

ON A SIMPLE PROOF OF SLIGHTLY CURVED SEQUENCES CONTAINING ARBITRARILY LONG ARITHMETIC PROGRESSIONS

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ABSTRACT. The author and Yoshida proved that a strictly increasing sequence $\{a(n)\}_{n \in A}$ of positive integers, which can be written as $a(n) = f(n) + O(1)$ for some function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $f''(x) = O(1/x^\alpha)$ for some $\alpha > 0$, must contain arbitrarily long arithmetic progressions for all $A \subset \mathbb{N}$ with positive upper Banach density. In this article, we get a simple proof and the same conclusion if we replace the condition $f''(x) = O(1/x^\alpha)$ to $f''(x) = o(1)$.

1. INTRODUCTION

In this article, we consider problems involving arithmetic progressions. Let $d \geq 1$ and $k \geq 3$ be integers. A sequence $\{a(j)\}_{j=0}^{k-1} \subset \mathbb{N}^d$ is called an *arithmetic progression (AP) of length k* if there exists $D \in \mathbb{N}^d$ such that

$$a(j) = a(0) + jD$$

for all $j = 0, 1, \dots, k-1$. Here $\mathbb{N}$ denotes the set of all positive integers. APs are taken interests from researchers studying number theory, arithmetic combinatorics, geometric measure theory, and fractal geometry. The author and Yoshida have found a new class of sets containing arbitrarily long APs, which is named a *slightly curved sequence*. Let $g : \mathbb{N} \rightarrow \mathbb{R}$ be an eventually positive function, and let $\mathbb{R}^+ = (0, \infty)$. A strictly increasing sequence $\{a(n)\}_{n=1}^\infty \subset \mathbb{N}$ is called a *slightly curved sequence with error $O(g(n))$* if there exists a twice differentiable function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$(1.1) \quad f''(x) = O(1/x^\alpha)$$

for some $\alpha > 0$, and

$$a(n) = f(n) + O(g(n)).$$

We define the *graph* of sequence $\{a(n)\}_{n \in A}$ as the set $\{(n, a(n)) : n \in A\}$. The author and Yoshida proved that if $\{a(n)\}_{n=1}^\infty$ is a slightly curved sequence with error $O(1)$ and $A \subset \mathbb{N}$ has positive upper Banach density, then $\{a(n)\}_{n \in A}$ contains arbitrarily long APs. Here we say that a set $A \subset \mathbb{N}$ has *positive upper Banach density* if the condition

$$\lim_{N \rightarrow \infty} \frac{\max_{M \in \mathbb{N}} |A \cap [M, M + N - 1]|}{N} > 0$$

holds. This result is contained Szemerédi's celebrated theorem:

Proposition 1.1 (Szemerédi [S]). For every $k \geq 3$ and $0 < \delta \leq 1$ there exists an integer $N(k, \delta) > 0$ such that if $N \geq N(k, \delta)$, then every set $A \subset \{1, 2, \dots, N\}$ with $|A| \geq \delta N$ contains an AP of length k .

Here $|X|$ denotes the cardinality of a finite set X . Note that the author and Yoshida obtained their result by using Szemerédi's theorem. Thus they do not give another proof of Szemerédi's theorem. As an application, the following result holds:

Proposition 1.2 ([SY, Corollary 1.5]). If a set $A \subset \mathbb{N}$ has positive upper Banach density, then the graph of $\{\lfloor n^a \rfloor\}_{n \in A}$ contains arbitrarily long APs for every $1 \leq a < 2$.

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We refer [SY] to the reader for more details. The goal of this article is to give a simple proof and to extend the condition (1.1). More precisely, we prove the following result:

Theorem 1.3. Suppose that a strictly increasing sequence $\{a(n)\}_{n=1}^{\infty} \subset \mathbb{N}$ satisfies that there exists a twice differentiable function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$(1.2) \quad f''(x) = o(1)$$

and

$$(1.3) \quad a(n) = f(n) + O(1).$$

Then the graph of $\{a(n)\}_{n \in A}$ contains arbitrarily long arithmetic progressions for every $A \subset \mathbb{N}$ with positive upper Banach density.

2. PREPARATION

In order to prove our theorem, let us define a semi-norm on the vector space $\mathcal{F} = \{f \mid f : [0, \infty) \rightarrow \mathbb{R}\}$. Let $k \geq 3$ be an integer and $P = \{b(j)\}_{j=0}^{k-1} \subset [0, \infty)$ be a strictly increasing sequence. We define

$$N_P(f) = \sum_{j=0}^{k-3} |\Delta^2[f \circ b](j)|,$$

for every $f \in \mathcal{F}$, where Δ denotes the difference operator, that is,

$$\Delta[f](x) = f(x+1) - f(x),$$

and $\Delta^2 := \Delta \circ \Delta$. We can find that N_P satisfies the following properties:

(N1) for every strictly increasing function $f \in \mathcal{F}$,

$$N_P(f) = 0 \text{ if and only if } f(P) \text{ is an AP of length } k;$$

(N2) $N_P(f) \geq 0$ for all $f \in \mathcal{F}$;

(N3) $N_P(f+g) \leq N_P(f) + N_P(g)$ for all $f, g \in \mathcal{F}$.

