

ARCHIMEDEAN SHINTANI FUNCTIONS ON $GL(2)$

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1. Introduction

Shintani functions for $GL(n)$ was defined by Murase and Sugano in the study of automorphic L -functions [5]. They proved the uniqueness and the existence of this function over a non-archimedean local field and obtained new kinds of integral formula for the standard L -functions as an application. Our aim in this note is the case study of archimedean Shintani functions on $GL(2)$, which is not studied in [5]. In §3, we define archimedean Shintani functions on $GL(2)$ generalizing that of Murase and Sugano. Also, our definition of this function can be considered as a generalization of the O_ξ model studied by Waldspurger [6]. Now we consider the following problems.

- (1) Decide the dimension of the space of archimedean Shintani functions.
- (2) Find an explicit formula of non-zero archimedean Shintani functions.

We will give an answer to these problems in §5.

2. Preliminaries

2.1. Groups and algebras. Throughout this note, E means either the field of real numbers $\mathbf{R}$ or that of complex numbers $\mathbf{C}$. Let G be the real reductive Lie group $GL(2, E)$ and θ be an involution defined by $\theta(g) = {}^t\bar{g}^{-1}$ ($g \in G$). We denote the set of fixed points of θ by K . Then K is a maximal compact subgroup of G and

$$K \simeq \begin{cases} O(2, \mathbf{R}) & \text{for } E = \mathbf{R}, \\ U(2) & \text{for } E = \mathbf{C}, \end{cases}$$

Moreover we define an involutive automorphism σ of G by $\sigma(g) = JgJ$ ($g \in G$), where $J = \text{diag}(-1, 1)$. Then $\theta\sigma = \sigma\theta$ and the set H of fixed points of σ is equal to

$$H = \{g \in G \mid \sigma(g) = g\} = \{\text{diag}(h_1, h_2) \in G \mid h_i \in E^\times\} \simeq E^\times \times E^\times.$$

In particular, H is abelian subgroup of G .

Let $\mathfrak{g} = \mathfrak{gl}(2, E)$ be the Lie algebra of G . If we denote the differentials of θ and σ , again by θ and σ , then we have $\theta(X) = -{}^t\bar{X}$ and $\sigma(X) = JXJ(X \in \mathfrak{g})$. Let us write the eigenspaces of θ and σ by

$$\begin{aligned}\mathfrak{k} &= \{X \in \mathfrak{g} | \theta(X) = X\}, & \mathfrak{p} &= \{X \in \mathfrak{g} | \theta(X) = -X\}, \\ \mathfrak{h} &= \{X \in \mathfrak{g} | \sigma(X) = X\}, & \mathfrak{q} &= \{X \in \mathfrak{g} | \sigma(X) = -X\}.\end{aligned}$$

Therefore we have the decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$. Remark that $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{h}$ is that of H . Let

$$A = \left\{ a_r = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix} \in G \mid r \in \mathbf{R} \right\}, \quad \mathfrak{a} = \text{Lie}(A).$$

Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$.

For a Lie algebra $\mathfrak{b}$, we denote by $\mathfrak{b}^{\mathbf{C}}$ the complexification $\mathfrak{b} \otimes_{\mathbf{R}} \mathbf{C}$ of $\mathfrak{b}$.

2.2. Representations. In this subsection, we recall parametrizations of the irreducible unitary representations of K , H and G .

Let us denote by $\hat{K}$ the set of the equivalence classes of irreducible finite dimensional representations of K . Since K is compact, the highest weight theory (cf. Knapp [4; Theorem 4.28]) gives a parametrization of $\hat{K}$ by the set

$$\Lambda = \begin{cases} \{(0, \varepsilon) \mid \varepsilon = 0, 1\} \cup \mathbf{N}, & \text{for } E = \mathbf{R}, \\ \{\lambda = (\lambda_1, \lambda_2) \mid \lambda_i \in \mathbf{Z}, \lambda_1 \geq \lambda_2\}, & \text{for } E = \mathbf{C}. \end{cases}$$

Let $(\tau_\lambda, V_\lambda) \in \hat{K}$ be the corresponding representation to $\lambda \in \Lambda$. Then we have

$$\dim V_\lambda = \begin{cases} 1, & \text{if } \lambda = (0, \varepsilon) \\ 2, & \text{if } \lambda \in \mathbf{N} \\ \lambda_1 - \lambda_2 + 1, & \text{for } E = \mathbf{C} \end{cases} \quad \text{for } E = \mathbf{R}$$

