

Construction of a single-peak solution of the Liouville-Gel'fand equation on a two-dimensional domain with a hole

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1 Introduction

We are concerned with the Liouville-Gel'fand equation

$$\begin{cases} \Delta u + \lambda e^u = 0 & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \quad (\text{LG})$$

Here $\lambda > 0$ is a parameter and $\Omega_\varepsilon \subset \mathbb{R}^2$ is a planar domain with a hole whose size is $\varepsilon > 0$. The precise definition of Ω_ε will be introduced later. What we discuss in this article is construction of a solution of (LG) caused by a hole in Ω_ε .

The equation (LG) has an interesting solution structure when a domain is non-simply connected. The case where Ω_ε is an annulus was investigated by S.-S. Lin [7] and Nagasaki and Suzuki [8]. They independently showed that radially symmetric solutions make a branch and it emanates from $(\lambda, u) = (0, 0)$, bends back once and blows up at each point in Ω_ε as $\lambda \downarrow 0$. Moreover, S.-S. Lin found that the branch has infinitely many secondary bifurcation points from which non-radially symmetric solutions emanate. Nagasaki and Suzuki also obtained non-radially symmetric solutions which have rotational symmetry of order k ($k \in \mathbb{N}$) and is large in some sense. Additionally, Dancer [2] showed that the set of bifurcating non-radially symmetric solutions are unbounded in the bifurcation diagram. These results indicate that bifurcating non-radially symmetric solutions connect to the large solutions obtained by Nagasaki and Suzuki. In [5, 6], suggestive evidence of this expectation was given provided that the inside radius of Ω_ε is small.

For a general non-simply connected domain, Chen and C.-C. Lin [1] revealed the existence of a solution whose mass is not equal to $8\pi k$ ($k \in \mathbb{N}$). Furthermore, del Pino, Kowalczyk and Musso [3] proved that for each $k \in \mathbb{N}$, (LG) has a solution blowing up at k different points as $\lambda \rightarrow 0$.

Our motivation is to obtain more detailed information on the solution structure for general non-simply connected domains by extending the results in [5, 6]. What we consider in particular is a solution with one maximum point. In this article, only by a formal argument, we explain how such a solution can be constructed.

2 Construction of a formal solution

We begin with the definition of the domain Ω_ε . Let Ω and $D \subset \mathbb{R}^2$ be bounded domains including the origin. Then, for small $\varepsilon > 0$, we define Ω_ε by

$$\Omega_\varepsilon := \Omega \setminus \overline{(\varepsilon D)} = \{x \in \Omega; \varepsilon^{-1}x \notin \overline{D}\}.$$

The following figure is an example of Ω_ε .

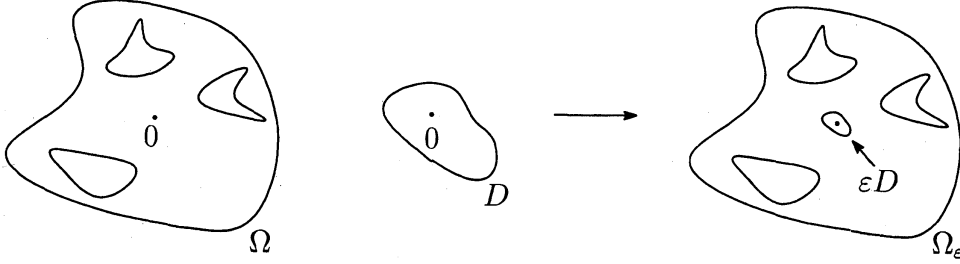


Figure : Domain Ω_ε

As will be seen below, an important factor to construct a formal solution is the regular part of a Green's function for Dirichlet Laplacian in Ω . We denote it by $H^\Omega = H^\Omega(x, y)$. Then, through this section, we assume that

$$\nabla_x H^\Omega(0, 0) \neq 0. \quad (2.1)$$

This assumption leads to success of argument.

