

The space of maps from a real projective space to a toric variety

Andrzej Kozłowski University of Warsaw
大野真裕 (Masahiro Ohno) 電気通信大学
山口耕平 (Kohhei Yamaguchi) 電気通信大学

Abstract

The main purpose of this note is consider the homotopy type of the space of algebraic maps from a real projective space to a projective smooth toric variety as in [14]. The main result of this paper (Theorem 1.1) is also regarded as one of generalizations of the previous work of the second and third authors [19].

An irreducible normal algebraic variety X (over $\mathbb{C}$) is called a *toric variety* if it has an algebraic action of algebraic torus $\mathbb{T}^r = (\mathbb{C}^*)^r$, such that the orbit $\mathbb{T}^r \cdot *$ of some point $*$ $\in X$ is dense in X and isomorphic to $\mathbb{T}^r$. A finite collection Σ of strongly convex rational polyhedral cones in $\mathbb{R}^n$ is called a *fan* if every face of element of Σ is belongs to Σ and the intersection of any two elements of Σ is a face of each. It is known that A toric variety X is completely characterized up to isomorphism by its fan Σ , and we denote by X_Σ the corresponding toric variety. For an n dimensional lattice polytope P , we denote by Σ_P the *normal fan* of P in $\mathbb{R}^n$. It is known that the toric variety X_Σ is projective if and only if $\Sigma = \Sigma_P$ for some n dimensional lattice polytope P in $\mathbb{R}^n$.

We shall use the symbols $\{z_k\}_{k=1}^r$ to denote variables of polynomials, and for $f_1, \dots, f_s \in \mathbb{C}[z_1, \dots, z_r]$, let $V(f_1, \dots, f_s)$ denote the affine variety $V(f_1, \dots, f_s) = \{\mathbf{x} \in \mathbb{C}^r \mid f_k(\mathbf{x}) = 0 \text{ for each } 1 \leq k \leq s\}$.

Let $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ denote the set of all one dimensional cones (or called a *ray*) in a fan Σ , and let $\mathbf{n}_k \in \mathbb{Z}^n$ denote the generator of $\rho_k \cap \mathbb{Z}^n$ called the *primitive element* of ρ_k for each $1 \leq k \leq r$. Define the affine variety $Z_\Sigma \subset \mathbb{C}^r$ by $Z_\Sigma = V(z^\sigma \mid \sigma \in \Sigma)$, where z^σ denotes the monomial given by $z^\sigma = \prod_{1 \leq k \leq r, \mathbf{n}_k \notin \sigma} z_k \in \mathbb{Z}[z_1, \dots, z_r]$ ($\sigma \in \Sigma$). Let $G_\Sigma \subset \mathbb{T}^r$ denote the subgroup consisting of all r -tuples $(\mu_1, \dots, \mu_r) \in \mathbb{T}^r$ such that $\prod_{k=1}^r \mu_k^{\langle \mathbf{m}, \mathbf{n}_k \rangle} =$

1 for any $\mathbf{m} \in \mathbb{Z}^n$, where we set $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^n x_k y_k$ for $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$. We say that a set of primitive elements $\{\mathbf{n}_{i_1}, \dots, \mathbf{n}_{i_s}\}$ is *primitive* if they do not lie in any cone in Σ but every proper subset does. It is known that

$$Z_\Sigma = \bigcup_{\{\mathbf{n}_{i_1}, \dots, \mathbf{n}_{i_s}\}: \text{primitive}} V(z_{i_1}, \dots, z_{i_s}).$$

Note that Z_Σ is a closed variety of dimension $2(r - r_{\min})$, where we set

$$r_{\min} = \min \{s \in \mathbb{Z}_{\geq 1} \mid \{\mathbf{n}_{i_1}, \dots, \mathbf{n}_{i_s}\} \text{ is primitive}\}.$$

It is also known that if the set $\{\mathbf{n}_1, \dots, \mathbf{n}_r\}$ spans $\mathbb{R}^n$, there is an isomorphism $X_\Sigma \cong (\mathbb{C}^r \setminus Z_\Sigma)/G_\Sigma$, where the group G_Σ acts on the complement $\mathbb{C}^r \setminus Z_\Sigma$ by the coordinate-wise multiplication.

