

Relative Recursive Enumerability of Generic Degrees

Dedicated to

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Introduction. Let ω be the set of natural numbers, i.e. $\{0,1,2,3,\dots\}$. A set $A \subseteq \omega$ is called n -generic if it is Cohen-generic for n -quantifier arithmetic. As characterized by Jockusch[4], this is equivalent to saying that for every Σ_n^0 set of strings S , there is a $\sigma \in A$ such that $\sigma \in S$ or $\forall v \geq \sigma (v \notin S)$. By degree we mean Turing degree (of unsolvability). We call a degree n -generic if it has an n -generic representative. For a degree a , $D(\leq a)$ shows a set of degrees recursive in a .

The relation between n -generic degrees and minimal degrees is widely studied in Chong[1], Chong and Jockusch[2], Haught[3], Jockusch[4], and Kumabe[5]. Jockusch[4] showed that for each $n \geq 2$, if a is n -generic and $0 < b \leq a$ then there is an n -generic degree c with $c \leq b$. From this and the fact that no n -generic degree is minimal, he showed that any n -generic degree bounds no minimal degree. Chong and Jockusch[2] showed the same result for 1-generic degrees below $0'$. Haught[3] showed a stronger result that if a is 1-generic below $0'$ and $0 < b < a$ then b is also 1-generic. On the other hand Chong[1] and Kumabe[5] independently showed that there is a 1-generic degree which bounds a minimal degree. Further Chong[1] showed by a different method that there is a 1-generic degree which bounds a minimal degree below $0'$. These results show an interesting downward homogeneity property of $D(\leq a)$ for n -generic degrees a with $n \geq 2$, but the same result does not hold for all 1-generic degrees.

As 1-generic degrees are not r.e., relative recursive enumerability of n -generic degrees is an interesting problem. Jockusch[4] showed if a is 1-generic there is a $c(\leq a)$ such that a is r.e. in c . So by the result of Haught[3] above, if a is a 1-generic degree below $0'$ then there is a 1-generic degree $b(\leq a)$

such that a is recursive enumerable in b . We show that for all $n \geq 1$ and for any n -generic degree a there is an n -generic degree $c(\langle a \rangle)$ such that a is r.e. in c . This answers to the question in Jockusch[4].

Our notation is standard. A string is a mapping from an initial segment of ω into $\{0,1\}$. Lower case Greek letters other than ω denote strings. For strings σ and ν , $\sigma \geq \nu$ denotes that σ extends ν , and in this case we say that ν is a substring of σ . Further σ and ν are said to be compatible if either extends the other. If σ and ν are incompatible we denote this by $\sigma \perp \nu$. We identify a set $A \subseteq \omega$ with its characteristic function. So $\sigma \leq A$ means that the characteristic function of A extends the string σ and in this case we say that σ is a beginning of A . We write $\sigma * \nu$ for the usual concatenation of σ and ν . We identify 0, 1 with the corresponding strings 0, 1 of length 1. We use i only for 0 or 1 and let $[i] = 1 - i$. Φ denotes the empty string. For each n , $i^{(n)}$ denotes the string σ of length n such that $\sigma(m) = i$ for all $m < n$. For a string σ , $|\sigma|$ denotes the length of σ and σ^- is the substring of σ such that $|\sigma^-| = |\sigma| - 1$. Further for a number $m \leq |\sigma|$, $\sigma[m]$ is the substring of σ of length m . For two strings σ and ν , $\sigma \cap \nu$ is the substring λ of σ and ν such that for all $m < |\lambda|$ $\sigma(m) = \nu(m)$, and $\sigma(|\lambda|) \neq \nu(|\lambda|)$ or at least one of them are not defined. Let Φ_n be n -th partial recursive operator for some fixed recursive enumeration of all the partial recursive operators. Let $\Phi_n(\sigma)(x) = y$ mean that the n -th partial recursive operator with oracle σ and input $x < |\sigma|$, yields output y in at most $|\sigma|$ steps and further that $\Phi_n(\sigma)(u)$ is defined for all $u < x$. Similarly, for an enumeration procedure Ξ , we say that $\Xi(\sigma)(k) = 1$ if there is a computation in Ξ with oracle σ enumerating k . Of course B is

recursive in A iff for some e , $\Phi_e(A)=B$. For two partial recursive operators (or enumeration operators) Ψ and Φ , $\Psi \geq \Phi$ denotes that for every string σ and every number n , $\Psi(\sigma)(n)=\Phi(\sigma)(n)$ whenever $\Phi(\sigma)(n)$ is defined. Strings σ and ν are called Φ_n (or n)-split if $\Phi_n(\sigma)$ and $\Phi_n(\nu)$ are incompatible.

The Result.

We first give two definitions and a lemma which will play an important role throughout the proof of the theorem.

Definition 1. Let Ψ be a partial recursive operator.

(1) σ is called Ψ -good if for any $\lambda \geq \Psi(\sigma)$ there is a $\tau \geq \sigma$ with $\Psi(\tau) \geq \lambda$.

