

Inequalities Involving Unitarily Invariant Norms and Operator Monotone Functions¹

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We consider square complex matrices. A norm $\|\cdot\|$ on the space of $n \times n$ matrices is called *unitarily invariant* if

$$\|UAV\| = \|A\| \quad \forall A, \forall \text{unitary } U, V.$$

Such a norm is determined by a symmetric gauge function Φ on $\mathbb{R}^n$:

$$\|A\| = \Phi(s_1(A), \dots, s_n(A))$$

where $s_1(A) \geq s_2(A) \geq \dots \geq s_n(A)$ are the singular values of A , that is, the eigenvalues of $|A| \equiv (A^*A)^{1/2}$.

Examples of unitarily invariant norms are:

Schatten p -norm $\|\cdot\|_p$ ($1 \leq p \leq \infty$) :

$$\|A\|_p \equiv \left\{ \sum_{j=1}^n s_j(A)^p \right\}^{1/p}.$$

Then $\|A\|_\infty = s_1(A)$ is the *spectral norm* and $\|A\|_2 = \{\sum_{i,j=1}^n |a_{ij}|^2\}^{1/2}$ is the *Frobenius norm*.

Fan k -norm $\|\cdot\|_{(k)}$ ($k = 1, 2, \dots, n$):

$$\|A\|_{(k)} \equiv \sum_{j=1}^k s_j(A).$$

For Hermitian matrices A, B , we write $A \geq B$ to mean that $A - B$ is positive semidefinite. In particular, $A \geq 0$ means that A is positive semidefinite.

We consider only continuous nonnegative functions on $[0, \infty)$. $f(t)$ is called *operator monotone* if

$$A \geq B \geq 0 \implies f(A) \geq f(B).$$

¹This paper appeared in Linear Algebra Appl. 341(2002) 151-169.

Here $f(A)$ is defined by the usual functional calculus via the spectral decomposition of A .

Examples of operator monotone functions are:

$$t^p \quad (0 < p \leq 1), \quad \log(t+1)$$

1. Convexity of certain functions involving unitarily invariant norms

Theorem 1. Given matrices $A, B \geq 0$, $\forall X$, real number $r > 0$, and any unitarily invariant norm, the function

$$\phi(t) = \| |A^t X B^{1-t}|^r \| \cdot \| |A^{1-t} X B^t|^r \|$$

is convex on the interval $[0, 1]$ and attains its minimum at $t = 1/2$. Consequently, it is decreasing on $[0, 1/2]$ and increasing on $[1/2, 1]$.

Corollary 2. For $0 \leq t \leq 1$,

$$\begin{aligned} \| |A^{1/2} X B^{1/2}|^r \|^2 &\leq \| |A^t X B^{1-t}|^r \| \cdot \| |A^{1-t} X B^t|^r \| \\ &\leq \| |AX|^r \| \cdot \| |XB|^r \| \end{aligned}$$

Note that this interpolates the known matrix Cauchy-Schwarz inequality

$$\| |A^{1/2} X B^{1/2}|^r \|^2 \leq \| |AX|^r \| \cdot \| |XB|^r \|.$$

Corollary 3. Let A, B be positive definite and X be arbitrary. For every $r > 0$ and every unitarily invariant norm, the function

$$g(s) = \| |A^s X B^s|^r \| \cdot \| |A^{-s} X B^{-s}|^r \|$$

is convex on $(-\infty, \infty)$, attains its minimum at $s = 0$, and hence it is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

The case $r = 1, X = B = I$ (the identity matrix) of this result says that the condition number

$$c(A^s) \equiv \|A^s\| \cdot \|A^{-s}\|$$

is increasing in $s > 0$, which is due to A. W. Marshall and I. Olkin (1965).

2. Norm inequalities for operator monotone functions with applications

A norm on $n \times n$ matrices is said to be *normalized* if $\|\text{diag}(1, 0, \dots, 0)\| = 1$.

All the Fan k -norms ($k = 1, \dots, n$) and Schatten p -norms ($1 \leq p \leq \infty$) are normalized.

Theorem 4. Let $f(t)$ be a nonnegative operator monotone function on $[0, \infty)$ and $\|\cdot\|$ be a normalized unitarily invariant norm. Then for every matrix A ,

$$f(\|A\|) \leq \|f(|A|)\|.$$

This inequality is reversed when the norm is normalized in another way.