In what follows, we find a formal expansion of a solution $(\lambda, u) = (\lambda_\varepsilon, u_\varepsilon)$ by using the method of matched asymptotic expansions. To do this we separate Ω_ε into three parts. Two of them are regions near the boundary ($|x| \sim 1$ and $|x| \sim \varepsilon$) and the other is a region between them. The latter region is supposed to be $|x| \sim \delta_\varepsilon$, where δ_ε has a property $\varepsilon \ll \delta_\varepsilon \ll 1$ ($\varepsilon \rightarrow 0$) and is determined later. To obtain the expansion in this region, it is convenient to perform the change of variables $x = \delta_\varepsilon y$ and $v_\varepsilon(y) = u_\varepsilon(x) + \log(\delta_\varepsilon^2 \lambda_\varepsilon)$. Then we see that v_ε satisfies

$$\Delta v_\varepsilon + e^{v_\varepsilon} = 0 \quad \text{in } (\delta_\varepsilon^{-1} \Omega) \setminus (\varepsilon \delta_\varepsilon^{-1} D).$$

Assuming that v_ε can be expanded as $v_\varepsilon(y) = v_0(y) + \delta_\varepsilon v_1(y) + \dots$, we have

$$\begin{aligned} \Delta v_0 + e^{v_0} &= 0, \\ \Delta v_1 + e^{v_0} v_1 &= 0 \end{aligned} \quad \text{in } \mathbb{R}^2 \setminus \{0\}.$$

Since a solution which we find has one peak, it is appropriate to choose v_0 as

$$v_0(y) = \log \frac{8(1 - \rho^2)}{(1 - \rho^2 + |y - \rho\omega|)^2},$$

or, in polar coordinates $y = (r \cos \theta, r \sin \theta)$,

$$v_0(y) = \log \frac{8(1 - \rho^2)}{r^2 \{r + r^{-1} - 2\rho \cos(\theta - \gamma)\}^2}. \quad (2.2)$$

Here $\rho \in (0, 1)$ and $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ are parameters and $\omega = (\cos \gamma, \sin \gamma)$. Substituting this into the equation for v_1 , we have

$$\Delta v_1 + \frac{8(1 - \rho^2)}{r^2 \{r + r^{-1} - 2\rho \cos(\theta - \gamma)\}^2} v_1 = 0 \quad \text{in } \mathbb{R}^2 \setminus \{0\}. \quad (2.3)$$

To determine v_1 , boundary conditions at the origin and infinity is needed. They are obtained as matching conditions, and therefore we consider the expansion near the boundary. First we treat the region $|x| \sim 1$. We formally expand $u_\varepsilon(x) = u_0(x) + \delta_\varepsilon u_1(x) + \dots$ as $\varepsilon \rightarrow 0$. Then, for $j = 0, 1$, we have

$$\begin{cases} \Delta u_j = 0 & \text{in } \Omega \setminus \{0\}, \\ u_j = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.4)$$

Since the maximum principle implies that u_ε is positive, u_0 must be nonnegative. Hence u_0 is given by

$$u_0(x) = c_0 G_0^\Omega(x). \quad (2.5)$$

Here c_0 is a nonnegative constant and G_0^Ω is a Green's function for the Dirichlet Laplacian in Ω with a singularity at the origin.

We substitute $x = \delta_\varepsilon^{\frac{1}{2}} \tilde{x}$ in (2.5) and $y = \delta_\varepsilon^{-\frac{1}{2}} \tilde{x}$ in (2.2), and compare the asymptotic behavior as $\varepsilon \rightarrow 0$. As $\varepsilon \rightarrow 0$,