(2) σ is called almost Ψ -good if there is a finite set F of strings such that

(2-i) for any $\tau \geq \sigma$ and $\delta \in F$, $\Psi(\tau) \not\geq \delta$,

(2-ii) there is a string $\nu \geq \Psi(\sigma)$ such that $\nu \not\geq \delta$ for any $\delta \in F$, and

(2-iii) For any string λ such that $\lambda \geq \Psi(\sigma)$ and $\lambda \not\geq \delta$ for any $\delta \in F$, there is a $\tau \geq \sigma$ with $\Psi(\tau) \geq \lambda$.

Definition 2. (1) A set S of strings is called dense if every string has an extension in S .

(2) A set P of strings is called strongly dense (s-dense) if for any nonrecursive set A and any beginning σ of A there is a beginning ν of A such that $\sigma \leq \nu$ and $\nu \in P$.

Clearly if σ is Ψ -good then σ is almost Ψ -good, and if a set P of strings is s -dense, P is dense. The next lemma corresponds to Lemma 4.6 in Jockusch[4].

Lemma 1. For all $n \geq 1$, if Ψ is a partial recursive operator and there is a dense Σ_n^0 (or s -dense) set P of almost Ψ -good strings, then $\Psi(A)$ is total and n -generic whenever A is n -generic.

Proof. Let F be the finite set of strings as defined in Definition 1-(2). To show that $\Psi(A)$ is total, let for each n , $S_n = \{\sigma : \Psi(\sigma)(n) \text{ is defined}\}$. Then S_n is a dense recursive set of strings. (In fact for any σ let v be such that $v \in P$ and $v \geq \sigma$, and let $v' \geq v$ be such that $|\Psi(v')| > n$.) Then by the 1-genericity of A , for each n there is a $\sigma \leq A$ such that $\sigma \in S_n$. So $\Psi(A)$ is total. Next let S be an arbitrary Σ_n^0 set of strings. Let T be the set of strings v such that $\Psi(v) \geq \lambda$ for some $\lambda \in S$. Then T is a Σ_n^0 set of strings. As A is n -generic, there is a $v \leq A$ such that $v \in T$ or no extension of v is in T . If there is a $v \leq A$ such that $v \in T$ then $\Psi(A)$ extends some string λ in S . If there is a $v \leq A$ such that no extension of v is in T then let $\delta \in P$ be a string such that $v \leq \delta \leq A$. (Such a δ exists because P is a dense Σ_n^0 (s -dense) set.) Since δ is almost Ψ -good, let λ be such that $\delta \leq \lambda \leq A$ and $\Psi(\lambda) \upharpoonright \tau$ for any $\tau \in F$ (such a λ exists as $\Psi(A)$ is total). As for any $\xi \geq \Psi(\lambda)$ there is a $u \geq \delta$ with $\Psi(u) \geq \xi$, it follows that no extension of $\Psi(\lambda)$ is in S . Since S was an arbitrary Σ_n^0 set of strings it follows that $\Psi(A)$ is n -generic. $\square$

Theorem. For any $n \geq 1$ and any n -generic degree a , there is an n -generic degree $c \leq a$ such that a is recursive enumerable in c .

Proof. Let A be an n -generic set of degree a . We construct ψ_s at stage s such that $\psi_s \geq \psi_{s-1}$ and $\bigcup_{s=0}^{\infty} \psi_s = \psi$ satisfies that $\psi(A)$ is a set of the desired degree c . Also we construct an enumeration procedure Ξ_s at stage s such that $\Xi_s \geq \Xi_{s-1}$ and $\bigcup_{s=0}^{\infty} \Xi_s = \Xi$ enumerates A relative to $\psi(A)$ (denote this by $\Xi(\psi(A)) = A$).

Before constructing ψ we give the abstract motivation of the construction. To prove the theorem we must construct a partial recursive operator ψ and an enumeration procedure Ξ which satisfy the following conditions:

- (1) $\Xi(\psi(A)) = A$.
- (2) $\psi(A)$ is n -generic, and
- (3) A is not recursive in $\psi(A)$.

Within the motivation, we use letters α, β, γ to refer to conditions on A , and σ, τ, δ to refer to conditions on $\psi(A)$. To satisfy (1), it is enough to arrange the following conditions:

- (1-i) if $\psi(\alpha) = \sigma$ and $\Xi(\sigma)(k) = 1$ then $\alpha(k) = 1$, and
- (1-ii) if $\alpha(k) = 1$ there is an extension β of α with $\Xi(\psi(\beta))(k) = 1$.

To satisfy (2), we construct a s -dense set G of ψ -good strings. (As a matter of fact, we construct a dense recursive set G of almost ψ -good strings, but here assume as above.) Then by Lemma 1 ψ preserves n -genericity. To satisfy (3), by the diagonal requirement for each n we must satisfy $\Phi_n(\psi(A)) \neq A$. In terms of

dense sets, it is enough to arrange that for any α , there is a k and a $\beta \geq \alpha$ satisfying

(3-i) $\Phi_n(\Psi(\beta))(k)=0$ and $\beta(k)=1$,

(3-ii) $\Phi_n(\Psi(\beta))(k)=1$ and $\beta(k)=0$, or

(3-iii) there is no extension of $\Psi(\beta)$ that makes Φ_n converge at k .