Next let us parametrize the totality $\hat{H}$ of the equivalence classes of irreducible unitary representations of H . To do this, we put

$$\mathcal{N}_E = \begin{cases} \{0, 1\}^2, & \text{for } E = \mathbf{R}, \\ \mathbf{Z}^2, & \text{for } E = \mathbf{C}. \end{cases}$$

For every $s = (s_1, s_2) \in \mathbf{C}^2$ and $k = (k_1, k_2) \in \mathcal{N}_E$, we define a representation η_s^k of H by

$$\eta_s^k(\text{diag}(h_1, h_2)) = h_1^{k_1} h_2^{k_2} |h_1|^{s_1 - k_1} |h_2|^{s_2 - k_2}, \quad \text{diag}(h_1, h_2) \in H.$$

Clearly $\hat{H} = \{\eta_s^k \mid s = (s_1, s_2) \in (\sqrt{-1}\mathbf{R})^2, k = (k_1, k_2) \in \mathcal{N}_E\}$.

Let $P = N_P A_P M_P$ be the Langlands decomposition of the upper triangular subgroup P of G . For every $z = (z_1, z_2) \in \mathbf{C}^2$ and $l = (l_1, l_2) \in \mathcal{N}_E$, we define σ_l on M_P and ν_z on $\mathfrak{a}_P = \text{Lie}(A_P)$ by

$$\begin{aligned}\sigma_l(\text{diag}(\varepsilon_1, \varepsilon_2)) &= \varepsilon_1^{l_1} \varepsilon_2^{l_2}, & \text{diag}(\varepsilon_1, \varepsilon_2) &\in M_P, \quad \varepsilon_i \in E^{(1)} \\ \nu_z(\text{diag}(t_1, t_2)) &= z_1 t_1 + z_2 t_2, & \text{diag}(t_1, t_2) &\in \mathfrak{a}_P, \quad t_i \in \mathbf{R}.\end{aligned}$$

Then we can construct a representation $\pi_z^l = \text{Ind}_P^G(1_{N_P} \otimes \exp \nu_z \otimes \sigma_l)$ of G which we call the non-unitary principal series representation. A dense subspace of the representation space is

$$\{f \in C^\infty(G) \mid f(namx) = e^{(\nu_z + \rho_E) \log a} \sigma_l(m) f(x)\}$$

with norm

$$\|f\|^2 = \int_K |f(k)|^2 dk,$$

and G acts by $\pi_z^l(g)f(x) = f(xg)$. Here ρ_E is the half sum of the roots of $(\mathfrak{a}_P, \mathfrak{g})$ positive for N_P .

If $z_i \in \sqrt{-1}\mathbf{R}$, then the representation π_z^l is irreducible and unitary. This representation is usually called the unitary principal series representation P_z^l of G . Now we put $\rho_{E,0} = \rho_E(\text{diag}(1, -1))$, i.e. $\rho_{\mathbf{R},0} = 1$ and $\rho_{\mathbf{C},0} = 2$. If the parameter (z, l) satisfies $z_1 + z_2 \in \sqrt{-1}\mathbf{R}$, $-\rho_{E,0} < z_1 - z_2 < 0$ and $l_1 = l_2$, π_z^l is irreducible and infinitesimally unitary. The unitary version of this representation is called the complementary series representation C_z^l of G . The representations π_z^l belonging to these two series have the following K -types from the Frobenius reciprocity theorem;

$$\begin{aligned} \pi_z^l|_K &= \begin{cases} \tau_0^{l_1} \oplus \sum_{n \in \mathbf{N}} \tau_{2n}, & \text{if } l_1 + l_2 \equiv 0 \pmod{2}, \\ \sum_{n \in \mathbf{N}} \tau_{2n-1}, & \text{if } l_1 + l_2 \equiv 1 \pmod{2}, \end{cases} & \text{for } E = \mathbf{R}, \\ \pi_z^l|_K &= \begin{cases} \sum_{j=0}^{\infty} \tau_{(l_1+j, l_2-j)}, & \text{if } l_1 \geq l_2, \\ \sum_{j=0}^{\infty} \tau_{(l_2+j, l_1-j)}, & \text{if } l_1 < l_2, \end{cases} & \text{for } E = \mathbf{C}. \end{aligned}$$

