

Nominal Sets, Equivariance Reasoning, and Variable Binding

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1 Equivariance reasoning

FM techniques are a methodology of thinking about syntax and in particular about syntax with binding. They take their name from the original FM (Fraenkel-Mostowski) theory of sets presented in my thesis [4]. For this document we use a different way of presenting FM techniques based on the idea of a **Nominal Set**, which is a set equipped with certain algebraic properties (just as a group, ring, or field, is a set equipped with certain properties), see Definition 1.2. First we need background machinery:

Definition 1.1. Fix a countably infinite set of atoms $\mathbb{A}$. Write typical elements $a, b, c, \dots \in \mathbb{A}$. For $a, b \in \mathbb{A}$ write $(a\ b)$ for the function $\mathbb{A} \rightarrow \mathbb{A}$ such that $a \mapsto b$ and $b \mapsto a$ and $n \mapsto n$ for all other atoms $n \neq a, b$. This is a bijection with inverse itself, write $P_{\mathbb{A}}$ for the group generated by $(a\ b)$ for all $a, b \in \mathbb{A}$ under functional composition $\circ$. Write typical elements $\pi, \pi' \in P_{\mathbb{A}}$ and $\text{Id} \in P_{\mathbb{A}}$ for the identity.

Definition 1.2. A **Nominal Set** is a pair $\langle X, \cdot \rangle$ of an underlying set X and **permutation action** $\cdot$ (written infix) of type $P_{\mathbb{A}} \times X \rightarrow X$ and satisfying the usual axioms, namely $\pi \cdot (\pi' \cdot x) = \pi \circ \pi' \cdot x$ and $\text{Id} \cdot x = x$. The permutation action also satisfies a finiteness condition omitted here.¹

The point is that finite labelled trees, and hence the standard model of syntax but also a natural model of *proofs* as trees, are Nominal Sets: the permutation action is given pointwise by the action on the labels. Thus for example natural numbers $\mathbb{N}$ satisfy a trivial permutation action given by $\pi \cdot n = n$ always. A datatype of trees for terms of the λ -calculus (using $\mathbb{A}$ as variable names)

$$\Lambda \cong \mathbb{A} + \Lambda \times \Lambda + \Lambda \times \Lambda \quad (1)$$

is also a Nominal Set with the permutation action given pointwise by the action on the atoms labelling the tree. Furthermore we can represent a theory of α -equivalence on these terms as a subset of “well-formed” trees in the inductively defined set

$$T \cong \mathbb{A} + T \times T + \Lambda \times T \times T, \quad (2)$$

namely those inductively constructed using the rules

$$a =_{\alpha} a \quad (\text{Var}) \quad \frac{t_1 =_{\alpha} t'_1 \quad t_2 =_{\alpha} t'_2}{t_1 t_2 =_{\alpha} t'_1 t'_2} \quad (\text{App}) \quad \frac{t\{c/a\} =_{\alpha} t'\{c/a'\}}{\lambda a t =_{\alpha} \lambda a' t'} \quad (\text{Lam}) \quad (3)$$

where in (Lam) there is a side-condition that $c \notin \{a, a'\} \cup n(t) \cup n(t')$ (thus ‘fresh’ for the conclusion). The usefulness of this way of looking at syntax and properties of syntax as Nominal Sets begins with the following trivial theorem:

Theorem 1.3. *If a property (on trees) is defined (inductively) using predicates whose validity is invariant under permuting atoms, then the property is invariant under permuting atoms.*

Here “invariant under permuting...” means “given some valid instance of the property, a permutation π uniformly applied to its arguments yields another valid instance”.

We have an example in the property of well-formedness of proof-trees of $=_{\alpha}$ given in (3). $\overline{a =_{\alpha} a}$ is a valid instance of (Var) and if we apply $(b\ a)$ to this we obtain $\overline{b =_{\alpha} b}$, which is also a valid instance of

¹See [3, eq. 3], [5, eq. 4], and ‘finite support’ in [2, Def. 3.3].