For connected spaces X and Y , let $\text{Map}(X, Y)$ be the space of all continuous maps $f : X \rightarrow Y$, and let $\text{Map}^*(X, Y)$ denote the corresponding subspace of all based continuous maps. If $m \geq 2$ and $g \in \text{Map}^*(\mathbb{RP}^{m-1}, X)$, let $F(\mathbb{RP}^m, X; g)$ denote the subspace of $\text{Map}^*(\mathbb{RP}^m, X)$ given by

$$F(\mathbb{RP}^m, X; g) = \{f \in \text{Map}^*(\mathbb{RP}^m, X) : f|_{\mathbb{RP}^{m-1}} = g\},$$

where we identify $\mathbb{RP}^{m-1} \subset \mathbb{RP}^m$ by putting $x_m = 0$. It is known that there is a homotopy equivalence $F(\mathbb{RP}^m, X; g) \simeq \Omega^m X$.

From now on, we assume that the following two conditions are satisfied:

- (1.1) Let Σ be a fan in $\mathbb{R}^n$, $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ be the set of all one-dimension cones in Σ , and all primitive elements $\{\mathbf{n}_1, \dots, \mathbf{n}_r\}$ of the fan Σ spans $\mathbb{R}^n$, where $\mathbf{n}_k \in \mathbb{Z}^n$ denotes the primitive element of ρ_k for $1 \leq k \leq r$.
- (1.2) Let $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ be an r -tuple of integers such that $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}$.

Then, we can identify $X_\Sigma = (\mathbb{C}^r \setminus Z_\Sigma)/G_\Sigma$ as above. For each $(a_1, \dots, a_r) \in \mathbb{C}^r \setminus Z_\Sigma$, we denote by $[a_1, \dots, a_r]$ the corresponding element of X_Σ . Let $\mathcal{H}_{d,m} \subset \mathbb{C}[z_0, \dots, z_m]$ denote the subspace consisting of all homogeneous polynomials of degree d . Let $A_D(m)$ denote the space

$$A_D(m) = \mathcal{H}_{d_1,m} \times \mathcal{H}_{d_2,m} \times \dots \times \mathcal{H}_{d_r,m}$$

and let $A_{D,\Sigma}(m) \subset A_D(m)$ denote the subspace consisting of all r -tuples $(f_1, \dots, f_r) \in A_D(m)$ such that $(f_1(\mathbf{x}), \dots, f_r(\mathbf{x})) \notin Z_\Sigma$ for any $\mathbf{x} \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}$. Let $x_0 \in X_\Sigma$ be the base point such that $x_0 = [x_{1,0}, \dots, x_{r,0}]$ for some fixed $(x_{1,0}, \dots, x_{r,0}) \in \mathbb{C}^r \setminus Z_\Sigma$. Then let $A_D(m, X_\Sigma) \subset A_{D,\Sigma}(m)$ denote

the subspace consisting of all r -tuples $(f_1, \dots, f_r) \in A_{D,\Sigma}(m)$ satisfying the condition $(f_1(\mathbf{e}_1), \dots, f_r(\mathbf{e}_1)) = (x_{1,0}, \dots, x_{r,0})$, where $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^{m+1}$, and let us choose $[\mathbf{e}_1] = [1 : 0 : \dots : 0]$ as the base-point of $\mathbb{RP}^m$. Define the natural map $j'_D : A_{D,\Sigma}(m) \rightarrow \text{Map}(\mathbb{RP}^m, X_\Sigma)$ by

$$j'_D(f_1, \dots, f_r)([x_0 : \dots : x_m]) = [f_1(\mathbf{x}), \dots, f_r(\mathbf{x})]$$

for $\mathbf{x} = (x_0, \dots, x_m) \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}$. Since the space $A_{D,\Sigma}(m)$ is connected, the image of j'_D lies in a connected component of $\text{Map}(\mathbb{RP}^m, X_\Sigma)$, which is denoted by $\text{Map}_D(\mathbb{RP}^m, X_\Sigma)$.

This also gives the natural map $j'_D : A_{D,\Sigma}(m) \rightarrow \text{Map}_D(\mathbb{RP}^m, X_\Sigma)$. Note that $j'_D(f_1, \dots, f_r) \in \text{Map}^*(\mathbb{RP}^m, X_\Sigma)$ if $(f_1, \dots, f_r) \in A_D(m, X_\Sigma)$. Hence, if we set $\text{Map}_D^*(\mathbb{RP}^m, X_\Sigma) = \text{Map}^*(\mathbb{RP}^m, X_\Sigma) \cap \text{Map}_D(\mathbb{RP}^m, X_\Sigma)$, we have the natural map $i_D = j'_D|_{A_D(m, X_\Sigma)} : A_D(m, X_\Sigma) \rightarrow \text{Map}_D^*(\mathbb{RP}^m, X_\Sigma)$.

