction.*

Before we state our result, we define some terminology of two-dimensional floating bodies. Consider a figure $D \subset \mathbb{R}^2$ whose perimeter ∂D is a simple closed curve, and take a number $0 < \rho < 1$. For a given angle $0 \leq \theta < 2\pi$, there is a directed line L_θ of slope angle θ which divides the area of D in the ratio $\rho : 1 - \rho$. In this paper, we assume the following three conditions:

(C1) ∂D is of class C^1 .

(C2) L_θ meets ∂D at exactly two points, say, P and Q .

(C3) Neither the tangent at P nor at Q is not parallel to the line PQ .

We call ρ the *density* of D , and the segment PQ the *water line* of slope angle θ . We denote by D_u and D_a the divided figures of area ratio $\rho : 1 - \rho$. We call D_u and D_a the *underwater* and *abovewater* parts of D , respectively. We denote by G_u and G_a the centroids of D_u and D_a , respectively. We say that D *floats in equilibrium* in direction $e_2(\theta) = (-\sin \theta, \cos \theta)$ if the line $G_u G_a$ is parallel to $e_2(\theta)$.

If the figure D of density ρ floats in equilibrium in every direction, we call $D \subset \mathbb{R}^2$ an *Auerbach figure* of an *Auerbach density* ρ . It is known that, if $D \subset \mathbb{R}^2$ is an Auerbach figure, then the water surface divides ∂D in constant ratio, say, $\sigma : 1 - \sigma$. See (ii) of Corollary 7. We call σ the *perimetral density* of the Auerbach figure D .

If D is an Auerbach figure of density $\rho = 1/2$, then the water lines L_θ and $L_{\theta+\pi}$ are the same but opposite directed lines. Thus it is of perimetral density $\sigma = 1/2$. In the proof of Theorem 1, the condition $\rho = 1/2$ is essential. It is difficult to make an Auerbach figures of density $\rho \neq 1/2$. So a question arises: *Is there a non-circular Auerbach figure of density $\rho \neq 1/2$?*

Recently, Wegner [10] gave an positive answer to this question. Wegner's examples exhibit more interesting fact. That is, for given integer $p \geq 3$, one of his examples has $(p - 2)$ different Auerbach densities. So one Auerbach figure can have many perimetral densities.

On the other hand, Bracho, Montejano and Oliberos [2] gave a following result.

Theorem 2. ([2]) *If there is an Auerbach figure $D \subset \mathbb{R}^2$ of perimetral density $\sigma = 1/3$ or $1/4$, then it is a circle.*

The purpose of the first part of this paper is to prove the following theorem.

Theorem 3. (1) If an Auerbach figure $D \subset \mathbb{R}^2$ has three perimetral densities σ_1, σ_2 and σ_3 , and if $\sigma_1 + \sigma_2 + \sigma_3 = 1$, then it is a circle. (These σ_i 's are not necessarily different.)

(2) If an Auerbach figure $D \subset \mathbb{R}^2$ has four perimetral densities $\sigma_1, \sigma_2, \sigma_3$ and σ_4 , and if $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 1$, then it is a circle. (These σ_i 's are not necessarily different.)

The above theorem is a generalization of Theorem 2. Certainly, putting $\sigma_1 = \sigma_2 = \sigma_3 = 1/3$ gives the 1/3 case of Theorem 2, and putting $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1/4$ gives the 1/4 case of Theorem 2.

2 Auerbach Figures

In this section, we give a short survey of Auerbach figures.

Theorem 4. ([1], [10]) If a figure $D \subset \mathbb{R}^2$ is Auerbach, then the water line is of constant length.

We give a proof of the above theorem in Section 5.