In the case of $E = \mathbf{R}$, π_z^l contains the discrete series representation $D_{j, z_1+z_2}^l$ as a subrepresentation if the parameters satisfy $z_1 + z_2 \in \sqrt{-1}\mathbf{R}$, $z_1 - z_2 = -j - 1$ for $j \in \mathbf{Z}_{\geq 0}$, and $l_1 + l_2 \equiv j \pmod{2}$. The K -types of $D_{j, z_1+z_2}^l$ are given by

$$D_{j, z_1+z_2}^l|_K = \sum_{n \in \mathbf{N}} \tau_{j+2n}.$$

Let us denote by $\hat{G}_\infty$ the set of the equivalence classes of irreducible unitary representations of G belonging to the above series. Then the unitary dual $\hat{G}$ of G consists of $\hat{G}_\infty$ together with the unitary characters (cf. Wallach [7]).

In the following sections, we use the same letter for a given representation and its underlying $(\mathfrak{g}^{\mathbf{C}}, K)$ -module. Also we use the notations $s' = s_1 - s_2$ and $z' = z_1 - z_2$, for brevity.

3. Shintani functions

3.1. Shintani function. Let $\eta \in \hat{H}$. Consider C^∞ -induced module $C^\infty \text{Ind}_H^G(\eta)$ with the representation space

$$C_\eta^\infty(H \backslash G) = \{F \in C^\infty(G) \mid F(hg) = \eta(h)F(g), (h, g) \in H \times G\}$$

on which G acts by the right translation. Then $C_\eta^\infty(H \backslash G)$ has structure of a smooth G -module and of a $(\mathfrak{g}^\mathbb{C}, K)$ -module.

On the other hand, let us take an irreducible Harish-Chandra module $\Pi^* \in \hat{G}_\infty$, and consider the intertwining space

$$\mathcal{I}_{\eta, \Pi} = \text{Hom}_{(\mathfrak{g}^\mathbb{C}, K)}(\Pi^*, C^\infty \text{Ind}_H^G(\eta))$$

and its image

$$\mathcal{S}_{\eta, \Pi} = \bigcup_{T \in \mathcal{I}_{\eta, \Pi}} \text{Image}(T).$$

Here $*$ means the contragredient $(\mathfrak{g}^\mathbb{C}, K)$ -module. We call $\varphi \in \mathcal{S}_{\eta, \Pi}$ a *Shintani function of type (η, Π)* .

For any finite dimensional K -module (τ, V_τ) , we define $C_{\eta, \tau}^\infty(H \backslash G/K)$ by the space of smooth functions $F : G \rightarrow V_\tau$ with the property

$$F(hgk) = \eta(h)\tau(k)^{-1}F(g), \quad (h, g, k) \in H \times G \times K.$$

Now let us take a finite dimensional K -module (τ, V_τ) and a K -equivariant map $i : \tau^* \rightarrow \Pi^*|_K$. Here τ^* is the contragredient representation of τ . Moreover let i^* be the pullback via i . Then the map

$$\mathcal{I}_{\eta, \Pi} \xrightarrow{i^*} \text{Hom}_K(\tau^*, C_\eta^\infty(H \backslash G)) \cong C_{\eta, \tau}^\infty(H \backslash G/K)$$

gives the restriction of $T \in \mathcal{I}_{\eta, \Pi}$ to τ^* which we denote by $T_i \in C_{\eta, \tau}^\infty(H \backslash G/K)$. Set

$$\mathcal{S}_{\eta, \Pi}(\tau) = \bigcup_i T_i, \quad T \in \mathcal{I}_{\eta, \Pi},$$

and we call $\varphi \in \mathcal{S}_{\eta, \Pi}(\tau)$ a *Shintani function of type $(\eta, \Pi; \tau)$* .