(**Var**). The case of (**App**) is simple. A permutation applied to a valid instance of (**Lam**) is also a valid instance of (**Lam**) because $c \notin \{a, a'\} \cup n(t) \cup n(t')$ if and only if $\pi \cdot c \notin \{\pi \cdot a, \pi \cdot a'\} \cup n(\pi \cdot t) \cup n(\pi \cdot t')$.²

We now have a very concrete demonstration that proofs of $=_\alpha$ are invariant under permutation; we permute the atoms in the proof as a tree. We can take this further. Consider proving transitivity of $=_\alpha$ by induction on proof-trees. The induction predicate is (in words) “given a valid proof-tree Π concluding in $t =_\alpha t'$, for all valid proof-trees Π' concluding in $t' =_\alpha t''$, there exists a valid proof-tree Π'' concluding in $t =_\alpha t''$ ”. This property is constructed using predicates invariant under permutations (validity of proofs of $=_\alpha$) and so is itself invariant under permutations. Thus from Theorem 1.3 we know if we have the inductive hypothesis of Π , we have it of $\pi \cdot \Pi$ for any permutation π .

We proceed by induction on Π . The case of (**Lam**) for $t = \lambda a s$ and $t' = \lambda a' s'$ causes problems: we may assume Π proves $s[c/a] =_\alpha s'[c/a']$ and Π' proves $s'[c'/a'] =_\alpha s''[c'/a'']$ and we assume the inductive predicate for Π , but we do not know $c = c'$ so we cannot proceed. However, we *can* apply a permutation $(d\ c)$ to Π , and $(d\ c')$ to Π' , for d chosen completely fresh. Now we have valid proofs $(d\ c) \cdot \Pi$ concluding in $s[d/a] =_\alpha s'[d/a']$ and $(d\ c') \cdot \Pi'$ concluding in $s'[d/a'] =_\alpha s''[d/a'']$, and also the inductive predicate for $(d\ c) \cdot \Pi$. We can now complete the proof of transitivity.

Just these ideas of permutations have already been adopted and put to use by other authors also in published work (see for example [6]).

2 Taking it further

There is an equivalence class of proofs concluding in $\lambda a t =_\alpha \lambda a' t'$, one for each fresh c ; we can take it as an object in its own right. This is an instance of FM abstraction $[A]X$ which exists for any Nominal Set X by virtue of the permutation action, which lets us rename atoms and construct an equivalence class in the general case (see [2, Section 5]). We can apply this to syntax as well as proofs:

$$\Lambda_\alpha \cong A + \Lambda_\alpha \times \Lambda_\alpha + [A]\Lambda_\alpha \quad (4)$$

is a datatype of λ -terms up to α -equivalence. An element of $[A]\Lambda_\alpha$ is (concretely) an equivalence class of pairs $\langle a, t \rangle$ for $a \in A$ a ‘bound atom’ fresh for the other atoms in the ‘body’ $t \in \Lambda_\alpha$.

There are various ways of taking this further; Nominal Sets form a category, the Schanuel Topos. Because it is a topos we can construct a general theory of abstractions and equivariance reasoning within it (this is FMCat in [5, Section 2], see also [2, p.21]). Nominal Sets are also a special case of a general theory of FM sets, see [4] and [2]. Nominal Sets can be axiomatised in first-order logic, see [7]. A team in Cambridge has developed FreshML, a programming language based on these principles in which we can program using abstractions and permutations, see [1]. Finally, I am currently implementing FM sets in Isabelle, see [4]. Further reading can be found in any of the references below, and my homepage www.cl.cam.ac.uk/~mjg1003.

References

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²This is not the case if we try to base the theorem on substitutions generated by $[b/a]$ instead of permutations generated by $(b\ a) = [b/a, a/b]$.