$$\begin{aligned} u_0(\delta_\varepsilon^{\frac{1}{2}} \tilde{x}) &= c_0 \left(\frac{1}{2\pi} \log \frac{1}{|\delta_\varepsilon^{\frac{1}{2}} \tilde{x}|} - H_0^\Omega(\delta_\varepsilon^{\frac{1}{2}} \tilde{x}) \right) \\ &\sim c_0 \left(\frac{1}{4\pi} \log \frac{1}{\delta_\varepsilon} + \frac{1}{2\pi} \log \frac{1}{|\tilde{x}|} - H_0^\Omega(0) - \delta_\varepsilon^{\frac{1}{2}} \nabla H_0^\Omega(0) \cdot \tilde{x} \right) \\ &= c_0 \left(\frac{1}{4\pi} \log \frac{1}{\delta_\varepsilon} + \frac{1}{2\pi} \log \frac{1}{\tilde{r}} - H_0^\Omega(0) - \delta_\varepsilon^{\frac{1}{2}} \mu \tilde{r} \cos(\tilde{\theta} - \tau) \right), \\ v_0(\delta_\varepsilon^{-\frac{1}{2}} \tilde{x}) &= \log \frac{8(1 - \rho^2)}{(\delta_\varepsilon^{-\frac{1}{2}} \tilde{r})^2 \{(\delta_\varepsilon^{-\frac{1}{2}} \tilde{r}) + (\delta_\varepsilon^{-\frac{1}{2}} \tilde{r})^{-1} - 2\rho \cos(\tilde{\theta} - \gamma)\}^2} \\ &\sim \log \{8(1 - \rho^2) \delta_\varepsilon^2\} + 4 \log \frac{1}{\tilde{r}} + \delta_\varepsilon^{\frac{1}{2}} \frac{4\rho \cos(\tilde{\theta} - \gamma)}{\tilde{r}} \\ &= \log \{8(1 - \rho^2) \delta_\varepsilon^2\} + 4 \log \frac{1}{\tilde{r}} + \delta_\varepsilon^{\frac{1}{2}} \frac{4\rho \tilde{x} \cdot \tilde{\omega}}{|\tilde{x}|^2}, \end{aligned}$$

where $H_0^\Omega(x) = H^\Omega(x, 0)$, $\nabla H_0^\Omega(0) = (\mu \cos \tau, \mu \sin \tau)$, $\tilde{x} = (\tilde{r} \cos \tilde{\theta}, \tilde{r} \sin \tilde{\theta})$ and $\tilde{\omega} = (\cos \gamma, \sin \gamma)$. By matching two expansions $\log 1/(\delta_\varepsilon^2 \lambda_\varepsilon) + v_0(z) + \delta_\varepsilon v_1(z) + \dots$ and $u_0(x) +$

$\delta_\varepsilon u_1(x) + \dots$ in the region $|x| \sim \delta_\varepsilon^{\frac{1}{2}}$, we have $c_0 = 8\pi$ and

$$u_1(x) = 4\rho \frac{x \cdot \tilde{\omega}}{|x|^2} + o\left(\frac{1}{|x|}\right) \quad \text{as } x \rightarrow 0. \quad (2.6)$$

(2.4) and (2.6) give

$$u_1(x) = 4\rho \left(\frac{x \cdot \tilde{\omega}}{|x|^2} - 2\pi \nabla_y H(x, 0) \cdot \tilde{\omega} \right) + c_1 G_0^\Omega(x),$$

where $c_1 \in \mathbb{R}$ is an undetermined constant. From this,

$$\begin{aligned} u_1(\delta_\varepsilon^{\frac{1}{2}} \tilde{x}) &\sim 4\rho \left(\delta_\varepsilon^{-\frac{1}{2}} \frac{\tilde{x} \cdot \omega}{|\tilde{x}|^2} - 2\pi \mu \omega \cdot \tilde{\omega} \right) + c_1 \left(\frac{1}{2\pi} \log \frac{1}{|\delta_\varepsilon^{\frac{1}{2}} \tilde{x}|} - H_0^\Omega(0) \right) \\ &= \delta_\varepsilon^{-\frac{1}{2}} \frac{4\rho \cos(\tilde{\theta} - \tau)}{\tilde{r}} + \frac{c_1}{4\pi} \log \frac{1}{\delta_\varepsilon} + \frac{c_1}{2\pi} \log \frac{1}{\tilde{r}} - 8\pi \rho \mu \cos(\gamma - \tau) - c_1 H_0^\Omega(0). \end{aligned}$$