Theorem 5. Let $f(t)$ be a nonnegative operator monotone function on $[0, \infty)$ and $\|\cdot\|$ be a unitarily invariant norm with $\|I\| = 1$. Then for every matrix A ,

$$f(\|A\|) \geq \|f(|A|)\|.$$

Given a unitarily invariant norm $\|\cdot\|$, for $p > 0$ define

$$\|X\|^{(p)} \equiv \| |X|^p \|^{1/p}.$$

Then it is known that when $p \geq 1$, $\|\cdot\|^{(p)}$ is also a unitarily invariant norm.

Corollary 6. Let $\|\cdot\|$ be a normalized unitarily invariant norm. Then for any matrix A , the function $p \mapsto \|A\|^{(p)}$ is decreasing on $(0, \infty)$ and

$$\lim_{p \rightarrow \infty} \|A\|^{(p)} = \|A\|_{\infty}.$$

The above limit formula remains valid without the normalization condition on $\|\cdot\|$.

We denote by $A \vee B$ the supremum of $A, B \geq 0$: $A \vee B = \lim_{p \rightarrow \infty} \{(A^p + B^p)/2\}^{1/p}$.

Theorem 7. Let A, B be positive semidefinite. For every unitarily invariant norm, the function $p \mapsto \|(A^p + B^p)^{1/p}\|$ is decreasing on $(0, 1]$. For every normalized unitarily invariant norm, the function $p \mapsto \|A^p + B^p\|^{1/p}$ is decreasing on $(0, \infty)$ and

$$\lim_{p \rightarrow \infty} \|A^p + B^p\|^{1/p} = \|A \vee B\|_{\infty}.$$

The above limit formula remains valid without the normalization condition.

3. Norm inequalities of Hölder and Minkowski types

Theorem 8. Let $1 \leq p, q \leq \infty$ with $p^{-1} + q^{-1} = 1$. For all matrices A, B, C, D and every unitarily invariant norm,

$$2^{-|\frac{1}{p} - \frac{1}{2}|} \|C^* A + D^* B\| \leq \| |A|^p + |B|^p \|^{1/p} \cdot \| |C|^q + |D|^q \|^{1/q}.$$

Moreover, the constant $2^{-|\frac{1}{p} - \frac{1}{2}|}$ is best possible.

Theorem 9. Let $1 \leq p < \infty$. For any A_i, B_i ($i = 1, 2$) and every unitarily invariant norm,

$$\begin{aligned} & 2^{-|\frac{1}{p} - \frac{1}{2}|} \| |A_1 + A_2|^p + |B_1 + B_2|^p \|^{1/p} \\ & \leq \| |A_1|^p + |B_1|^p \|^{1/p} + \| |A_2|^p + |B_2|^p \|^{1/p}. \end{aligned}$$

Main Ingredients of the Proofs

• *Integral representation:* A nonnegative operator monotone function $f(t)$ on $[0, \infty)$ is represented as

$$f(t) = \alpha + \beta t + \int_0^{\infty} \frac{st}{s+t} d\mu(s)$$

where $\alpha, \beta \geq 0$ and $\mu(\cdot)$ is a positive measure on $[0, \infty)$.

• *Dual norm:* Given a norm $\|\cdot\|$ on $n \times n$ matrices, the dual norm of $\|\cdot\|$ with respect to the Frobenius inner product is

$$\|A\|^D \equiv \max \{ |\operatorname{tr} AX^*| : \|X\| = 1 \}.$$

If $\|\cdot\|$ is a unitarily invariant norm and $A \geq 0$, then by the duality theorem we have

$$\|A\| = \max \{ \operatorname{tr} AB : B \geq 0, \|B\|^D = 1 \}.$$

• *Theorem* [conjectured by F. Hiai and proved by T. Ando and X. Zhan, Math. Ann. 315 (1999)]: Let $A, B \geq 0$, and $\|\cdot\|$ be a unitarily invariant norm. If $f(t)$ nonnegative operator monotone on $[0, \infty)$, then

$$\|f(A + B)\| \leq \|f(A) + f(B)\|.$$

If $g(t)$ is strictly increasing on $[0, \infty)$ with $g(0) = 0$, $g(\infty) = \infty$ and the inverse function g^{-1} on $[0, \infty)$ is operator monotone, then

$$\|g(A + B)\| \geq \|g(A) + g(B)\|.$$