

REMARKS ON POSITIVE MAPS ON SELFDUAL CONES

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ここではヒルベルト空間における selfdual cone を保存する意味での正值写像および作用素の順序 ($\leq$) に関する基本的な性質を考える。内容は [MI] を部分的に含む。

§1. INTRODUCTION

Let $\mathcal{H}$ be a separable complex Hilbert space with an inner product $(\cdot, \cdot)$. A convex cone $\mathcal{H}^+$ in $\mathcal{H}$ is said to be selfdual if $\mathcal{H}^+ = \{\xi \in \mathcal{H} | (\xi, \eta) \geq 0 \ \forall \eta \in \mathcal{H}^+\}$. The set of all bounded operators is denoted by $L(\mathcal{H})$. For a fixed selfdual cone $\mathcal{H}^+$, we shall write

$$A \leq B \quad \text{if} \quad (B - A)(\mathcal{H}^+) \subset \mathcal{H}^+, A, B \in L(\mathcal{H}).$$

Since $\mathcal{H}$ is algebraically spanned by $\mathcal{H}^+$, the relation ' $\leq$ ' defines the partial order on $L(\mathcal{H})$.

Recall a selfdual cone associated with a standard von Neumann algebra in the sense of Haagerup [H], which appears in the form $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ where $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$ and J is an isometric involution related to a selfdual cone $\mathcal{H}^+$ in $\mathcal{H}$. For example, $\ell^2_+ = \{\xi = \{\lambda_n\} | \lambda_n \geq 0\}$ is a selfdual cone associated with an abelian standard von Neumann algebra ℓ^∞ . Then, for $A = (\lambda_{ij}) \in L(\ell^2)$, $A \geq 0$ if and only if $\lambda_{ij} \geq 0$ for $i, j = 1, 2, \dots$.

Moreover, suppose that $(\mathcal{H}, \mathcal{H}_n^+, n \in \mathbf{N})$ and $(\tilde{\mathcal{H}}, \tilde{\mathcal{H}}_n^+, n \in \mathbf{N})$ are matrix ordered Hilbert spaces. Here $\mathcal{H}_n^+$ denotes a selfdual cone in $\mathcal{H}_n = M_n(\mathcal{H})$. A linear map A of $\mathcal{H}$ into $\tilde{\mathcal{H}}$ is said to be n -positive (resp. n -co-positive) when the multiplicity map $A_n (= A \otimes \text{id}_n)$ satisfies $A_n \mathcal{H}_n^+ \subset \tilde{\mathcal{H}}_n^+$ (resp. ${}^t(A_n \mathcal{H}_n^+) \subset \tilde{\mathcal{H}}_n^+$). Here ${}^t(\cdot)$ denotes a set of all transposed matrices. When A is n -positive (resp. n -co-positive) for all

$n \in \mathbb{N}$, A is said to be completely positive (resp. completely co-positive). Put, for $A \in L(\mathcal{H})$

$$\hat{A}\xi = AJAJ\xi, \quad \xi \in \mathcal{H}.$$

It is known that if, in a matrix ordered standard form $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ as introduced in [SW2], $A \in \mathcal{M}$ then $\hat{A}$ is completely positive, and we shall write $\hat{A} \succeq_{cp} O$.

§2. POSITIVE MAPS ASSOCIATED WITH SELFDUAL CONES

We obtain the following proposition for a general selfdual cone in a finite dimensional Hilbert space. In particular, when $\mathcal{H}^+$ is associated with an abelian von Neumann algebra, that is, a matrix is entrywise positive, it is known as the Peron theorem (see, example [HJ, Corollary 8.2.6]).

(2.1). *Let $\mathcal{H}$ be an n -dimensional Hilbert space with a selfdual cone $\mathcal{H}^+$. If A is an injective linear operator on $\mathcal{H}$ satisfying $A \succeq O$, then there exist a number $\lambda > 0$ and a non-zero element $\xi_0 \in \mathcal{H}^+$ such that $A\xi_0 = \lambda\xi_0$.*