We omit the proof of all the properties (N1), (N2), and (N3) because they are trivial. The semi-norm $N_P(\cdot)$ first appeared in [SY].

3. PROOF

Proof of Theorem 1.3. Fix a set $A \subset \mathbb{N}$ with positive Banach upper density and $k \geq 3$. We show that $N_P(a) = 0$ for some arithmetic progression $P = \{b_j\}_{j=0}^{k-1} \subset A$ of length k . Let $R(x) := a(x) - f(x)$. Then there exists a positive integer $M > 0$ such that $|R(x)| < M$ for every $x \in \mathbb{N}$ since $a(x) = f(x) + O(1)$. Let

$$\delta := \overline{\lim}_{N \rightarrow \infty} \frac{\max_{M \in \mathbb{N}} |A \cap [M, M+N-1]|}{N},$$

and let

$$L := N \left(\frac{\delta}{2}, N \left(\frac{1}{4kM}, k \right) \right).$$

Assume that there exists $j_0 > 0$ such that for every $m \geq m_0$ we have

$$|A \cap [1 + (m-1)L, mL]| < L\delta/2.$$

Let M be a parameter of a positive integer. If $M < 1 + (m_0 - 1)L$ holds, then we obtain that

$$\begin{aligned} |A \cap [M, M+N-1]| &\leq |A \cap [1, (m_0 - 1)L]| + |A \cap [1 + (m_0 - 1)L, (m_0 - 1)L + N - 1]| \\ &\leq N\delta/2 + O_{m_0, L}(1) \end{aligned}$$

On the other hand, if $M \geq 1 + (m_0 - 1)L$ holds, then we obtain that

$$|A \cap [M, M+N-1]| \leq N\delta/2 + O_{m_0, L}(1).$$

Therefore we have

$$\lim_{N \rightarrow \infty} \frac{\max_{M \in \mathbb{N}} |A \cap [M, M + N - 1]|}{N} \leq \delta/2,$$

which is a contradiction. Hence there exists a infinite sequence $0 < m_1 < m_2 < \dots$ of integers such that for every $s = 1, 2, \dots$

$$|A \cap [1 + (m_s - 1)L, m_s L]| \geq L\delta/2$$

holds. Let $I_s := [1 + (m_s - 1)L, m_s L]$ for every $s = 1, 2, \dots$. We can find an arithmetic progression $P' \subseteq A \cap I_s$ of length $N(1/(4kM), k)$ by Szemerédi's theorem (Proposition 1.1). Let

$$S_j := \left[-M + \frac{j-1}{2k}, -M + \frac{j}{2k} \right), \quad B_j = \{x \in P' \mid R(x) \in S_j\}$$

for every $j = 1, 2, \dots, 4kM$. We partition the arithmetic progression P' into small $4kM$ sets B_j . Since at least one B_j satisfies $|B_j| \geq |P'|/(4kM)$, there exists an integer $q \in \{1, 2, \dots, 4kM\}$ such that B_q contains at least one arithmetic progression P of length k by Szemerédi's theorem (Proposition 1.1). Let $P = \{b(j)\}_{j=0}^{k-1}$. From the triangle inequality (N3), it follows that

$$(3.1) \quad N_P(a) = N_P(f - R) \leq N_P(f) + N_P(R).$$

From $P \subseteq B_q$, the inequality $|\Delta[R \circ b](j)| \leq 1/2k$ holds for all $j = 0, 1, \dots, k-1$. Thus the second term can be bounded as follows:

$$N_P(R) = \sum_{j=0}^{k-3} |\Delta^2[R \circ b](j)| \leq \sum_{j=0}^{k-3} (|\Delta[R \circ b](j+1)| + |\Delta[R \circ b](j)|) \leq (k-2) \frac{1}{k} = 1 - \frac{2}{k}.$$

The remaining part is to estimate the first term on the right hand side of (3.1). Let $b(j) = dj + e$ for some $d, e \in \mathbb{N}$. By the mean value theorem, for every $j = 0, 1, \dots, k-1$ there exists $\eta_j, \theta_j \in [0, 1)$ such that

$$\begin{aligned} \Delta^2[f \circ b](j) &= (f \circ b(j+2) - f \circ b(j+1)) - (f \circ b(j+1) - f \circ b(j)) \\ &= d(f' \circ b(j + \eta_j + 1) - f' \circ b(j + \eta_j)) \\ &= d^2 f'' \circ b(j + \eta_j + \theta_j). \end{aligned}$$

Since $b(j) \in P \subseteq P' \subseteq A \cap I_s$ holds, we obtain $e \geq m_s$ and $d \leq L$. Hence we have

$$N_P(f) = \sum_{j=0}^{k-3} |\Delta^2[f \circ b](j)| = \sum_{j=0}^{k-3} d^2 f'' \circ b(j + \eta_j + \theta_j) \leq L^2(k-2) \times o(1) \rightarrow 0$$

as $s \rightarrow \infty$. Therefore if s is sufficiently large, then the following inequality holds:

$$0 \leq N_P(a) \leq N_P(f) + N_P(R) \leq L^2(k-2) \times o(1) + 1 - \frac{2}{k} < 1.$$

Therefore $N_P(a) = 0$. Hence $a(P)$ is an arithmetic progression of length k . $\square$

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