Theorem 5. If a figure $D \subset \mathbb{R}^2$ is Auerbach, and if PQ is the water line of slope angle θ , then there is a 2π -periodic function f of class C^2 such that the position vectors of P and Q are given by

$$\mathbf{p}(\theta) = -f(\theta)\mathbf{e}_2(\theta) + (f'(\theta) - l)\mathbf{e}_1(\theta), \quad \mathbf{q}(\theta) = -f(\theta)\mathbf{e}_2(\theta) + (f'(\theta) + l)\mathbf{e}_1(\theta), \quad (1)$$

where $\mathbf{e}_1(\theta) = (\cos \theta, \sin \theta)$, $\mathbf{e}_2(\theta) = (-\sin \theta, \cos \theta)$, and l is half the length of PQ .

Proof. Assume that D is an Auerbach figure. Then by Theorem 4, the waterline is of constant length. Since $\{\mathbf{e}_1(\theta), \mathbf{e}_2(\theta)\}$ is a basis of $\mathbb{R}^2$, we can represent the position vectors of the points P and Q as follows:

$$\mathbf{p}(\theta) = -f(\theta)\mathbf{e}_2(\theta) + g(\theta)\mathbf{e}_1(\theta), \quad \mathbf{q}(\theta) = -f(\theta)\mathbf{e}_2(\theta) + (g(\theta) + 2l)\mathbf{e}_1(\theta). \quad (2)$$

Suppose that the chord P^*Q^* of C is the water line of slope angle $\theta + h$. Then the position vector of the intersection H of the chords PQ and P^*Q^* are given by

$$\overrightarrow{OH} = -f(\theta)\mathbf{e}_2(\theta) + \lambda\mathbf{e}_1(\theta) = -f(\theta + h)\mathbf{e}_2(\theta + h) + \mu\mathbf{e}_1(\theta + h). \quad (3)$$

By taking the inner product of (3) and $\mathbf{e}_2(\theta + h)$, we have that $f(\theta + h) = \lambda \sin h + f(\theta) \cos h$. Thus we obtain that

$$f'(\theta) = \frac{f(\theta + h) - f(\theta)}{h} + o(1) = \lambda \frac{\sin h}{h} - f(\theta) \frac{1 - \cos h}{h} + o(1) = \lambda + o(1). \quad (4)$$

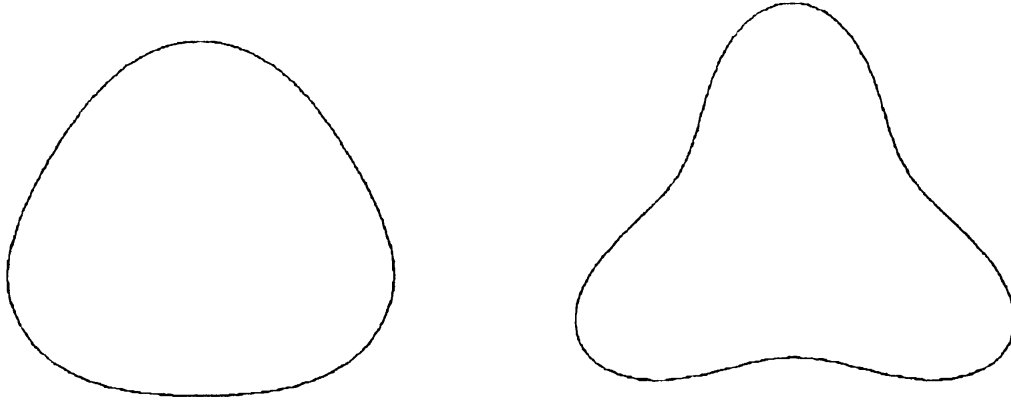
We can evaluate the areas of the sectors HPP^* and HQQ^* by

$$\begin{aligned} \frac{1}{2}HP^2 h + o(h) &= \frac{1}{2}|g(\theta) - f'(\theta)|^2 h + o(h) \quad \text{and} \\ \frac{1}{2}HQ^2 h + o(h) &= \frac{1}{2}|g(\theta) - f'(\theta) + 2l|^2 h + o(h), \end{aligned} \quad (5)$$

respectively. Since these two areas are equal, we obtain that $g(\theta) = f'(\theta) - l$. Hence we have proved (1). By taking the inner product of (1) and $\mathbf{e}_1(\theta)$, we have that $f'(\theta) = \mathbf{p}(\theta) \cdot \mathbf{e}_1(\theta) + l$. Thus the function $f(\theta)$ is of class C^2 . $\square$

The following result is a "proof" of Theorem 1.