The construction is organized in terms of strategies. During the course of executing a strategy we may take one of the following actions.

(A) Enumerate axioms into one or both of Ψ and Ξ .

(B) Prohibit such enumeration by strategies of lower priority. We restrain the enumeration of k above σ by prohibiting the enumeration of any axiom to the effect of $\Xi(\tau)(k)=1$ with $\tau \geq \sigma$. Similarly, we restrain Ψ away from σ above α by prohibiting the enumeration of any axiom $\Psi(\beta)=\tau$ with $\tau \geq \sigma$ and $\beta \geq \alpha$.

Note that restraint above α implies restraint above any extension of α .

There are four types of strategies to be considered here. Three of them are designed to satisfy the three types of requirements mentioned above. The remaining strategy is a global constraint imposed on the construction to simplify the analysis of the forcing relation during a typical step. The crux of the problem is, for each α , to understand what axioms enumerated so far imply about the values of Ψ on A or Ξ on $\Psi(A)$ when A extends α . In other words, given the axioms so far, what is the forcing

relation for Ψ and Ξ . The analysis can be made very manageable by the following.

(I) For each stage and each condition α maintain the property that α has infinitely many extensions for which there are no axioms in Ψ other than those that already apply to α . Similarly, for each σ maintain the condition that σ has infinitely many extensions for which there are no axioms in Ξ other than those that already apply to σ .

These two property imply that at each stage s the axioms enumerated into Ψ and Ξ do no more than the following.

$\alpha \Vdash \Psi(A) \text{ extends } \sigma \iff \Psi(\alpha) \text{ extends } \sigma.$

$\sigma \Vdash F \subseteq A \iff \bigvee_{k \in F} [\Xi(\sigma)(k) = 1].$

(II) To satisfy $\Xi(\Psi(A)) = A$ impose:

(II-i) $\Psi(\alpha) \geq \sigma$ and $\Xi(\sigma)(k) = 1$ implies $\alpha(k) = 1$.

(II-ii) If $\alpha(k) = 1$ then the enumeration of k cannot be restrained above α .

Assuming that the construction respects the conditions mentioned so far, for any stage of the construction and for any α , we are free to extend Ψ and Ξ so that there is an extension β of α with $\Psi(\beta) = \sigma$ and $\Xi(\sigma)(k) = 1$. We can enumerate the relevant axioms and respect (I) by choosing β and σ to be sufficiently long length. Combining (I), (II) and the possibility of global

restraint we obtain the following analysis of the forcing relation.

$\alpha \Vdash \neg \Psi(A)$ does not extend σ $\iff$ one of:

- (a) $\Psi(\alpha)$ is incompatible with σ .
- (b) $\exists k [\alpha(k)=0 \ \& \ \Xi(\sigma)(k)=1]$.
- (c) Ψ is restraint away from σ above α .

These combine with earlier observation on the forcing relation to give a complete analysis. Both of the above strategies have a constant effect on the construction. In the case of (II), the strategy impose a global restraint and a stage by stage enumeration of axioms into Ψ and Ξ . However, it does not impose any coherent pattern to the length or distribution of these axioms.

(III) The third strategy is used to produce a Ψ -good condition extending α . First extend α to β and enumerate axioms into Ψ and Ξ so that if $\beta(k)=1$ then $\Xi(\Psi(\beta))(k)=1$. Note that there is no reason that the relevant axioms in Ξ cannot all have the same use, namely the length of $\Psi(\beta)$. We work under the assumption that no higher priority strategy imposes any restraints on the values of Ψ above β to restrain them away from extensions of $\Psi(\beta)$ and also that no higher priority strategy restrains the enumeration of any number greater than the length of β above $\Psi(\beta)$. For each n , impose the restraints that

(III-i) no axiom with use $\beta * 1^{(n)}$ may be enumerated into Ψ . However axioms with use extending $\beta * 1^{(n)} * 0$ may be enumerated,

(III-ii) if a strategy of lower priority restrains ψ away from σ above $\beta * 1^{(n)}$ and $\sigma \geq \psi(\beta)$ then that same strategy provides a mechanism by which the range of ψ on the conditions extending β is dense below σ , and

(III-iii) similarly, if $k \geq |\beta|$ and a strategy of lower priority restrains the enumeration of k above σ then that strategy provides a mechanism by which the range of ψ on the conditions extending β is dense below σ .

Suppose that $\sigma \geq \psi(\beta)$. Providing that the construction respects these conditions, either there is a τ extending σ such that we can enumerate an axiom $\psi(\beta * 1^{(n)} * 0) = \tau$ or we can invoke a provided mechanism that enumerates an axiom putting an extension of σ in the range of ψ above β . Note that it is always safe to enumerate the axiom mentioned, for large enough n , and respect (II) since every number in $\Xi(\sigma)$ is already in $\beta * 1^{(n)}$.