Suppose that $m \geq 2$ and let us choose a fixed element $(g_1, \dots, g_r) \in A_D(m-1, X_\Sigma)$. For each $1 \leq k \leq r$, let $B_k = \{g_k + z_m h : h \in \mathcal{H}_{d_k-1, m}\}$. Then define the subspace $A_D(m, X_\Sigma; g) \subset A_D(m, X_\Sigma)$ by

$$A_D(m, X_\Sigma; g) = A_D(m, X_\Sigma) \cap (B_1 \times B_2 \times \dots \times B_r).$$

It is easy to see that $i_D(f_1, \dots, f_r)|_{\mathbb{RP}^{m-1}} = g$ if $(f_1, \dots, f_r) \in A_D(m, X_\Sigma; g)$, where g denotes the map in $\text{Map}_D^*(\mathbb{RP}^{m-1}, X_\Sigma)$ given by

$$g([x_0 : \dots : x_{m-1}]) = [g_1(\mathbf{x}), \dots, g_r(\mathbf{x})] \quad \text{for } \mathbf{x} = (x_0, \dots, x_{m-1}) \in \mathbb{R}^m \setminus \{\mathbf{0}\}.$$

Then, define the map $i'_D : A_D(m, X_\Sigma; g) \rightarrow F(\mathbb{RP}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma$ by the restriction $i'_D = i_D|_{A_D(m, X_\Sigma; g)}$. Now define the equivalence relation " $\sim$ " on $A_{D,\Sigma}(m)$ by $(f_1, \dots, f_r) \sim (g_1, \dots, g_r)$ if there exists some element $\lambda \in \mathbb{R}^*$ such that $f_k = \lambda^{d_k} g_k$ for any $1 \leq k \leq r$. We denote by $\widetilde{A}_D(m, X_\Sigma)$ the quotient space $\widetilde{A}_D(m, X_\Sigma) = A_{D,\Sigma}(m)/\sim$. Then define the map $j_D : \widetilde{A}_D(m, X_\Sigma) \rightarrow \text{Map}_D(\mathbb{RP}^m, X_\Sigma)$ by $j_D([f_1, \dots, f_r])([x_0, \dots, x_r]) = [f_1(\mathbf{x}), \dots, f_r(\mathbf{x})]$ for $\mathbf{x} = (x_0, \dots, x_m) \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}$.

A map $f : \mathbb{RP}^m \rightarrow X_\Sigma$ is called an *algebraic map of degree D* if it can be represented as a rational map (or regular map) of the form

$$f = j'_D(f_1, \dots, f_r) = [f_1, \dots, f_r] \quad \text{for some } (f_1, \dots, f_r) \in A_{D,\Sigma}(m).$$

We denote by $\text{Alg}_D(\mathbb{RP}^m, X_\Sigma)$ the space of all algebraic maps $f : \mathbb{RP}^m \rightarrow X_\Sigma$ of degree D . Consider the natural projection $\Gamma'_D : A_{D,\Sigma}(m) \rightarrow \text{Alg}_D(\mathbb{RP}^m, X_\Sigma)$ given by $\Gamma'_D(f_1, \dots, f_r) = j'_D(f_1, \dots, f_r) = [f_1, \dots, f_r]$. Then it clearly induces a natural projection $\Gamma_D : \widetilde{A}_D(m, X_\Sigma) \rightarrow \text{Alg}_D(\mathbb{RP}^m, X_\Sigma)$.