Example 6. Put $f(\theta) = -k \cos 3\theta$ in Equation (1). Then the curve is rotational symmetric with respect to the angle $2\pi/3$. So it surrounds an Auerbach figure of density 1/2. See Theorem 11. The figures of $k/l = 0.03$ and $k/l = 0.1$ are drawn as follows:



The following result gives geometric properties of Auerbach figures.

Corollary 7. *If a figure $D \subset \mathbb{R}^2$ is Auerbach, and if PQ is the water line of slope angle θ , then:*

- (i) *The vectors $\mathbf{p}'(\theta)$ and $\mathbf{q}'(\theta)$ are symmetric with respect to the line PQ .*
- (ii) *The arc PQ of ∂D is of constant length.*

Proof. By differentiating (1), we have that

$$\mathbf{p}'(\theta) = s(\theta)\mathbf{e}_1(\theta) - l\mathbf{e}_2(\theta), \quad \mathbf{q}'(\theta) = s(\theta)\mathbf{e}_1(\theta) + l\mathbf{e}_2(\theta), \quad (6)$$

where $s(\theta) = f(\theta) + f''(\theta)$. Since the line PQ is parallel to the vector $\mathbf{e}_1(\theta)$, we have proved (i).

(ii) By (6), we have that $|\mathbf{p}'(\theta)| = |\mathbf{q}'(\theta)| = \sqrt{s(\theta)^2 + l^2}$. This implies that the points P and Q move at the same speed along ∂D . Thus we have proved (ii). $\square$

Remark. By integrating (6), we have that

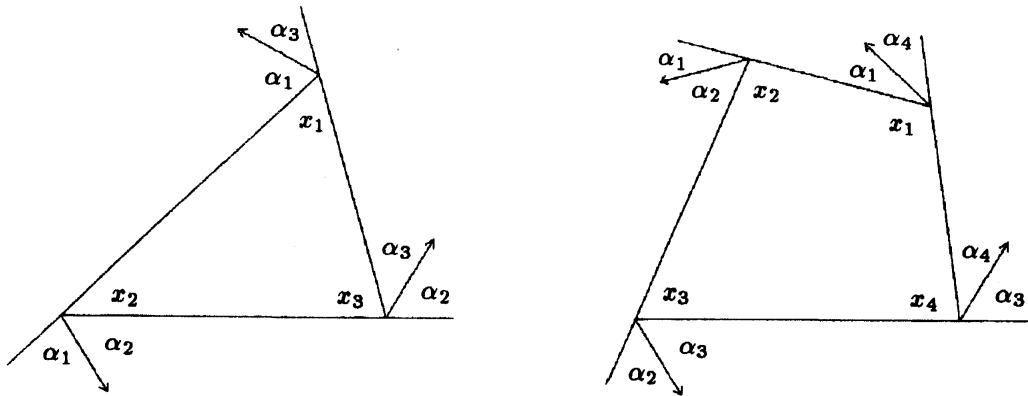
$$\mathbf{p}(\theta) = \mathbf{c} + \int_0^\theta s(\phi)\mathbf{e}_1(\phi) d\phi - l\mathbf{e}_1(\theta), \quad \mathbf{q}(\theta) = \mathbf{c} + \int_0^\theta s(\phi)\mathbf{e}_1(\phi) d\phi + l\mathbf{e}_1(\theta), \quad (7)$$

where $\mathbf{c}$ is a constant vector. These formulas are same as those given in Section 2 of [10].