Thus it is appropriate to impose the condition

$$v_1(y) = -c_0 \mu r \cos(\theta - \tau) + \frac{c_1}{2\pi} \log \frac{1}{r} + a_1 + o(1) \quad \text{as } r \rightarrow \infty. \quad (2.7)$$

Here a_1 is a constant determined later. Moreover,

$$\begin{aligned} &\frac{c_0}{4\pi} \log \frac{1}{\delta_\varepsilon} - c_0 H_0^\Omega(0) + \frac{c_1}{2\pi} \delta_\varepsilon \log \frac{1}{\delta_\varepsilon} - \delta_\varepsilon (8\pi \rho \mu \cos(\gamma - \tau) + c_1 H_0^\Omega(0)) \\ &= \log \frac{1}{\delta_\varepsilon^2 \lambda_\varepsilon} + \log \{8(1 - \rho^2) \delta_\varepsilon^2\} - \delta_\varepsilon a_1, \end{aligned}$$

which gives

$$\begin{aligned} \lambda_\varepsilon &= 8(1 - \rho^2) \delta_\varepsilon^2 \exp \left[c_0 H_0^\Omega(0) + \frac{c_1}{2\pi} \delta_\varepsilon \log \frac{1}{\delta_\varepsilon} \right. \\ &\quad \left. + \delta_\varepsilon \{a_1 - 8\pi \rho \mu \cos(\gamma - \tau) - c_1 H_0^\Omega(0)\} \right]. \end{aligned} \quad (2.8)$$

Next we consider the expansion in $|x| \sim \varepsilon$. Performing the change of variables $x = \varepsilon z$ and putting $w_\varepsilon(z) = u_\varepsilon(x)$, we have

$$\begin{cases} \Delta w_\varepsilon + \varepsilon^2 \lambda_\varepsilon e^{w_\varepsilon} = 0 & \text{in } \varepsilon^{-1} \Omega_\varepsilon = (\varepsilon^{-1} \Omega) \setminus \overline{D}, \\ w_\varepsilon = 0 & \text{on } \partial(\varepsilon^{-1} \Omega_\varepsilon). \end{cases}$$

Hence the formal expansion $w_\varepsilon(z) = w_0(z) + \delta_\varepsilon w_1(z) + \dots$ gives

$$\begin{cases} \Delta w_j = 0 & \text{in } \mathbb{R}^2 \setminus D, \\ w_j = 0 & \text{on } \partial D \end{cases}$$

for $j = 0, 1$. To find a solution of the above equation, we perform the Kelvin transform

$$w_j^*(z^*) = w_j(z), \quad z^* = \frac{z}{|z|^2}.$$

Then w_j^* satisfies

$$\begin{cases} \Delta w_j^* = 0 & \text{in } D^* \setminus \{0\}, \\ w_j^* = 0 & \text{on } \partial D^*, \end{cases}$$

where $D^* := \{z^* = z/|z|^2; z \in D\}$. Since w_0^* is nonnegative, we see that $w_0^*(z^*) = d_0 G_0^{D^*}(z^*)$ for some constant $d_0 \geq 0$. Thus

$$w_0(z) = d_0 G_0^{D^*}(z^*) = d_0 G_0^{D^*}(z/|z|^2).$$

If $d_0 > 0$, this function has logarithmic growth at $z = \infty$, while v_0 has no such a singularity at $y = 0$. This implies that $d_0 = 0$, that is, $w_0 \equiv 0$. Since w_1 satisfies the same equation as w_0 and must be nonnegative, we have

$$w_1(z) = d_1 G_0^{D^*}(z^*) = d_1 G_0^{D^*}(z/|z|^2), \quad (2.9)$$

where $d_1 \geq 0$ is some undetermined constant.