(IV) The final strategy is used to make the conditions forcing $\Phi_n(\psi(A)) \neq A$ dense for all n as in the statement (3-i), (3-ii), (3-iii). Begin with α and move to β as in (III). Let w be the length of β . Enumerate the axiom

$$\psi(\beta * 0) = \psi(\beta) * 0.$$

(IV)-(A) While there is no $\sigma \geq \psi(\beta * 0)$ with $\Phi_n(\sigma)(w) = 0$, then

(1) restrain ψ away from $\psi(\beta * 0)$ above any γ incompatible with $\beta * 0$.

(2) restrain the enumeration of w above $\psi(\beta * 0)$,

(3) restrain all Ψ -axioms with use $B*1^{(n)}$ beyond those applying to B , and

(4) use the strategy described in (III) to make $\Psi(B*0)$ a Ψ -good condition.

(IV)-(B) When the first $\sigma \geq \Psi(B*0)$ is discovered with $\Phi_n(\sigma)(w)=0$, then drop the above restraints and extend Ψ and Ξ so that there is an n and a τ extending σ such that $\Psi(B*1^{(n)}*0)=\tau$.

Interference between strategies occurs when a diagonal strategy of type (IV) moves from condition (A) to (B). For example, the Ψ -good strategies (III) are injured in this case. Namely, when $\Psi(B*1^{(n)}*0)=\tau$ is enumerated as above and a strategy S , of type (III), was attempting to make some γ with $\gamma \geq B*0$ and $\Psi(B*0) \leq \Psi(\gamma) \leq \tau$, a Ψ -good condition, S cannot be successful. By (II), τ must have some extension τ' with $\Xi(\tau')(w)=1$. But then also by (II), every condition extending $B*0$ is prohibited from being mapped by Ψ to such a τ' . Similarly, the restraint imposed by a type (IV) strategy may also be injured by a type (IV) strategy of higher priority. Luckily, a strategy of type (IV) acts at most one time if not itself injured. (Hence γ will be almost Ψ -good.)

During a stage s of the construction, we work to make sure that each condition of length less than s has an extension with an active strategy for each of the first s many requirements. Since the set of actions in the construction is Σ_1^0 , any 1-generic set must meet this set infinitely often. By a Friedberg style finite injury argument, for any nonrecursive path (not necessarily generic) every requirement has infinitely many initial segments above which a strategy relevant to that

requirement is active and never injured. But there is an important fact. The string γ in the previous paragraph is almost Ψ -good, and so all the strings are almost Ψ -good. To satisfy (2), by Lemma 1, it is enough to construct a dense recursive set of almost Ψ -good strings. So such a finite injury argument does not need. By the notion of "almost Ψ -good", the construction and the proof become extremely simple.

We now give the construction.

Construction.

Stage 0. Let $\Xi_0 = \Psi_0 = \Phi$. We call 0 maximal string at stage 0.

Stage $n+1$. For a string v and a number m with $m < |v|$, let $\text{Sub}(v, m)$ be the substring δ of v of length m , if any, such that $\delta * 0 \leq v^-$. For a maximal string σ at stage n , we say that σ needs m -attention at stage $n+1$ if

(1) $\text{Sub}(\sigma, m)$ is defined and it is not m -satisfied by the end of stage n , and

(2) $\Phi_m(\Psi_n(\sigma))(m) = 0$.

If σ needs m -attention at stage $n+1$, let m_{n+1} be the least such number m , and let σ_{n+1} be the least such string σ in some fixed recursive enumeration of all the strings. We say $\text{Sub}(\sigma_{n+1}, m_{n+1})$ is m_{n+1} -satisfied at stage $n+1$. Let τ_{n+1} be the maximal string at stage n such that $(\tau_{n+1})^- = \text{Sub}(\sigma, m) * 1^{(k)}$ for some $k \geq 0$. Enumerate the axioms:

$$\Psi_{n+1}(\sigma_{n+1}) = \Psi_n(\sigma_{n+1}) * 0, \quad \Psi_{n+1}(\tau_{n+1} * 0) = \Psi_{n+1}(\tau_{n+1}) * 0,$$

$$\Psi_{n+1}(\tau_{n+1}^- * 1 * 0 * 0) = \Psi_n(\sigma_{n+1}) * 1,$$

$$\psi_{n+1}(\tau_{n+1}^{-} * 1 * 1 * 0) = \psi_n(\tau_{n+1}) * 1,$$

$$\Xi_{n+1}(\psi_{n+1}(\tau_{n+1}^{-} * 1 * 0 * 0))(w) = 1 \text{ for any } w \text{ such that}$$

$$m_{n+1} \leq w \leq |\tau_{n+1}^{-}|.$$

We call σ_{n+1} , $\tau_{n+1} * 0$, $\tau_{n+1}^{-} * 1 * 0 * 0$ and $\tau_{n+1}^{-} * 1 * 1 * 0$ maximal strings at stage $n+1$. For any maximal string δ at stage n such that $\delta \neq \sigma_{n+1} \cdot \tau_{n+1}$ if such σ_{n+1} and τ_{n+1} exist, enumerate the axioms:

$$\psi_{n+1}(\delta * 0) = \psi_n(\delta) * 0,$$

$$\psi_{n+1}(\delta^{-} * 1 * 0) = \psi_n(\delta) * 1, \quad \Xi_{n+1}(\psi_{n+1}(\delta^{-} * 1 * 0))(|\delta^{-}|) = 1.$$