3 Proof of Theorem 3

Proof of Theorem 3. (i) Let P_1, P_2 and P_3 be three points of ∂D such that for each $i = 1, 2, 3$, the line $P_i P_{i+1}$ can be a water surface of perimetral density σ_i . (The indices are taken cyclic in modulo 3.) For each $i = 1, 2, 3$, we denote by $\mathbf{p}_i(\theta)$ the position vector of P_i , by x_i the angle $\angle P_{i-1} P_i P_{i+1}$ and by α_i the angle between $\mathbf{p}'_i(\theta)$ and $P_i P_{i+1}$. By (i) of Corollary 7, the angle between $P_{i-1} P_i$ and $\mathbf{p}'_i(\theta)$ is equal to α_i . So we obtain that $x_1 + \alpha_3 + \alpha_1 = \pi$, $x_2 + \alpha_1 + \alpha_2 = \pi$ and $x_3 + \alpha_2 + \alpha_3 = \pi$. Since $x_1 + x_2 + x_3 = \pi$, we have that $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. See the figure below left. So we obtain that $\alpha_1 = \alpha_3$. By the converse of Alternate Segment Theorem, $\mathbf{p}'_1(\theta)$ tangents to the circumcircle of the triangle $P_1 P_2 P_3$. Thus P_1 varies on the circumcircle. Hence D is a circle.

(ii) Let P_1, P_2, P_3 and P_4 be four points of ∂D such that for each $i = 1, 2, 3, 4$, the line $P_i P_{i+1}$ can be a water surface of perimetral density σ_i . (The indices are taken cyclic in modulo 4.) By the same notation and argument used in (i) of this theorem, we obtain that $x_1 + \alpha_4 + \alpha_1 = \pi$, $x_2 + \alpha_1 + \alpha_2 = \pi$, $x_3 + \alpha_2 + \alpha_3 = \pi$ and $x_4 + \alpha_3 + \alpha_4 = \pi$. See the figure below right. Since $x_1 + x_2 + x_3 + x_4 = 2\pi$, we have that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \pi$. So we obtain that $x_1 + x_3 = \pi$. By the converse of Inscribed Quadrangle Theorem, the quadrangle $P_1 P_2 P_3 P_4$ inscribes to a circle. Thus $P_3 P_1$ is of constant length, and therefore, it can be a water line of perimetral density $\sigma_3 + \sigma_4$. Hence, by (i) of this theorem, D is a circle. $\square$



4 On formulas for 3-dimensional floating bodies

Suppose that a solid $\mathcal{F} \subset \mathbb{R}^3$ has a volume V and a uniform density ρ ($0 < \rho < 1$) and that it floats on water so that a unit vector $\mathbf{n} \in \mathbb{R}^3$ is the vertical and upward direction. The *water surface* is a plane which is orthogonal to $\mathbf{n}$ and cuts $\mathcal{F}$ in ratio $1 - \rho : \rho$. By Archimedes Principle, the volume of the undersurface part of $\mathcal{F}$ is equal to ρV . In the paper, we assume that the boundary of $\mathcal{F}$ is sufficiently differentiable and do not tangent to the water surface. From now, we consider that $\mathcal{F}$ is fixed and $\mathbf{n}$ varies, that is, we consider that $\mathcal{F}$ is not inclined but the water surface is inclined. We call $\mathbf{n}$ the *vertical vector* and orthogonal vectors to $\mathbf{n}$ *horizontal vectors*.

We can solve the stability problem of floating bodies by analyzing the relation between the positions of its center of gravity and its center of buoyancy. The *center of buoyancy* is the center of gravity of the underwater part of the floating body. The center of gravity G does not depend on $\mathbf{n}$, but the center of buoyancy B is a function of $\mathbf{n}$.

The gravity acts vertically downward on G and the buoyancy acts vertically upward on B . If G and B do not lie on same vertical line, then the floating body rotates. We say that $\mathcal{F}$ is in *equilibrium state* with respect to the direction $\mathbf{n}$ if G and B lie on the same vertical line, that is, $(\mathbf{B} - \mathbf{G}) \cdot \mathbf{u} = 0$ for every horizontal vector $\mathbf{u}$.