3.2. Radial part. Let us write the centralizer and the normalizer of $\mathfrak{a}$ in $K \cap H$ by $Z_{K \cap H}(\mathfrak{a})$ and $N_{K \cap H}(\mathfrak{a})$, respectively. If we put $w_0 = \text{diag}(1, -1)$, then the quotient group $W = N_{K \cap H}(\mathfrak{a})/Z_{K \cap H}(\mathfrak{a})$ has the unique nontrivial element $w_0 Z_{K \cap H}(\mathfrak{a})$.

For each pair of $\eta \in \hat{H}$ and a finite dimensional K -module (τ, V_τ) , let us denote by $C_W^\infty(A; \eta, \tau)$ the space of smooth functions $\varphi : A \rightarrow V_\tau$ satisfying the following conditions;

$$\begin{cases} (1) & (\eta(m)\tau(m))\varphi(a) = \varphi(a), & m \in Z_{K \cap H}(\mathfrak{a}), \ a \in A, \\ (2) & (\eta(w_0)\tau(w_0))\varphi(a) = \varphi(a^{-1}), & a \in A, \\ (3) & (\eta(l)\tau(l))\varphi(1) = \varphi(1), & l \in K \cap H. \end{cases}$$

Lemma 3.1. (Flensted-Jensen [1; Theorem 4.1])

- (1) $G = HAK = HA^+K$, where $A^+ = \{a_r \in A \mid r > 0\}$.
- (2) The set $C_{\eta, \tau}^\infty(H \backslash G/K)$ is in bijective correspondence, via restriction A , with the set $C_W^\infty(A; \eta, \tau)$.

Let (τ, V_τ) and $(\tau', V_{\tau'})$ be two finite dimensional K -modules. For each $\mathbb{C}$ -linear map $u : C_{\eta, \tau}^\infty(H \backslash G/K) \rightarrow C_{\eta, \tau'}^\infty(H \backslash G/K)$, we have a unique $\mathbb{C}$ -linear map $\mathcal{R}(u) : C_W^\infty(A; \eta, \tau) \rightarrow C_W^\infty(A; \eta, \tau')$ with the property $(uf)|_A = \mathcal{R}(u)(f|_A)$ for $f \in C_{\eta, \tau}^\infty(H \backslash G/K)$. We call $\mathcal{R}(u)$ the *radial part of u* .

4.Characterization

4.1. Shift operator. The vector space $\mathfrak{p}^{\mathbb{C}}$ becomes a K -module via the adjoint representation. Let $\mathfrak{p}^{\mathbb{C}} = \mathfrak{p}_S \oplus \mathfrak{p}_Z$ be the irreducible decomposition of $\mathfrak{p}^{\mathbb{C}}$ as a K -module, where $\mathfrak{p}_Z = (\mathfrak{p} \cap Z_{\mathfrak{g}})^{\mathbb{C}}$, $Z_{\mathfrak{g}}$ is the center of $\mathfrak{g}$, and $\mathfrak{p}_S \simeq V_{\beta}$ with $\beta = 2$ for $E = \mathbf{R}$ or $\beta = (1, -1)$ for $E = \mathbf{C}$.

Take an orthonormal basis $\{X_i\}$ of $\mathfrak{p}_S$ with respect to the Killing form. For a given $\eta_s^k \in \hat{H}$ and $(\tau_{\lambda}, V_{\lambda}) \in \hat{K}$, we define a first order gradient type differential operator $\nabla_{\eta_s^k, \tau_{\lambda}}^S : C_{\eta_s^k, \tau_{\lambda}}^{\infty}(H \backslash G/K) \rightarrow C_{\eta_s^k, \tau_{\lambda} \otimes \text{Ad}_{\mathfrak{p}_S}}^{\infty}(H \backslash G/K)$ by

$$\nabla_{\eta_s^k, \tau_{\lambda}}^S f = \sum_i R_{X_i} f \otimes X_i, \quad f \in C_{\eta_s^k, \tau_{\lambda}}^{\infty}(H \backslash G/K),$$

where

$$R_X f(g) = \left. \frac{d}{dt} f(g \cdot \exp(tX)) \right|_{t=0}, \quad \text{for } X \in \mathfrak{g}^{\mathbb{C}}, \quad g \in G.$$