For $g \in \text{Alg}_D^*(\mathbb{RP}^{m-1}, X_\Sigma)$, let $\text{Alg}_D^*(\mathbb{RP}^m, X_\Sigma)$ and $\text{Alg}^*(\mathbb{RP}^m, X_\Sigma; g)$ denote the subspaces of $\text{Alg}_D(\mathbb{RP}^m, X_\Sigma)$ given by

$$\begin{cases} \text{Alg}_D^*(\mathbb{RP}^m, X_\Sigma) &= \text{Alg}_D(\mathbb{RP}^m, X_\Sigma) \cap \text{Map}^*(\mathbb{RP}^m, X_\Sigma) \\ \text{Alg}_D^*(\mathbb{RP}^m, X_\Sigma; g) &= \text{Alg}_D(\mathbb{RP}^m, X_\Sigma) \cap F(\mathbb{RP}^m, X_\Sigma; g) \end{cases}$$

Then the projection Γ'_D induces the projection maps by the restrictions

$$\begin{cases} \Psi_D : A_D(m, X_\Sigma) \rightarrow \text{Alg}_D^*(\mathbb{RP}^m, X_\Sigma) \\ \Psi'_D : A_D(m, X_\Sigma; g) \rightarrow \text{Alg}_D^*(\mathbb{RP}^m, X_\Sigma; g) \end{cases}$$

Let

$$\begin{cases} j_{D,\mathbb{C}} : \text{Alg}_D(\mathbb{RP}^m, X_\Sigma) \hookrightarrow \text{Map}_D(\mathbb{RP}^m, X_\Sigma) \\ i_{D,\mathbb{C}} : \text{Alg}_D^*(\mathbb{RP}^m, X_\Sigma) \hookrightarrow \text{Map}_D^*(\mathbb{RP}^m, X_\Sigma) \\ i'_{D,\mathbb{C}} : \text{Alg}_D^*(\mathbb{RP}^m, X_\Sigma; g) \hookrightarrow F(\mathbb{RP}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma \end{cases}$$

denote the inclusions. It is easy to see that the following equalities hold:

$$\begin{cases} j_D = j_{D,\mathbb{C}} \circ \Gamma_D : \widetilde{A}_D(m, X_\Sigma) \rightarrow \text{Map}_D(\mathbb{RP}^m, X_\Sigma) \\ i_D = i_{D,\mathbb{C}} \circ \Psi_D : A_D(m, X_\Sigma) \rightarrow \text{Map}_D^*(\mathbb{RP}^m, X_\Sigma) \\ i'_D = i'_{D,\mathbb{C}} \circ \Psi'_D : A_D(m, X_\Sigma; g) \rightarrow F(\mathbb{RP}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma \end{cases}$$

Let $g \in \text{Alg}_D^*(\mathbb{RP}^{m-1}, X_\Sigma)$ be any fixed algebraic map of degree D and we choose an element $(g_1, \dots, g_r) \in A_D(m-1, X_\Sigma)$ such that $g = [g_1, \dots, g_r]$.

Now we can state the our main result as follows.

Theorem 1.1 ([14]). *Let $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ and let Σ be a complete smooth fan in $\mathbb{R}^n$ satisfying the above conditions (1.1) and (1.2). Then if $2 \leq m \leq 2(r_{\min} - 1)$ and X_Σ is a smooth compact toric variety, the maps*

$$\begin{cases} j_D : \widetilde{A}_D(m, X_\Sigma) \rightarrow \text{Map}_D(\mathbb{RP}^m, X_\Sigma) \\ i_D : A_D(m, X_\Sigma) \rightarrow \text{Map}_D^*(\mathbb{RP}^m, X_\Sigma) \\ i'_D : A_D(m, X_\Sigma; g) \rightarrow F(\mathbb{RP}^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma \end{cases}$$

are homology equivalences through dimension $D(d_1, \dots, d_r; m)$, where the number $D(d_1, \dots, d_r; m)$ is given by

$$D(d_1, \dots, d_r; m) = (2r_{\min} - m - 1) \min\{d_1, d_2, \dots, d_r\} - 2. \quad \square$$

Remark. A map $f : X \rightarrow Y$ is called a *homology equivalence through dimension N* if the induced homomorphism $f_* : H_k(X, \mathbb{Z}) \rightarrow H_k(Y, \mathbb{Z})$ is an isomorphism for any $k \leq N$.