Suppose that the floating body $\mathcal{F}$ is in equilibrium state with respect to $\mathbf{n}_0$. Then fix a horizontal vector $\mathbf{t}$ and consider the vector $\mathbf{u}_0$ which is made by the vector product $\mathbf{u}_0 = \mathbf{n}_0 \times \mathbf{t}$. Rotate the vectors $\mathbf{u}_0$ and $\mathbf{n}_0$ counterclockwise with respect to $\mathbf{t}$, that is,

$$\mathbf{u} = \mathbf{u}_0 \cos \theta + \mathbf{n}_0 \sin \theta, \quad \mathbf{n} = -\mathbf{u}_0 \sin \theta + \mathbf{n}_0 \cos \theta. \quad (8)$$

It means that the water line inclines counterclockwise by the angle θ , that is, the floating body inclines clockwise by the angle θ . We call $\mathbf{t}$ the *vector of rotation axis*.

Assume that the floating body inclines by a sufficiently small angle θ . If the function $F(\theta) = (\mathbf{B} - \mathbf{G}) \cdot \mathbf{u}$ is monotone increasing, then the gravity and buoyancy act as the floating body returns to the original state. If the function $F(\theta)$ is monotone decreasing, then the gravity and buoyancy act as the floating body inclines more. So we define as follows:

Definition. We say that a floating body $\mathcal{F} \subset \mathbb{R}^3$ in equilibrium is *stable* (*unstable*) with respect to a rotation axis $\mathbf{t}$ if $F(\theta)$ is monotone increasing (decreasing). We say that $\mathcal{F}$ is *stable* if it is stable with respect to every rotation axis $\mathbf{t}$.

We denote by E the cross section of $\mathcal{F}$ cut by the water surface. We make a coordinate plane whose origin is the center of gravity of E and x - and z -axis have the same direction as $\mathbf{u}$ and $\mathbf{t}$, respectively. Then we can regard E as the figure in the xz -plane. Consider the following quantity:

$$I(\mathbf{n}, \mathbf{t}) = \iint_E x^2 dx dz. \quad (9)$$

We call it the *moment of inertia* with respect to $\mathbf{n}$ and $\mathbf{t}$. From now, we fix vectors $\mathbf{n}_0$ and $\mathbf{t}$, and consider the moment of inertia as a function of θ , that is, we denote $I(\theta) = I(\mathbf{n}, \mathbf{t})$.

Suppose that a floating body $\mathcal{F}$ is in equilibrium with respect to the direction $\mathbf{n}_0$. Then the following formula holds:

$$F(\theta) = \int_0^\theta \left(\frac{I(\phi)}{\rho V} - \overline{GB}_0 \right) \cos(\theta - \phi) d\phi, \quad (\text{F1})$$

where B_0 is the center of buoyancy of $\mathcal{F}$ with respect to the direction $\mathbf{n}_0$. Remark that $\overline{GB}_0$ is a distance between G and B_0 , and so, a constant number. We can approximate (F1) as follows:

$$F(\theta) = \left(\frac{I(0)}{\rho V} - \overline{GB}_0 \right) \theta + O(\theta^2). \quad (10)$$

By using (10), we deduce the following corollary:

Corollary 8. *If $I(0) > \rho V \overline{GB}_0$, then $\mathcal{F}$ is stable with respect to the axis $\mathbf{t}$. If $I(0) < \rho V \overline{GB}_0$, then $\mathcal{F}$ is unstable with respect to the axis $\mathbf{t}$.*

Set $U(\theta) = (\mathbf{G} - \mathbf{B}) \cdot \mathbf{n}$. We call it the *potential function* of the floating body $\mathcal{F}$. Suppose that a floating body $\mathcal{F}$ is in equilibrium with respect to the direction $\mathbf{n}_0$. Then the following formula holds:

$$U(\theta) = \int_0^\theta \left(\frac{I(\phi)}{\rho V} - \overline{GB}_0 \right) \sin(\theta - \phi) d\phi \quad (\text{F2})$$

By (F1) and (F2), we find that

$$U'(\theta) = F(\theta), \quad F'(\theta) = -U(\theta) + \frac{I(\theta)}{\rho V} - \overline{GB}_0. \quad (11)$$

By the first formula of (11), we obtain the following corollary:

Corollary 9. *The floating body $\mathcal{F}$ is stable with respect to the axis $\mathbf{t}$ if and only if the function $U(\theta)$ takes a local minimum at $\theta = 0$. The floating body $\mathcal{F}$ is unstable with respect to the axis $\mathbf{t}$ if and only if the function $U(\theta)$ takes a local maximum at $\theta = 0$.*

Proof of (F1) and (F2). Rotate $\mathbf{u}$ and $\mathbf{n}$ by a small angle ε with respect to the axis $\mathbf{t}$. Denote them by $\mathbf{u}_\varepsilon$ and $\mathbf{n}_\varepsilon$, that is,

$$\mathbf{u}_\varepsilon = \mathbf{u} \cos \varepsilon + \mathbf{n} \sin \varepsilon, \quad \mathbf{n}_\varepsilon = -\mathbf{u} \sin \varepsilon + \mathbf{n} \cos \varepsilon. \quad (12)$$

We denote by $\mathcal{W}$ and $\mathcal{W}_\varepsilon$ the under surface parts of $\mathcal{F}$ with respect to the vertical vectors $\mathbf{n}$ and $\mathbf{n}_\varepsilon$, respectively, and by B and B_ε the centers of buoyancy of them. By the definition, B and B_ε are the center of gravity of $\mathcal{W}$ and $\mathcal{W}_\varepsilon$, respectively. Then put

$$\mathcal{W}_3 = \mathcal{W} \cap \mathcal{W}_\varepsilon, \quad \mathcal{W}_1 = \mathcal{W} \setminus \mathcal{W}_3 \quad \text{and} \quad \mathcal{W}_2 = \mathcal{W}_\varepsilon \setminus \mathcal{W}_3, \quad (13)$$

and denote by B_3 , B_1 and B_2 the centers of gravity of them, respectively. Since the volumes of $\mathcal{W}$ and $\mathcal{W}_\varepsilon$ are equal to ρV , the volumes of $\mathcal{W}_1$ and $\mathcal{W}_2$ are equal, putting V_1 . Then by the property of center of gravity, we obtain that

$$\mathbf{B}_\varepsilon = \left(1 - \frac{V_1}{\rho V}\right) \mathbf{B}_3 + \frac{V_1}{\rho V} \mathbf{B}_2, \quad \mathbf{B} = \left(1 - \frac{V_1}{\rho V}\right) \mathbf{B}_3 + \frac{V_1}{\rho V} \mathbf{B}_1. \quad (14)$$

By the above equality, we obtain that

$$\mathbf{B}_\varepsilon - \mathbf{B} = \frac{V_1}{\rho V} (\mathbf{B}_2 - \mathbf{B}_1). \quad (15)$$

We cut the cross section E into the following two parts:

$$E_2 = E \cap \mathcal{W}_2, \quad E_1 = E \cap \mathcal{W}_1. \quad (16)$$

We take the center of gravity of E as the origin, and take x , y and z -axis so that they have the same directions as $\mathbf{u}$, $\mathbf{n}$ and $\mathbf{t}$, respectively. We denote by $x = \delta$ the border line of E_2 and E_1 , and set

$$\begin{aligned} \tilde{E}_2 &= \{(r \cos \theta + \delta, r \sin \theta, z) \mid (r + \delta, z) \in E_2, 0 \leq \theta \leq \varepsilon\}, \\ \tilde{E}_1 &= \{(-r \cos \theta + \delta, -r \sin \theta, z) \mid (-r + \delta, z) \in E_1, 0 \leq \theta \leq \varepsilon\}. \end{aligned} \quad (17)$$

Then we obtain that

$$\begin{aligned} V_1 &= \iiint_{\mathcal{W}_2} dx dy dz = \iiint_{\tilde{E}_2} r dr d\theta dz + O(\varepsilon^2) \\ &= \varepsilon \iint_{(r+\delta, z) \in E_2} r dr dz + O(\varepsilon^2) \\ &= \varepsilon \iint_{E_2} (x - \delta) dx dz + O(\varepsilon^2). \end{aligned} \quad (18)$$