We compare the expansions in the region $|x| \sim \varepsilon^{\frac{1}{2}} \delta_\varepsilon^{\frac{1}{2}}$. By putting $z = \varepsilon^{-\frac{1}{2}} \delta_\varepsilon^{\frac{1}{2}} \hat{x}$ in (2.9) and $y = \varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \hat{x}$ in (2.2), we have

$$\begin{aligned} w_1(\varepsilon^{-\frac{1}{2}} \delta_\varepsilon^{\frac{1}{2}} \hat{x}) &= d_1 \left(\frac{1}{2\pi} \log |\varepsilon^{-\frac{1}{2}} \delta_\varepsilon^{\frac{1}{2}} \hat{x}| - H_0^{D^*} \left(\varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \frac{\hat{x}}{|\hat{x}|^2} \right) \right) \\ &\sim d_1 \left(\frac{1}{2\pi} \log(\varepsilon^{-1} \delta_\varepsilon) + \frac{1}{2\pi} \log |\varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \hat{x}| - H_0^{D^*}(0) \right), \\ v_0(\varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \hat{x}) &= \log \frac{8(1 - \rho^2)}{(\varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \hat{r})^2 \{ (\varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \hat{r}) + (\varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \hat{r})^{-1} - 2\rho \cos(\hat{\theta} - \gamma) \}^2} \\ &\sim \log \{ 8(1 - \rho^2) \}. \end{aligned}$$

Thus, assuming that two expansions $\log 1/(\delta_\varepsilon^2 \lambda_\varepsilon) + v_0(y) + \delta_\varepsilon v_1(y) + \dots$ and $w_0(z) + \delta_\varepsilon w_1(z) + \dots$ give the same expansion in $|x| \sim \varepsilon^{\frac{1}{2}} \delta_\varepsilon^{\frac{1}{2}}$, we deduce

$$v_1(y) = \frac{d_1}{2\pi} \log r + a_2 + o(1) \quad \text{as } r \rightarrow 0 \quad (2.10)$$

and

$$\log \frac{8(1 - \rho^2)}{\delta_\varepsilon^2 \lambda_\varepsilon} - \delta_\varepsilon a_2 = d_1 \delta_\varepsilon \left(\frac{1}{2\pi} \log(\varepsilon^{-1} \delta_\varepsilon) - H_0^{D^*}(0) \right). \quad (2.11)$$

Here $a_2 \in \mathbb{R}$ is a constant determined later.

Now we solve (2.3) under the conditions (2.7) and (2.10). First we observe that the functions

$$\begin{aligned}\Phi_{\rho,\gamma,1}(z) &= \frac{r - r^{-1}}{r + r^{-1} - 2\rho \cos(\theta - \gamma)}, \\ \Phi_{\rho,\gamma,2}(z) &= \frac{2\cos(\theta - \gamma) - \rho(r + r^{-1})}{r + r^{-1} - 2\rho \cos(\theta - \gamma)}, \\ \Phi_{\rho,\gamma,3}(z) &= \frac{\sin(\theta - \gamma)}{r + r^{-1} - 2\rho \cos(\theta - \gamma)}\end{aligned}$$

are bounded solutions of (2.3). Furthermore, every bounded solution of (2.3) is linear combination of these solutions (see [4], [5]). We also observe what is necessary to solve the equation. Suppose that (2.3), (2.10) and (2.7) has a solution. By a simple computation, we have

$$\begin{aligned}\Phi_{\rho,\gamma,j}(z) &= \begin{cases} 1 + 2\rho r^{-1} \cos(\theta - \gamma) + O(r^{-2}) & (j = 1) \\ -\rho\{1 - 2(\rho^{-1} - \rho)r^{-1} \cos(\theta - \gamma)\} + O(r^{-2}) & (j = 2) \\ r^{-1} \cos(\theta - \gamma) + O(r^{-2}) & (j = 3) \end{cases} \quad \text{as } r \rightarrow \infty, \\ \Phi_{\rho,\gamma,j}(z) &= \begin{cases} -1 + 2\rho r \cos(\theta - \gamma) + O(r^2) & (j = 1) \\ -\rho\{1 - 2(\rho^{-1} - \rho)r \cos(\theta - \gamma)\} + O(r^2) & (j = 2) \\ r \cos(\theta - \gamma) + O(r^2) & (j = 3) \end{cases} \quad \text{as } r \rightarrow 0.\end{aligned}$$

