

On the Grossman representations of the automorphism groups of free groups

東京理科大学理学部第二部数学科 佐藤 隆夫*

Satoh, Takao

Faculty of Science Division II, Tokyo University of Science

Abstract

In this announcement, we show that for any $k \geq 2$ the subgroup of the automorphism group of a free group consisting of all automorphisms which act trivially on the $(k+1)$ -st derived quotient of the free group is not finitely generated in some stable range. In order to show it, we consider a slight generalization of results of Grossman in [6], and a sequence of descending filtrations of the IA-automorphism group of a free group.

Let F_n be the free group of rank $n \geq 2$ with basis $x_1, \dots, x_n$, and $\text{Aut } F_n$ the automorphism group of F_n . Let F_n^{ab} be the abelianization of F_n . The kernel of the natural homomorphism $\text{Aut } F_n \rightarrow \text{Aut } F_n^{\text{ab}}$ is called the IA-automorphism group of F_n , and is denoted by $\text{IA}(F_n)$, or simply IA_n . Over the last three decades, IA_n has actively been studied with the Andreadakis-Johnson filtration and the Johnson homomorphisms by many authors. By using verbal subgroups of F_n , Grossman [6] introduced a descending filtration of IA_n , which is essentially different from the Andreadakis-Johnson filtration. For any $k \geq 1$, let $V_{n,k}$ be the verbal subgroups of F_n defined by $V_{n,1} := F_n$ and $V_{n,k} := V_{n,k-1}(X^2)$ for $k \geq 2$. (For details for verbal subgroups, see Subsection ??.) Denote by $K_{n,k}$ by the kernel of the natural homomorphism $\text{Aut } F_n \rightarrow \text{Aut } V_{n,k}^{\text{ab}}$, where for a group G , the abelianization of G is written as G^{ab} . Then Grossman showed that the subgroups $K_{n,k}$ defines a descending filtration $\text{IA}_n \supset K_{n,1} \supset K_{n,2} \supset \dots$, and they have the trivial intersection $\bigcap_{k \geq 1} K_{n,k} = \{1\}$. Since each of $V_{n,k}^{\text{ab}}$ is a free abelian group of finite rank, the homomorphisms $\text{Aut } F_n \rightarrow \text{Aut } V_{n,k}^{\text{ab}}$ are considered as linear representations of $\text{Aut } F_n$. As is well-known due to Formanek and Procesi [5], $\text{Aut } F_n$ is not linear. Grossman's result $\bigcap_{k \geq 1} K_{n,k} = \{1\}$ shows that $\text{Aut } F_n$ is residually linear.

In this announcement, to begin with, we slightly generalize Grossman's results with the same arguments as hers by replacing 2-torsion with d -torsion. We crucially use the d -torsion generalization to prove Theorem 4. For any $d \geq 2$ and $k \geq 1$, let $V_{n,k}$ be the verbal subgroups of F_n defined by

$$V_{n,d,1} := F_n, \quad V_{n,d,k} := V_{n,d,k-1}(X^d, XYX^{-1}Y^{-1}) \quad (k \geq 2).$$

*e-address: takao@rs.tus.ac.jp

Denote by $K_{n,d,k}$ by the kernel of the natural homomorphism $\rho_{n,d,k} : \text{Aut } F_n \rightarrow \text{Aut } V_{n,d,k}^{\text{ab}}$. Grossman's subgroup $K_{n,k}$ is nothing but $K_{n,2,k}$. We call $\rho_{n,d,k}$ the Grossman representation of $\text{Aut } F_n$. Remark that for a group G the verbal subgroup $G(X^2)$ contains the commutator subgroup of G . For $d \geq 3$, the verbal subgroup $G(X^d)$ does not always contain the commutator subgroup of G . By using the same arguments as those by Grossman, we obtain the following.

Theorem 1. (1) For any $n, d \geq 2$,

$$\text{IA}_n = K_{n,d,1} \supset K_{n,d,2} \supset K_{n,d,3} \supset \cdots$$

(2) For any $n, d \geq 2$,

$$\bigcap_{k \geq 1} K_{n,d,k} = \{1\}.$$

We remark that the successive quotient group $K_{n,d,k}/K_{n,d,k+1}$ is not abelian in general.

Since the successive quotient group $V_{n,d,k}/V_{n,d,k+1}$ for any $k \geq 1$ is a finite abelian group consisting of d -torsions, each $V_{n,d,k}$ is a free group of finite rank. We see the faithfulness of the action of $\text{Aut } F_n$ on $V_{n,d,k}$.

Theorem 2. For any $n \geq 2$, $d \geq 2$ and $k \geq 1$, the map $\text{Aut } F_n \rightarrow \text{Aut } V_{n,d,k}$ is injective.

From this theorem, we can consider $\text{Aut } F_n$ as a subgroup of $\text{Aut } V_{n,d,k}$. From this viewpoint, as a corollary, we show that $K_{2,d,k}$ coincide with the inner automorphism group of $V_{2,d,k}$. Moreover, by using the first Johnson homomorphism of $\text{Aut } V_{n,d,k}$, we show that the abelianization of $K_{n,d,k}$ contains a free abelian group which is isomorphic to $V_{n,d,k}^{\text{ab}}$.

In general, the action of $\text{Aut } F_n$ on $V_{n,d,k}^{\text{ab}}$ does not factor through $\text{Aut } F_n^{\text{ab}}$. In other words, IA_n does not act on $V_{n,d,k}^{\text{ab}}$ trivially. In order to describe the action of $\text{Aut } F_n$, we need a basis of $V_{n,d,k}$. However, the rank of $V_{n,d,k}$ is too large to write down a basis of $V_{n,d,k}$ in general. In this announcement, we consider the case where $k = 2$ as a first step. We study the $\text{Aut } F_n$ -module structure of $V_{n,d,2}^{\text{ab}}$. In particular, we obtain the following.

Theorem 3. For any $n \geq 2$ and $d \geq 2$, the module $V_{n,d,2}^{\text{ab}}$ decomposes to $W_{n,d} \oplus W_{n,d}^\perp$ as an $\text{Aut } F_n$ -module. Moreover, the module $W_{n,d}$ is isomorphic to F_n^{ab} .

We have obtained explicit basis of $W_{n,d}$ and $W_{n,d}^\perp$.

Let $F_n = D_n(1) \supset D_n(2) \supset D_n(3) \supset \cdots$ be the derived series of F_n . Denote by $\mathcal{K}_n^{(k)}$ the kernel of the natural homomorphism $\text{Aut } F_n \rightarrow \text{Aut } (F_n/D_n(k+1))$ for any $k \geq 2$. Notice that $\mathcal{K}_n^{(1)} = \text{IA}_n$ and $\mathcal{K}_n^{(k)} \subset K_{n,d,k}$ for any $d \geq 2$. We show the following theorem.

Theorem 4. If $\binom{n-1}{2} \geq 2^{k-1}$, then $(\mathcal{K}_n^{(k)})^{\text{ab}}$ contains a free abelian group of infinite rank. In particular, $\mathcal{K}_n^{(k)}$ is not finitely generated.

In order to show the above theorem, we use the first Johnson homomorphism of $\text{Aut } V_{n,d,k}$ to detect infinitely many linearly independent elements in $(\mathcal{K}_n^{(k)})^{\text{ab}}$. In our previous paper [15], we obtained the result in the case where $k = 2$. This theorem is a generalization of our result in [15] for general $k \geq 2$. At the present stage, we cannot give any comment on whether the statement in Theorem 4 holds or not in the unstable range.

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