Similarly, we obtain that

$$V_1 = \iiint_{\mathcal{W}_2} dx dy dz = \varepsilon \iint_{E_1} (\delta - x) dx dz + O(\varepsilon^2). \quad (19)$$

By (18) and (19), we obtain that

$$\varepsilon \left(\iint_E x dx dz - \delta \iint_E dx dz \right) = \varepsilon \iint_{E_2} (x - \delta) dx dz - \varepsilon \iint_{E_1} (\delta - x) dx dz = O(\varepsilon^2). \quad (20)$$

Since we take the center of gravity of E as the origin, we obtain that $\iint_E x dx dz = 0$. Putting it into (20), we obtain that $\delta = O(\varepsilon)$. Since $\mathbf{u}$ is the unit vector of direction x -axis, we obtain that

$$\begin{aligned} V_1 \mathbf{B}_2 \cdot \mathbf{u} &= \iiint_{\mathcal{W}_2} x dx dy dz = \iiint_{\tilde{E}_2} r(r \cos \theta + \delta) dr d\theta dz + O(\varepsilon^2) \\ &= \iint_{(r+\delta, z) \in E_2} r(r \sin \varepsilon + \delta \varepsilon) dr dz + O(\varepsilon^2) \\ &= \varepsilon \iint_{E_2} x(x - \delta) dx dz + O(\varepsilon^2) = \varepsilon \iint_{E_2} x^2 dx dz + O(\varepsilon^2). \end{aligned} \quad (21)$$

Similarly, we obtain that

$$V_1 \mathbf{B}_1 \cdot \mathbf{u} = -\varepsilon \iint_{E_1} x^2 dx dz + O(\varepsilon^2), \quad (22)$$

$$V_1 \mathbf{B}_2 \cdot \mathbf{n} = O(\varepsilon^2), \quad V_1 \mathbf{B}_1 \cdot \mathbf{n} = O(\varepsilon^2). \quad (23)$$

By using (15), (21) and (22), we obtain that

$$(\mathbf{B}_\varepsilon - \mathbf{B}) \cdot \mathbf{u} = \frac{\varepsilon}{\rho V} \iint_E x^2 dx dz + O(\varepsilon^2) = \frac{\varepsilon I(\theta)}{\rho V} + O(\varepsilon^2). \quad (24)$$

Similarly, we obtain that

$$(\mathbf{B}_\varepsilon - \mathbf{B}) \cdot \mathbf{n} = O(\varepsilon^2). \quad (25)$$

By dividing (24) and (25) by ε , and taking $\varepsilon \rightarrow 0$, we obtain that

$$\mathbf{B}'(\theta) \cdot \mathbf{u} = \frac{I(\theta)}{\rho V}, \quad \mathbf{B}'(\theta) \cdot \mathbf{n} = 0. \quad (26)$$

By putting (8) into (26), we obtain that

$$\mathbf{B}'(\theta) \cdot \mathbf{u}_0 = \frac{I(\theta)}{\rho V} \cos \theta, \quad \mathbf{B}'(\theta) \cdot \mathbf{n}_0 = \frac{I(\theta)}{\rho V} \sin \theta. \quad (27)$$

By replacing θ of (27) by ϕ , and by integrating it on the interval $0 \leq \phi \leq \theta$, we obtain that

$$(\mathbf{B} - \mathbf{B}_0) \cdot \mathbf{u}_0 = \frac{1}{\rho V} \int_0^\theta I(\phi) \cos \phi \, d\phi, \quad (\mathbf{B} - \mathbf{B}_0) \cdot \mathbf{n}_0 = \frac{1}{\rho V} \int_0^\theta I(\phi) \sin \phi \, d\phi. \quad (28)$$

By (8) and (28), we obtain that

$$\begin{aligned} (\mathbf{B} - \mathbf{B}_0) \cdot \mathbf{u} &= \frac{1}{\rho V} \int_0^\theta I(\phi) (\cos \theta \cos \phi + \sin \theta \sin \phi) \, d\phi \\ &= \frac{1}{\rho V} \int_0^\theta I(\phi) \cos(\theta - \phi) \, d\phi. \end{aligned} \quad (29)$$