Hence, as $r \rightarrow \infty$,

$$\begin{aligned}& r \left(\frac{\partial w_1}{\partial r} \Phi_{\rho,\gamma,j} - w_1 \frac{\partial \Phi_{\rho,\gamma,j}}{\partial r} \right) \\ &= \begin{cases} -c_0 \mu r \cos(\theta - \tau) - \frac{c_1}{2\pi} - 4c_0 \rho \mu \cos(\theta - \tau) \cos(\theta - \gamma) + o(1) & (j = 1) \\ -\rho\{-c_0 \mu r \cos(\theta - \tau) - \frac{c_1}{2\pi} \\ \quad + 4c_0(\rho^{-1} - \rho)\mu \cos(\theta - \tau) \cos(\theta - \gamma)\} + o(1) & (j = 2) \\ -2c_0 \mu \cos(\theta - \tau) \sin(\theta - \gamma) + o(1) & (j = 3) \end{cases},\end{aligned}$$

and as $r \rightarrow 0$,

$$r \left(\frac{\partial w_1}{\partial r} \Phi_{\rho,\gamma,j} - w_1 \frac{\partial \Phi_{\rho,\gamma,j}}{\partial r} \right) = \begin{cases} -d_1/(2\pi) + o(1) & (j = 1) \\ -(\rho d_1)/(2\pi) + o(1) & (j = 2) \\ o(1) & (j = 3) \end{cases}.$$

Thus multiplying both sides of (2.3) by $\Phi_{\rho,\gamma,j}$ and integrating give

$$0 = \left[\int_0^{2\pi} r \left(\frac{\partial w_1}{\partial r} \Phi_{\rho,\gamma,j} - w_1 \frac{\partial \Phi_{\rho,\gamma,j}}{\partial r} \right) d\theta \right]_{r=0}^{\infty}$$

$$= \begin{cases} -c_1 - 4\pi c_0 \rho \mu \cos(\gamma - \tau) + d_1 & (j=1) \\ -\rho \{-c_1 + 4\pi c_0 (\rho^{-1} - \rho) \mu \cos(\gamma - \tau) - d_1\} & (j=2) \\ 2\pi c_0 \mu \sin(\gamma - \tau) & (j=3) \end{cases}.$$

Note that $\mu > 0$ from (2.1) and $d_1 \geq 0$. Therefore the above relations yield

$$\begin{aligned} \gamma &= \tau, \\ c_1 &= 2\pi c_0 \mu \left(\frac{1}{\rho} - 2\rho \right) = 16\pi^2 \mu \left(\frac{1}{\rho} - 2\rho \right), \\ d_1 &= \frac{2\pi c_0 \mu}{\rho} = \frac{16\pi^2 \mu}{\rho}. \end{aligned}$$

Conversely, it can be checked that the function

$$V(y) = -c_0 \mu \left\{ \left(\frac{1}{\rho} - \rho \right) \Phi_{\rho, \tau, 1}(y) \log r + \Phi_{\rho, \tau, 1}(y) \log r - \frac{1}{\rho} + r \cos(\theta - \tau) \right\}$$

is a solution of (2.3), (2.10), (2.7) provided that γ, c_1 and d_1 satisfy the above relations. Thus, by setting $a_2 = a_3 = (c_0 \mu)/\rho$, we see that v_1 is given by

$$v_1(y) = V(y) + \alpha \Phi_{\rho, \tau, 3}(y),$$

where $\alpha \in \mathbb{R}$ is an arbitrary constant.