Proof. Put

$$\mathcal{V} = \text{co}\{\xi \in \mathcal{H}^+ \mid \|\xi\| = 1\},$$

where co denotes the convex hull. Consider the map r defined by

$$r(\xi) = \frac{A\xi}{\|A\xi\|}, \quad \xi \in \mathcal{V}.$$

By assumption r maps $\mathcal{V}$ to itself. Note that $0 \notin \mathcal{V}$. Because, by the Carathéodory theorem (see, for example [La, Theorem 2.23]) any element $\xi \in \mathcal{V}$ can be expressed as

$$\xi = \lambda_1 \xi_1 + \cdots + \lambda_s \xi_s,$$

where $\lambda_1, \dots, \lambda_s > 0$, $\xi_1, \dots, \xi_s \in \mathcal{H}^+$ with $\|\xi_1\| = \cdots = \|\xi_s\| = 1$ and $1 \leq s \leq n+1$. It follows that $\xi \geq \lambda_1 \xi_1(\mathcal{H}^+)$, and so $\|\xi\| \geq \|\lambda_1 \xi_1\| = |\lambda_1| > 0$. Since a convex hull of a compact set is compact [La, Theorem 2.30], it follows from Schauder's fixed point theorem [Sd, Satz I] that there exists an element $\xi_0 \in \mathcal{V}$ satisfying $r(\xi_0) = \xi_0$. Hence $A\xi_0 = \|A\xi_0\| \xi_0$. $\square$

The following fundamental proposition is valid for a general selfdual cone. It says that the order ' $\preceq$ ' is different from the usual order ' $\leq$ ' based on positivity of hermitian operators in point of compatibility with product.

(2.2). (cf. [IM, Proposition 1]) Let $\mathcal{H}$ be a Hilbert space with a selfdual cone $\mathcal{H}^+$. Then for bounded operators on $\mathcal{H}$ we have the following properties:

- (1) If $O \leq A_1 \leq B_1$ and $O \leq A_2 \leq B_2$, then $O \leq A_1 A_2 \leq B_1 B_2$. In particular, if $O \leq A \leq B$, then $A^n \leq B^n$ for every natural number n .
- (2) If $O \leq A \leq B$, then $O \leq A^* \leq B^*$.
- (3) If $A, A^{-1}, B, B^{-1} \geq O$ and $A \leq B$, then $B^{-1} \leq A^{-1}$.
- (4) If $O \leq A \leq B$, then $\|A\| \leq \|B\|$.

Proof. We sketch a proof which is similar to [IM].

(1) By assumption $A_i(\mathcal{H}^+) \subset \mathcal{H}^+$ and $(B_i - A_i)(\mathcal{H}^+) \subset \mathcal{H}^+$ hold for $i = 1, 2$. Since $B_1 B_2 - A_1 A_2 = B_1(B_2 - A_2) + (B_1 - A_1)A_2$, we obtain the desired inequality.

(2) Let $A(\mathcal{H}^+) \subset \mathcal{H}^+$. Then we have $(A^* \xi, \eta) = (\xi, A\eta) \geq 0$ for all $\xi, \eta \in \mathcal{H}^+$. The selfduality of $\mathcal{H}^+$ shows that $A^* \geq O$. Exchanging the role of A and $B - A$ we obtain the desired property.

(3) If $A \leq B$, then $B^{-1} = A^{-1} A B^{-1} \leq A^{-1} B B^{-1} = A^{-1}$ from (1).

(4) For $A \geq O$, put $\|A\|_+ = \sup\{\|A\xi\|; \|\xi\| \leq 1, \xi \in \mathcal{H}^+\}$. Suppose $O \leq A \leq B$. Note that if $\eta - \xi \in \mathcal{H}^+$ for $\xi, \eta \in \mathcal{H}^+$, then $\|\xi\| \leq \|\eta\|$. Since $\|A\|_+ \leq \|B\|_+$, it suffices to show $\|\cdot\|_+ = \|\cdot\|$. It is known that any element $\xi \in \mathcal{H}$ can be written as $\xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4)$, $\xi_1 \perp \xi_2, \xi_3 \perp \xi_4$, for some $\xi_i \in \mathcal{H}^+$. Then $\|\xi\|^2 = \sum_{i=1}^4 \|\xi_i\|^2$. Noticing that $A \geq O$, we see that

$$\begin{aligned} \|A\xi\|^2 &= \sum_{i=1}^4 \|A\xi_i\|^2 - 2(A\xi_1, A\xi_2) - 2(A\xi_3, A\xi_4) \\ &\leq \|A(\xi_1 + \xi_2)\|^2 + \|A(\xi_3 + \xi_4)\|^2 \leq \|A\|_+^2 \|\xi\|^2. \end{aligned}$$

It follows that $\|A\| \leq \|A\|_+$. The converse inequality is trivial. $\square$

(2.3). Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ be a standard form of a von Neumann algebra. For a selfadjoint element $A \in \mathcal{M} \cup \mathcal{M}'$, the following conditions are equivalent:

- (1) $A \geq O$.
- (2) $A \in Z(\mathcal{M})$ and $A \geq O$.