We call $\delta * 0$ and $\delta^{-} * 1 * 0$ maximal strings at stage $n+1$. For any λ and k let

$\psi_{n+1}(\lambda) = \cup \{ \psi_m(\lambda') \mid \exists m \leq n+1 [\lambda' \leq \lambda \text{ \& } \psi_m(\lambda') \text{ is explicitly defined at stage } m] \}$

$\Xi_{n+1}(\lambda)(k) = 1$ if for some $\lambda' \leq \lambda$ and $m \leq n+1$, $\Xi_m(\lambda')(k) = 1$ is explicitly defined at stage m ,

$\psi(\lambda) = \cup \{ \psi_m(\lambda') \mid \exists m [\lambda' \leq \lambda \text{ \& } \psi_m(\lambda') \text{ is explicitly defined at stage } m] \}$, and

$\Xi(\lambda)(k) = 1$ if for some $\lambda' \leq \lambda$ and m , $\Xi_m(\lambda')(k) = 1$ is explicitly defined at stage m .

This completes the construction.

The next lemma follows directly from the construction.

Lemma 2. Let δ be a maximal string at stage $n+1$.

(1) If σ_{n+2} is defined and $\sigma_{n+2}=\delta$, then δ is a maximal string at stage $n+2$ and $\psi_{n+2}(\delta)=\psi_{n+1}(\delta)*0$. If σ_{n+2} is defined and $\tau_{n+1}=\delta$, then $\delta*0$, $\delta^-*1*0*0$ and $\delta^-*1*1*0$ are maximal strings at stage $n+2$. $\psi_{n+2}(\delta*0)=\psi_{n+1}(\delta)*0$, $\psi_{n+2}(\delta^-*1*0*0)=\psi_{n+1}(\sigma_{n+2})*1$, and $\psi_{n+2}(\delta^-*1*1*0)=\psi_{n+1}(\delta)*1$. Otherwise then $\delta*0$ and δ^-*1*0 are maximal strings at stage $n+2$, $\psi_{n+2}(\delta*0)=\psi_{n+1}(\delta)*0$, and $\psi_{n+2}(\delta^-*1*0)=\psi_{n+1}(\delta)*1$.

(2) $\delta(|\delta|-1)=0$.

(3) $|\psi_{n+1}(\delta)|=n+1$.

(4) If λ is a maximal string at stage $n+1$ then $\delta|\lambda$ iff $\delta^-|\lambda^-$ iff $\delta \neq \lambda$ iff $\psi_{n+1}(\lambda) \neq \psi_{n+1}(\delta)$.

(5) If $\lambda < \delta$ then $\psi_{n+1}(\delta) > \psi_{n+1}(\lambda) = \psi_m(\lambda)$ for all $m \geq n$ (so $\psi_n(\lambda) = \psi(\lambda)$).

(6) If $\lambda \geq \delta^-*1$ then $\psi_{n+1}(\delta) > \psi_{n+1}(\lambda) = \psi_n(\lambda)$.

(7) If $\lambda < \delta$ then there is unique maximal string τ at stage $n+1$ such that $\tau^- = \lambda*1^{(k)}$ for some $k \geq 0$.

(8) α is a maximal string at stage $n+1$ iff $\psi_{n+1}(\alpha)$ is explicitly defined at stage $n+1$ iff for any $\beta \geq \alpha$ $\psi_{n+1}(\beta) = \psi_{n+1}(\alpha)$ and $\psi_{n+1}(\alpha^-) < \psi_{n+1}(\alpha)$.

(9) $\delta(k)=1$ iff $\Xi_{n+1}(\psi_{n+1}(\delta))(k)=1$.

(10) If $\text{Sub}(\sigma_n, m_n)$ is m_n -satisfied at stage n then for any $s > n$, $\text{Sub}(\sigma_s, m_s) \neq \text{Sub}(\sigma_n, m_n)$ whenever σ_s is defined.

(11) If σ_{n+2} is not defined or it is defined and $\delta \neq \sigma_{n+2}$ then δ is not a maximal string at stage $n+2$.

(12) For any string λ and any number n , there is a maximal string τ at stage n such that λ and τ^- are compatible.

(13) For each n , if $\Xi_n(\sigma)(k)=1$ is explicitly defined at stage n then $|\sigma|=n$, and there is unique maximal string α at stage

n such that $\alpha(k)=1$ and $\psi_n(\alpha)=\sigma$. So for each m with $n \geq m$, no axiom of the form $\Xi_m(\sigma)(k)=1$ with $|\sigma| > n$ is enumerated at stage m .

(14) If $\psi_n(\alpha) \geq \sigma$ and $\Xi(\sigma)(k)=1$ for some α then $\Xi_n(\sigma)(k)=1$.