Similarly, we obtain that

$$\begin{aligned} (\mathbf{B} - \mathbf{B}_0) \cdot \mathbf{n} &= -\frac{1}{\rho V} \int_0^\theta I(\phi) (\sin \theta \cos \phi - \cos \theta \sin \phi) \, d\phi \\ &= -\frac{1}{\rho V} \int_0^\theta I(\phi) \sin(\theta - \phi) \, d\phi. \end{aligned} \quad (30)$$

Hence we obtain that

$$\begin{aligned} F(\theta) &= (-(\mathbf{G} - \mathbf{B}_0) + (\mathbf{B} - \mathbf{B}_0)) \cdot \mathbf{u} \\ &= -(\mathbf{G} - \mathbf{B}_0) \cdot (\mathbf{u}_0 \cos \theta + \mathbf{n}_0 \sin \theta) + (\mathbf{B} - \mathbf{B}_0) \cdot \mathbf{u} \\ &= -\overline{G\mathbf{B}_0} \sin \theta + \frac{1}{\rho V} \int_0^\theta I(\phi) \cos(\theta - \phi) \, d\phi \\ &= \int_0^\theta \left(\frac{I(\phi)}{\rho V} - \overline{G\mathbf{B}_0} \right) \cos(\theta - \phi) \, d\phi. \end{aligned} \quad (31)$$

Similarly, we obtain that

$$U(\theta) = \int_0^\theta \left(\frac{I(\phi)}{\rho V} - \overline{G\mathbf{B}_0} \right) \sin(\theta - \phi) \, d\phi. \quad (32)$$

Hence we have proved (F1) and (F2). $\square$

5 Application to Ulam's problem

Firstly, we consider three dimensional case. Remark that a floating body $\mathcal{F} \in \mathbb{R}^3$ is in equilibrium in every direction if and only if $F(\theta) = 0$ for every rotation axis $\mathbf{t}$. If $F(\theta) = 0$, then by (11), we have that $I(\mathbf{n}, \mathbf{t}) = I(\theta)$ is constant. Hence we have proved the following theorem:

Theorem 10. *If a floating body $\mathcal{F}$ is in equilibrium in every direction, then the moment of inertia $I(\mathbf{n}, \mathbf{t})$ is constant for every $\mathbf{n}$ and $\mathbf{t}$.*

Secondly, we consider two dimensional case. Consider a rod of base $D \subset \mathbb{R}^2$ and height h . Suppose that it floats so that the rotation axis $\mathbf{t}$ is parallel to the axis of the rod and fixed. In this case, denote by $\ell = \ell(\theta)$ the length of the water line. Then we obtain that

$$I(\theta) = \int_{-h/2}^{h/2} \left\{ \int_{-\ell/2}^{\ell/2} x^2 \, dx \right\} dz = \frac{h}{12} \ell^3. \quad (33)$$

By Theorem 10 and (33), we have proved Theorem 4.

The converse of Theorem 4 holds under a rotational symmetry condition.

Theorem 11. *If a figure $D \subset \mathbb{R}^2$ is rotational symmetric with respect to some angle, and if the water line is of constant length, then it is Auerbach.*

Proof. By (33), the moment of inertia $I(\theta)$ is constant. So by (28), we obtain that

$$B = G + \left(\frac{I(0)}{\rho V} - \overline{GB_0} \right) \mathbf{n}_0 + \frac{I(0)}{\rho V} (\mathbf{u}_0 \sin \theta - \mathbf{n}_0 \cos \theta). \quad (34)$$

So the locus of B is a circle. Since D is rotational symmetric, so is the locus of B . Thus we obtain that

$$B = G + \frac{I(0)}{\rho V} (\mathbf{u}_0 \sin \theta - \mathbf{n}_0 \cos \theta). \quad (35)$$

By (35), we obtain that $F(\theta) = 0$. Hence D is Auerbach. $\square$

References

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