This differential operator $\nabla_{\eta_s^k, \tau_{\lambda}}^S$ is called *the Schmid operator*. Now let us assume that $\lambda \in \mathbf{N}_{\geq 3}$ for $E = \mathbf{R}$ or $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ with $\lambda_1 - \lambda_2 \geq 2$ for $E = \mathbf{C}$. Then we can define *the minus shift operator*

$$\nabla_{\eta_s^k, \tau_{\lambda}}^{-} : C_{\eta_s^k, \tau_{\lambda}}^{\infty}(H \backslash G/K) \rightarrow C_{\eta_s^k, \tau_{\lambda} - \beta}^{\infty}(H \backslash G/K)$$

as the compositions of $\nabla_{\eta_s^k, \tau_{\lambda}}^S$ with the projector from $V_{\lambda} \otimes \mathfrak{p}_S$ into an irreducible component $V_{\lambda - \beta}$.

4.2. System of differential equations. Let $\Pi^* \in \hat{G}_{\infty}$, and let $(\tau_{\lambda}, V_{\lambda}) \in \hat{K}$ be the minimal K -type of Π . Moreover, let $\eta_s^k \in \hat{H}$ with $s = (s_1, s_2) \in (\sqrt{-1}\mathbf{R})^2$ and $k = (k_1, k_2) \in \mathcal{N}_E$. We consider a characterization of the space $\mathcal{S}_{\eta_s^k, \Pi}(\tau_{\lambda})$ of Shintani functions of type $(\eta_s^k, \Pi; \tau_{\lambda})$ by some differential equations.

Let $Z(\mathfrak{g}^{\mathbb{C}})$ be the center of the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$. It is well known that each element $u \in Z(\mathfrak{g}^{\mathbb{C}})$ acts on Π^* , hence on $\mathcal{S}_{\eta_s^k, \Pi}(\tau_{\lambda})|_A$, as a scalar operator χ_u called an infinitesimal character (cf. Knapp [4; Chap.VIII §6]). Therefore we have the differential equation

$$(4.1) \quad \mathcal{R}(u)\varphi(a_r) = \chi_u\varphi(a_r)$$

for each $\varphi \in \mathcal{S}_{\eta_s^k, \Pi}(\tau_{\lambda})|_A$ and $u \in Z(\mathfrak{g}^{\mathbb{C}})$.

Now let us assume that $\Pi^* = D_{j, z_1 + z_2}^l$ if $E = \mathbf{R}$ or $\Pi^* = P_z^l$ such that $|l_1 - l_2| \geq 2$ if $E = \mathbf{C}$. Since τ_{λ} is the minimal K -type of Π , then $\tau_{\lambda - \beta}$ does not occur in the K -type of Π . Thus any element in $\mathcal{S}_{\eta_s^k, \Pi}(\tau_{\lambda})$ is annihilated by the action of the minus shift operator $\nabla_{\eta_s^k, \tau_{\lambda}}^{-} : C_{\eta_s^k, \tau_{\lambda}}^{\infty}(H \backslash G/K) \rightarrow C_{\eta_s^k, \tau_{\lambda} - \beta}^{\infty}(H \backslash G/K)$, and hence, the differential equation

$$(4.2) \quad \mathcal{R}(\nabla_{\eta_s^k, \tau_{\lambda}}^{-})\varphi(a_r) = 0$$

holds for each $\varphi \in \mathcal{S}_{\eta_s^k, \Pi}(\tau_{\lambda})|_A$.

The above differential equations for $\varphi \in C_W^{\infty}(A; \eta_s^k, \tau_{\lambda})$ are necessary conditions for belonging to the space $\mathcal{S}_{\eta_s^k, \Pi}(\tau_{\lambda})|_A$. But we can prove the following theorem which says that the above equations are also sufficient conditions.

Theorem 4.1. ([2; Proposition 6.1], [3; Theorem 5.3], [8; Theorem 2.4])

Let $\eta_s^k \in \hat{H}$, $\Pi^* \in \hat{G}_\infty$, and let $(\tau_\lambda, V_\lambda) \in \hat{K}$ be the minimal K -type of Π . Then the following system of differential equations characterizes the space $\mathcal{S}_{\eta_s^k, \Pi}(\tau_\lambda)|_A \subset C_W^\infty(A; \eta_s^k, \tau_\lambda)$ of Shintani functions of type $(\eta_s^k, \Pi; \tau_\lambda)$.