Proof. (1), ..., (11) Clear by the construction using induction on stage n .

(12) Clear by (1) and the construction by using induction on stage n .

(13) Clear by (3) and (4).

(14) Clear by (3), (8) and (13). $\square$

By Lemma 2-(10) for each σ let $F(\sigma)$ be the least stage n such that for any stage $s \geq n$, if σ_s is defined then $\text{Sub}(\sigma_s, m_s) \neq \sigma$. Clearly Ξ is consistently defined.

Lemma 3. ψ is consistently defined, i.e. for all n ,

- (1) if σ_n is defined then τ_n is also defined, and
- (2) for any strings λ, τ if $\lambda \geq \tau$ then $\psi_n(\lambda) \geq \psi_n(\tau)$.

Proof. (1) is clear by Lemma 2-(7).

(2) We prove (2) by induction on n . Assume the lemma holds for n . Let λ and τ be such that $\lambda \geq \tau$. By Lemma 2-(12) let δ be a maximal string at stage $n+1$ such that δ^- is compatible with λ . If $\delta^- > \lambda$ then by Lemma 2-(5) $\psi_{n+1}(\lambda) = \psi_n(\lambda) \geq \psi_n(\tau) = \psi_{n+1}(\tau)$. If $\lambda \geq \delta * 0 (= \delta) > \tau$ then by Lemma 2-(5)(8) $\psi_{n+1}(\lambda) \geq \psi_{n+1}(\tau) = \psi_n(\tau)$. If $\lambda \geq \delta^- * 1 > \tau$ then by Lemma 2-(5)(6) $\psi_{n+1}(\lambda) = \psi_n(\lambda) \geq \psi_n(\tau) = \psi_{n+1}(\tau)$. If $\tau \geq \delta^- * 0 (= \delta)$ then by Lemma 2-(8) $\psi_{n+1}(\lambda) = \psi_{n+1}(\tau)$. Finally if $\tau \geq \delta^- * 1$ then by lemma 2-(6) $\psi_{n+1}(\lambda) = \psi_n(\lambda) \geq \psi_n(\tau) = \psi_{n+1}(\tau)$. $\square$

Lemma 4. Let σ be an arbitrary string. If $\sigma \neq \emptyset$ and $\sigma(|\sigma|-1)=0$. then

- (i) $\sigma*0$ and σ^-*1*0 are maximal strings at some stage, or
- (ii) $\sigma*0$, $\sigma^-*1*0*0$ and $\sigma^-*1*1*0$ are maximal strings at some stage.

and if $\sigma = \emptyset$, or $\sigma \neq \emptyset$ and $\sigma(|\sigma|-1)=1$ then

- (iii) $\sigma*0$ is a maximal string at some stage, or
- (iv) $\sigma*0*0$ and $\sigma*1*0$ are maximal strings at some stage.

So there is a maximal string λ at some stage with $\lambda \geq \sigma$.

Proof. We proceed by induction on the length of σ . First by the construction, 0 is a maximal string at stage 0 . So the lemma holds for empty string. Let σ be an arbitrary string with $|\sigma| \geq 1$. If $|\sigma| \geq 2$ let $\sigma(|(\sigma^-)^-|)=i$.

If (1) $|\sigma|=1$, (2) $|\sigma| \geq 2$ and $i=0$, or (3) $|\sigma| \geq 2$, $i=1$ and (iii) holds for σ^- , then by the inductive hypothesis σ^-*0 is a maximal string at some stage s . Let $\delta = \sigma^-*0$. If δ is a maximal string at stage t for any $t \geq s$, then for any $t > s$, σ_t is defined and $\sigma_t = \delta$ by Lemma 2-(10). So $\text{Sub}(\sigma_t, m_t) \leq \delta$. But if $t \geq F(\delta)$ this is a contradiction to the assumption on $F(\delta)$. So let $t > s$ be the least stage such that δ is not a maximal string at stage t . Then by Lemma 2-(1),

(A) $\delta = \tau_t$, and $\delta*0$, $\delta^-*1*0*0$ and $\delta^-*1*1*0$ are maximal strings at stage t , or

(B) $\delta*0$ and δ^-*1*0 are maximal strings at stage t .

If (A) is the case and $\sigma = \sigma^- * 0 (= \delta)$ then (ii) holds for σ . If (A) is the case and $\sigma = \sigma^- * 1 (= \delta^- * 1)$ then (iv) holds for σ . If (B) is the case and $\sigma = \sigma^- * 0 (= \delta)$ then (i) holds for σ . If (B) is the case and $\sigma = \sigma^- * 1 (= \delta^- * 1)$ then (iii) holds for σ . In all cases, the lemma holds.

Next assume $|\sigma| \geq 2$, $i=1$ and (iv) holds for σ^- , i.e. $\sigma^- * 0 * 0$ and $\sigma^- * 1 * 0$ are maximal strings at some stage s . If $\sigma = \sigma^- * 0$ then (i) holds for σ . If $\sigma = \sigma^- * 1$ then (iii) holds for σ . In all cases the lemma holds. $\square$

Definition 3. We say that a string σ is almost ψ_{s+1} -good at stage $s+1$ if for any maximal string δ at stage s with $\delta \geq \sigma$, there are maximal strings λ_0, λ_1 at stage $s+1$ such that $\lambda_1 \geq \sigma$ and $\psi_{s+1}(\lambda_1) = \psi_s(\delta) * i$ for each i .