From (2.8) and (2.11), it can be shown that

$$\delta_\varepsilon = \frac{\rho}{2\pi\mu} \frac{\log \log \frac{1}{\varepsilon}}{\log \frac{1}{\varepsilon}} (1 + o(1))$$

as $\varepsilon \rightarrow 0$. Hence setting $\eta_\varepsilon = 2\pi\mu\delta_\varepsilon/\rho$, we have

$$\begin{aligned} \eta_\varepsilon &= \frac{\log \log \frac{1}{\varepsilon}}{\log \frac{1}{\varepsilon}} (1 + o(1)), \\ \lambda_\varepsilon &= \frac{4\rho^2(1 - \rho^2)e^{8\pi H_0^\Omega(0)}}{\mu\pi} \eta_\varepsilon^2 (1 + o(1)). \end{aligned}$$

This indicates that u_ε appears through a saddle-node bifurcation when $\rho \sim 1/\sqrt{2}$.

Finally we discuss how the constant α is determined. From the formal expansion obtained above, the solution u_ε is expected to expand as

$$u_\varepsilon(x) = \log \frac{1}{\delta_\varepsilon^2 \lambda_\varepsilon} + v_0(y) + \delta_\varepsilon V(y) + \alpha \delta_\varepsilon \Phi_{\rho, \tau, 3}(y) + (h.o.t.)$$

provided that $|y| \sim 1$. This expansion is valid only in the region $|y| \sim 1$, and therefore we add a correction term to obtain an approximation in the whole region of Ω_ε . We define a

correction function v_c as a solution of

$$\begin{cases} \Delta v_c = 0 & \text{in } \delta_\varepsilon^{-1}\Omega_\varepsilon, \\ v_c = -\log \frac{1}{\delta_\varepsilon^2 \lambda_\varepsilon} - v_0 - \delta_\varepsilon V & \text{on } \partial(\delta_\varepsilon^{-1}\Omega_\varepsilon). \end{cases}$$

Then one can show that

$$|v_c(y)| \leq C (\varepsilon r^{-1} + \delta_\varepsilon^2 r^2)$$

for all $y \in \delta_\varepsilon^{-1}\Omega_\varepsilon$, and

$$v_c(y) = \delta_\varepsilon^2 \xi(y) + o(\delta_\varepsilon^2)$$

locally uniformly for $y \in \mathbb{R} \setminus \{0\}$ as $\varepsilon \rightarrow 0$. Here $C > 0$ is a constant independent of ε and ξ is a function determined by the regular part of a Green's function in Ω (we omit the detail of ξ). Consequently, we obtain the expansion

$$u_\varepsilon(x) = \log \frac{1}{\delta_\varepsilon^2 \lambda_\varepsilon} + U_\varepsilon(y) + \alpha \delta_\varepsilon \Phi_{\rho,\tau,3}(y) + r_\varepsilon(y),$$

where $U_\varepsilon = v_0 + \delta_\varepsilon V + v_c$ and r_ε is a remainder term. r_ε is expected to be small on whole domain Ω_ε in some appropriate topology.

We set $\eta_\varepsilon(y) = \alpha \delta_\varepsilon \Phi_{\rho,\tau,3}(y) + r_\varepsilon(y)$ and substitute the above expansion into (LG). Then the equation is rewritten as

$$\mathcal{L}(\eta_\varepsilon) + F(\eta_\varepsilon) + R_\varepsilon = 0,$$

where

$$\begin{aligned} \mathcal{L}(\eta_\varepsilon) &= \Delta \eta_\varepsilon + e^{U_\varepsilon} \eta_\varepsilon, \\ F(\eta_\varepsilon) &= e^{U_\varepsilon} (e^{\eta_\varepsilon} - 1 - \eta_\varepsilon), \\ R_\varepsilon &= \Delta U_\varepsilon + e^{U_\varepsilon}. \end{aligned}$$