- (1) If $\Pi^* = P_z^l$ or C_z^l , the equations (4.1) for all $u \in Z(\mathfrak{g}^{\mathbb{C}})$.
- (2) If $E = \mathbf{R}$ and $\Pi^* = D_{j, z_1+z_2}^l$, the equations (4.1) for $u = I$ and (4.2).

5. Results

In view of Theorem 4.1, the space $\mathcal{S}_{\eta_s^k, \Pi}(\tau_\lambda)|_A$ of Shintani functions is the solution space of some system of differential equations in $C_W^\infty(A; \eta_s^k, \tau_\lambda)$. By the systems of equations in Theorem 4.1 and the constructions of Shintani functions via the Poisson integrals [3; §6], we can prove the following theorem.

Theorem 5.1. ([2; Theorem 6.2, 6.3], [3; Theorem 7.1])

Let $\eta_s^k \in \hat{H}$ and $\Pi^* \in \hat{G}_\infty$. Then the space $\mathcal{S}_{\eta_s^k, \Pi}$ of Shintani functions of type (η_s^k, Π) is non zero if and only if $\eta_s^k|_{Z_G} = \Pi|_{Z_G}$. Here Z_G is the center of G . Moreover, for such pair of representations (η_s^k, Π) we have

$$\dim \mathcal{S}_{\eta, \Pi} = \begin{cases} 2, & \text{if } E = \mathbf{R}, \Pi^* = P_z^l, \text{ and } l_1 \neq l_2 \\ 1, & \text{otherwise.} \end{cases}$$

Moreover we can state an explicit formula of Shintani functions of type $(\eta_s^k, \Pi; \tau_\lambda)$ for some special cases.

Theorem 5.2. ([2; Theorem 6.2], [3; Theorem 7.2])

Let $\eta_s^k \in \hat{H}$ and $\Pi^* = P_z^l$ or $C_z^l \in \hat{G}_\infty$ with $l_1 = l_2$. Then the minimal K -type $(\tau_\lambda, V_\lambda) \in \hat{K}$ of Π is 1-dimensional. If the parameters s, z, k , and l satisfy the equations

$$s_1 + s_2 = z_1 + z_2, \quad k_1 + k_2 \equiv l_1 + l_2 \pmod{2} \quad (E = \mathbf{R}), \quad k_1 + k_2 = l_1 + l_2 \quad (E = \mathbf{C}),$$

then the space $\mathcal{S}_{\eta_s^k, \Pi}(\tau_\lambda)$ has a base whose radial part is given by

$$x^{\frac{s}{4}}(1-x)^{\frac{z'+\rho_{E,0}}{4}} {}_2F_1\left(\frac{z'+s'+\rho_{E,0}+\delta}{4}, \frac{z'-s'+\rho_{E,0}+\delta}{4}; \frac{\rho_{E,0}+\delta}{2}; x\right) v_0^\lambda,$$

with $\delta = 2|k_1 - l_1|$, $v_0^\lambda \in V_\lambda$, and the variable $x = \tanh^2 2r$. Here ${}_2F_1(a, b; c; x)$ is the Gauss's hypergeometric function.

Theorem 5.3. ([2; Theorem 6.3])

Let $\eta_s^k \in \hat{H}$ and $\Pi^* = D_{j, z_1+z_2}^l \in \hat{G}_\infty$. Then $(\tau_\lambda, V_\lambda) \in \hat{K}$ with $\lambda = j+2$ is the minimal K -type of Π . If the parameters satisfy the equations

$$s_1 + s_2 = z_1 + z_2, \quad k_1 + k_2 \equiv l_1 + l_2 \equiv j \pmod{2},$$

the space $\mathcal{S}_{\eta_s^k, \Pi}(\tau_\lambda)$ has a base whose radial part is given by

$$u_{j+2}^s(y)v_{j+2} + (-1)^{k_2} u_{-j-2}^s(y)v_{-j-2}.$$

Here $\{v_{-j-2}, v_{j+2}\}$ is the standard basis of V_{j+2} , and

$$u_{\pm(j+2)}^s(r) = (-y)^{\frac{\pm s' - j - 2}{4}} (1-y)^{\frac{j+2}{2}}$$

with the variable $y = \left(\frac{e^{2r} - \sqrt{-1}}{e^{2r} + \sqrt{-1}}\right)^2$.

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