References

- [1] M. Adamaszek, A. Kozłowski and K. Yamaguchi, Spaces of algebraic and continuous maps between real algebraic varieties, *Quart. J. Math.* **62** (2011), 771–790.
- [2] C. P. Boyer, J. C. Hurtubise, B. M. Mann and R. J. Milgram, The topology of the space of rational maps into generalized flag manifolds, *Acta Math.* **173** (1994), 61–101.
- [3] J. Bochnak, M. Coste and M-F. Roy, Real algebraic geometry, A series of Modern Surveys in Math. **36**, Springer-Verlag, 1991.
- [4] V.M. Buchstaber and T.E. Panov, Torus actions and their applications in topology and combinatorics, Univ. Lecture Note Series **24**, Amer. Math. Soc. Providence, 2002.
- [5] R.L. Cohen, J.D.S. Jones and G.B. Segal, Stability for holomorphic spheres and Morse Theory, *Contemporary Math.* **258** (2000), 87–106.
- [6] D.A. Cox, J.B. Little and H.K. Schenck, Toric varieties, Graduate Studies in Math. **124**, Amer. Math. Soc., 2011.
- [7] M. Goresky and R. MacPherson, Stratified Morse theory, A Series of Modern Surveys in Math., Springer-Verlag, 1980.
- [8] M.A. Guest, Configuration spaces and the space of rational curves on a toric variety, *Bull. Amer. Math. Soc.* **31** (1994), 191–196.
- [9] M.A. Guest, The topology of the space of rational curves on a toric variety, *Acta Math.* **174** (1995), 119–145.
- [10] M.A. Guest, A. Kozłowski and K. Yamaguchi, The topology of spaces of coprime polynomials, *Math. Z.* **217** (1994), 435–446.
- [11] M.A. Guest, A. Kozłowski and K. Yamaguchi, Spaces of polynomials with roots of bounded multiplicity, *Fund. Math.* **116** (1999), 93–117.
- [12] R. Hartshorne, Algebraic geometry, Graduate Texts in Math. **52**, Springer-Verlag, 1977.
- [13] J. Havlicek, On spaces of holomorphic maps from two copies of the Riemann sphere to complex Grassmannians, *Contemporary Math.* **182** (1995), 83–116.

- [14] A. Kozłowski, M. Ohno and K. Yamaguchi, Spaces of algebraic maps to toric varieties, preprint.
- [15] A. Kozłowski and K. Yamaguchi, Topology of complements of discriminants and resultants, *J. Math. Soc. Japan* **52** (2000), 949–959.
- [16] A. Kozłowski and K. Yamaguchi, Spaces of holomorphic maps between complex projective spaces of degree one, *Topology Appl.* **132** (2003), 139–145.
- [17] A. Kozłowski and K. Yamaguchi, Spaces of algebraic maps from real projective spaces into complex projective spaces, *Contemporary Math.* **519** (2010), 145–164.
- [18] A. Kozłowski and K. Yamaguchi, Simplicial resolutions and spaces of algebraic maps between real projective spaces, *Topology Appl.* **160** (2013), 87–98.
- [19] A. Kozłowski and K. Yamaguchi, Spaces of equivariant algebraic maps from real projective spaces into complex projective spaces, *RIMS Kôkyûroku Bessatsu*, **B39** (2013), 51–61.
- [20] A. Kozłowski and K. Yamaguchi, Spaces of algebraic maps to real toric varieties, preprint.
- [21] J. Mostovoy, Spaces of rational maps and the Stone-Weierstrass Theorem, *Topology* **45** (2006), 281–293.
- [22] J. Mostovoy, Truncated simplicial resolutions and spaces of rational maps, *Quart. J. Math.* **63** (2012), 181–187.
- [23] J. Mostovoy and E. Munguia-Villanueva, Spaces of morphisms from a projective space to a toric variety, preprint.
- [24] G.B. Segal, The topology of spaces of rational functions, *Acta Math.* **143** (1979), 39–72.
- [25] V.A. Vassiliev, Complements of discriminants of smooth maps, *Topology and Applications*, Amer. Math. Soc., Translations of Math. Monographs **98**, 1992 (revised edition 1994).
- [26] K. Yamaguchi, Complements of resultants and homotopy types, *J. Math. Kyoto Univ.* **39** (1999), 675–684.

Andrzej Kozłowski
Institute of Applied Mathematics and Mechanics
University of Warsaw
Banacha 2, 02-097 Warsaw, Poland
(akoz@mimuw.edu.pl)

Masahiro Ohno
Department of Mathematics
University of Electro-Communications
Chofu, Tokyo 182-8585, Japan
(masahiro-ohno@uec.ac.jp)

Kohhei Yamaguchi
Department of Mathematics
University of Electro-Communications
Chofu, Tokyo 182-8585, Japan
(kohhei.yamag@uec.ac.jp)