To determine the constant α , we multiply the above equation by $\Phi_{\rho,\tau,3}$ and integrate over $\delta_\varepsilon^{-1}\Omega_\varepsilon$. Then we have

$$\begin{aligned} \int_{\delta_\varepsilon^{-1}\Omega_\varepsilon} \mathcal{L}(\eta_\varepsilon) \Phi_{\rho,\tau,3} dx &\sim \int_{\delta_\varepsilon^{-1}\Omega_\varepsilon} \eta_\varepsilon \mathcal{L}(\Phi_{\rho,\tau,3}) dx \\ &\sim \alpha \delta_\varepsilon \int_{\delta_\varepsilon^{-1}\Omega_\varepsilon} (e^{U_\varepsilon} - e^{v_0}) \Phi_{\rho,\tau,3}^2 dx \\ &\sim \alpha \delta_\varepsilon^2 \int_{\mathbb{R}^2} e^{v_0} V \Phi_{\rho,\tau,3}^2 dx \\ &= \frac{4\pi^2 \mu}{\rho} \alpha \delta_\varepsilon^2, \\ \int_{\delta_\varepsilon^{-1}\Omega_\varepsilon} F(\eta_\varepsilon) \Phi_{\rho,\tau,3} dx &\sim \int_{\delta_\varepsilon^{-1}\Omega_\varepsilon} e^{U_\varepsilon} \eta_\varepsilon^2 \Phi_{\rho,\tau,3} dx \end{aligned}$$

$$\begin{aligned} &\sim \alpha^2 \delta_\varepsilon^2 \int_{\mathbb{R}^2} e^{v_0} \Phi_{\rho,\tau,3}^3 dx \\ &= 0. \end{aligned}$$

From the definition of U_ε , we see that

$$R_\varepsilon = e^{v_0} (e^{\delta_\varepsilon V + v_c} - 1 - \delta_\varepsilon V).$$

Hence

$$\begin{aligned} \int_{\delta_\varepsilon^{-1} \Omega_\varepsilon} R_\varepsilon \Phi_{\rho,\tau,3} dx &\sim \int_{\delta_\varepsilon^{-1} \Omega_\varepsilon} e^{v_0} \{v_c + (\delta_\varepsilon V + v_c)^2\} \Phi_{\rho,\tau,3} dx \\ &\sim \delta_\varepsilon^2 \int_{\mathbb{R}^2} e^{v_0} (\xi + V^2) \Phi_{\rho,\tau,3} dx \\ &= \delta_\varepsilon^2 \int_{\mathbb{R}^2} e^{v_0} \xi \Phi_{\rho,\tau,3} dx. \end{aligned}$$

Thus α is given by

$$\alpha = \frac{\rho}{4\pi^2 \mu} \int_{\mathbb{R}^2} e^{v_0} \xi \Phi_{\rho,\tau,3} dx.$$

At the end, we summarize what we obtained.

Main Result 1. Assume (2.1). Then, for small ε and $\rho \in (0, 1)$, we can construct a “formal” solution $(\lambda_\varepsilon, u_\varepsilon)$ of (LG) with the following expansion :

$$\begin{aligned} \lambda_\varepsilon &\sim \frac{4\rho^2(1-\rho^2)e^{8\pi H_0^\Omega(0)}}{\mu\pi} \left(\frac{\log \log \frac{1}{\varepsilon}}{\log \frac{1}{\varepsilon}} \right)^2, \\ u_\varepsilon(x) &\sim \log \frac{1}{\delta_\varepsilon^2 \lambda_\varepsilon} + v_0(\delta_\varepsilon^{-1}x) + \delta_\varepsilon v_1(\delta_\varepsilon^{-1}x) + v_c(\delta_\varepsilon^{-1}x) \end{aligned} \quad \text{as } \varepsilon \rightarrow 0.$$

Here constants and functions are chosen suitably as discussed above.

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