Lemma 5. $\psi(A)$ is total and n -generic.

Proof. Let σ be an arbitrary string. By Lemma 1 it suffices to show that σ is almost ψ -good. By Lemma 4 let n be such that there is a maximal string λ at stage n with $\lambda > \sigma$. Let $x = \max\{F(\sigma), n\}$.

Let F be the set of strings τ such that $|\tau| = x$ and $\tau \leq \psi_x(\lambda)$ for any maximal string λ at stage x with $\lambda \geq \sigma$. Clearly F is finite. By Lemma 2-(3), $|\psi_x(\lambda)| = x$ for any maximal string λ at stage x . So any string u of length x is either an element of F or $u = \psi(\lambda)$ for some maximal string λ at stage x . By Lemma 2-(1), for any $v > x$ and any maximal string λ at stage v , $\psi_v(\lambda) = \psi_{v-1}(\lambda') * i$ for some i and maximal string λ' at stage $v-1$.

We first show that $\lambda' > \sigma$ whenever $\lambda > \sigma$ (*)

Assume $\lambda > \sigma$. (1) If $\lambda' = \sigma_v$ then $\lambda = \sigma_v$, so $\lambda' > \sigma$. (2) If $\lambda' = \tau_v$ then $\lambda = \lambda' * 0$, $(\lambda')^- * 1 * 0 * 0$, or $(\lambda')^- * 1 * 1 * 0$. (2-i) If $\lambda = \lambda' * 0$ then λ' and σ are compatible. As λ' is a maximal string at stage $v-1 \geq x$, $\lambda' > \sigma$ by the assumption on x . (2-ii) If $\lambda = (\lambda')^- * 1 * 0 * 0$ or $(\lambda')^- * 1 * 1 * 0$ then assume for a contradiction that $\lambda' \not> \sigma$. By the assumption on x , $\lambda' \neq \sigma$. As $\lambda' = (\lambda')^- * 0$ by Lemma 2-(2), $\sigma \geq (\lambda')^- * 1$. By Lemma 2-(3)(6), no string extending $(\lambda')^- * 1$ is a maximal string at stage $v-1 (\geq x)$. This is a contradiction to the assumption on x . (3) Otherwise $\lambda = \lambda' * 0$ or $(\lambda')^- * 1 * 0$. If $\lambda = \lambda' * 0$ then the proof is exactly same as (2-i). If $\lambda = (\lambda')^- * 1 * 0$ then the proof is exactly same as (2-ii).

If for some $v > x$, $\lambda > \sigma$, and $\tau \in F$, $\psi_v(\lambda) \geq \tau$ then let v be the least such stage. Further let λ' be the least substring of λ such that $\psi_v(\lambda) = \psi_v(\lambda')$. Then by the construction and Lemma 2-(8), $\psi_v(\lambda')$ is explicitly defined at stage v and λ' is a maximal string at stage v . Then by (*), $\psi_v(\lambda') = \psi_{v-1}(\delta) * i$ for some i and maximal string δ at stage $v-1$ with $\delta > \sigma$. By Lemma 2-(3), $|\psi_{v-1}(\delta)| = v-1 \geq x$, so $\psi_{v-1}(\delta) \geq \tau (\in F)$. By the assumption on v , $v-1 = x$. But this is a contradiction to the definition on F . Hence for any $v > x$, $\lambda > \sigma$, and $\tau \in F$, $\psi_v(\lambda) \not\geq \tau$. So it suffices to show that σ is almost ψ_s -good at stage s for all $s > x$. Let s be an arbitrary number with $s \geq x$, and δ be an arbitrary maximal string at stage s with $\delta > \sigma$. If σ_{s+1} is defined at stage $s+1$ then $\text{Sub}(\sigma_{s+1}, m_{s+1}) \not\geq \sigma$ by the assumption on $F(\sigma)$. (A) If $\delta = \sigma_{s+1}$ then, as $\sigma_{s+1} \geq \text{Sub}(\sigma_{s+1}, m_{s+1})$, $(\tau_{s+1})^- \geq \text{Sub}(\sigma_{s+1}, m_{s+1}) \geq \sigma$. Further by Lemma 2-(1), $\psi_{s+1}(\delta) = \psi_s(\delta) * 0$, $\psi_{s+1}((\tau_{s+1})^- * 1 * 0 * 0) = \psi_s(\delta) * 1$, and δ and $(\tau_{s+1})^- * 1 * 0 * 0$ are maximal strings at stage $s+1$. (B) If $\delta = \tau_{s+1}$ then by Lemma 2-(1), $\psi_{s+1}(\delta * 0) = \psi_s(\delta) * 0$, $\psi_{s+1}(\delta^- * 1 * 1 * 0) = \psi_s(\delta) * 1$, and $\delta * 0$ and $\delta^- * 1 * 1 * 0$ are maximal strings at stage $s+1$. (C)

Otherwise by Lemma 2-(1), $\psi_{s+1}(\delta*0)=\psi_s(\delta)*0$,

$\psi_{s+1}(\delta^-*1*0)=\psi_s(\delta)*1$, and $\delta*0$ and $\delta^-*1*0*0$ are maximal strings at stage $s+1$. In all cases, σ is almost ψ_{s+1} -good at stage $s+1$. $\square$

Lemma 6. $\Xi(\psi(A))=A$.

Proof. It suffices to show that for any numbers s, k and any strings α, σ ,

(1) if $\psi_s(\alpha) \geq \sigma$ and $\Xi_s(\sigma)(k)=1$ then $\alpha(k)=1$, and

(2) if $\alpha(k)=1$ then there is an extension β of $\alpha[k+1]$ such that $\Xi(\psi(\beta))(k)=1$.

(1) We proceed by induction on stage s . Assume $\psi_s(\alpha) \geq \sigma$ and $\Xi_s(\sigma)(k)=1$. If $\psi_{s-1}(\alpha) \geq \sigma$ then by Lemma 2-(14), $\Xi_{s-1}(\sigma)(k)=1$, so by the inductive hypothesis $\alpha(k)=1$. If $\psi_{s-1}(\alpha) < \sigma$ then $\psi_{s-1}(\alpha) < \psi_s(\alpha)$. By Lemma 2-(12) let α_0 be a maximal string at stage s such that $(\alpha_0)^-$ and α are compatible. If $(\alpha_0)^- \geq \alpha$ then by Lemma 2-(5) $\psi_{s-1}(\alpha) = \psi_s(\alpha)$, which is a contradiction. If $\alpha \geq (\alpha_0)^- * 1$ then by Lemma 2-(6) $\psi_{s-1}(\alpha) = \psi_s(\alpha)$, also a contradiction. So $\alpha \geq (\alpha_0)^- * 0 (= \alpha_0$ by Lemma 2-(2)). Hence $|\psi_s(\alpha)| = s$ and $|\psi_{s-1}(\alpha)| = s-1$ by Lemma 2-(3)(8). So $\psi_s(\alpha) = \sigma$ and $\psi_{s-1}(\alpha) = \sigma^-$. If $\Xi_{s-1}(\sigma)(k)=1$ then $\Xi_{s-1}(\sigma^-)(k) = \Xi_{s-1}(\psi_{s-1}(\alpha))(k) = 1$ by Lemma 2-(13). So by the inductive hypothesis $\alpha(k)=1$. If $\Xi_{s-1}(\sigma)(k)$ is not defined then $\Xi_s(\sigma)(k)=1$ is explicitly defined at stage s and $\alpha_0(k)=1$ by Lemma 2-(13) and the fact that $\psi_s(\alpha_0) = \sigma$. As $\alpha \geq \alpha_0$, $\alpha(k) = \alpha_0(k) = 1$.

(2) Assume $\alpha(k)=1$. By Lemma 4, let n be such that β is a maximal string at stage n for some $\beta \geq \alpha[k+1]$. Then by Lemma 2-(9) $\Xi_n(\psi_n(\beta))(k)=1$. $\square$

Lemma 7. A is not recursive in $\psi(A)$.

Proof. It suffices to show that $\phi_n(\psi(A)) \neq A$ for all n . Let n be an arbitrary number. Let R be a infinite recursive set of numbers such that $\phi_n = \phi_m$ whenever $m \in R$. Let m and δ be such that $m \in R$ and $|\delta| = m$. By the 1-genericity of A it suffices to show that

- (1) for any $\lambda \geq \delta * 0$, $\phi_m(\psi(\lambda))(m)$ is not defined, or
- (2) for some $\lambda \geq \delta * 1$, $\phi_m(\psi(\lambda))(m) = 0$.

Assume for a contradiction that for some $\lambda' \geq \delta * 0$, $\phi_m(\psi(\lambda'))(m) = 0$ and there is no string $\mu \geq \delta * 1$ with $\phi_m(\psi(\mu))(m) = 0$. Let t' be such that $t' \geq \max\{F(\delta) : |\delta| = m\}$ and $\phi_m(\psi_{t'}(\lambda'))(m) = 0$. By Lemma 4 let t and λ be such that $t \geq t'$, $\lambda \geq \lambda'$ and λ is a maximal string at stage t . Then λ needs m -attention at stage $t+1$, and m is the least such number by the assumption on t . So $\text{Sub}(\lambda, m) = \delta$ is m -satisfied at stage $t+1$ and $\psi_{t+1}((\tau_{t+1})^{-1} * 1 * 0 * 0) \geq \psi_t(\lambda) \geq \psi_t(\lambda')$. Hence $\phi_m(\psi_{t+1}((\tau_{t+1})^{-1} * 1 * 0 * 0))(m) = 0$. But $(\tau_{n+1})^{-1} * 1 \geq \text{Sub}(\lambda', m) (= \delta) * 1$. This is a contradiction. $\square$

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