Proof. (1) $\Rightarrow$ (2): Since $A \geq O$ if and only if $JAJ \geq O$, it suffices to investigate the case $A \in \mathcal{M}$. Suppose $A \geq O, A \in \mathcal{M}$. Since any element of $\mathcal{H}$ can be written

as $\xi + i\eta$ with $J\xi = \xi$, $J\eta = \eta$, it follows that for such elements ξ, η

$$JAJ(\xi + i\eta) = JA(\xi - i\eta) = JA\xi + iJA\eta = A(\xi + i\eta).$$

Hence $A \in Z(\mathcal{M})$ and $A^* = JAJ = A$. Choose an arbitrary element $\xi \in \mathcal{H}$. Then one can write as $\xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4)$, $\xi_i \in \mathcal{H}^+$ such that $\mathcal{M}\xi_1 \perp \mathcal{M}\xi_2$, $\mathcal{M}\xi_3 \perp \mathcal{M}\xi_4$. We then have

$$\begin{aligned} (A\xi, \xi) &= (A\xi_1 - A\xi_2 + i(A\xi_3 - A\xi_4), \xi_1 - \xi_2 + i(\xi_3 - \xi_4)) \\ &= \sum_{i=1}^4 (A\xi_i, \xi_i) \geq 0 \end{aligned}$$

because $(A\xi_1, \xi_2) = (A\xi_3, \xi_4) = 0$ and $((A(\xi_1 - \xi_2), \xi_3 - \xi_4))$ is a real number. Hence $A \geq 0$.

(2) $\Rightarrow$ (1): It is immediate. $\square$

(2.4). Suppose that $A \in L(\mathcal{H})^+$ has a closed range in which $A\mathcal{H}^+$ is a selfdual cone. Then we obtain the following properties:

- (1) Under the condition that $\mathcal{H}^+$ is a facially homogeneous selfdual cone in $\mathcal{H}$, if $A \geq 0$, then for all $\lambda \in \mathbb{R}$, $A^\lambda \geq 0$.
- (2) For a matrix ordered standard form $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$, if $A \geq 0$ and the support projection of A is completely positive, then for all $\lambda \in \mathbb{R}$, $A^\lambda \geq_{cp} 0$.

Here the inverse for a not invertible A is taken as reduced by the support projection of A .

Proof. (1) Let P denote the support projection of A . By assumption we obtain that $P \geq 0$ and $P\mathcal{H}^+ = A\mathcal{H}^+$. Hence, by [I, Proposition II.1.6], $P\mathcal{H}^+$ is facially homogeneous. Since $A = PA = AP$ and PA maps $P\mathcal{H}^+$ onto itself, it follows from [I, Corollary II.3.2] that there exists a derivation $\delta \in D(P\mathcal{H}^+)^+$ such that $PA|_{P\mathcal{H}} = e^\delta$. Hence

$$A^\lambda = Pe^{\lambda\delta}P \geq 0$$

for every real number λ .

(2) Put $\mathcal{N} = P\mathcal{M}|_{P\mathcal{H}}$. Since P is completely positive, we see from [MN, Lemma 3] that $(\mathcal{N}, P\mathcal{H}, P_n\mathcal{H}_n^+)$ is a matrix ordered standard form. It follows

from [C, Theorem 3.3] that there exists an element $B \in \mathcal{N}^+$ such that $PA = BJ_{P\mathcal{H}+}BJ_{P\mathcal{H}+}P$. Hence

$$A^\lambda = B^\lambda J_{P\mathcal{H}+}B^\lambda J_{P\mathcal{H}+}P \succeq_{cp} O$$

for every real number λ . $\square$

A simple counter-example can show that it is essential in the above proposition for $A\mathcal{H}^+$ to be selfdual. In fact, we obtain the following remark:

Remark. In the case $\mathbf{C}^{n+}$ (non-negative entries), a necessary and sufficient condition for $A \in M_n^+$ to enjoy $A\mathbf{C}^{n+} = \mathbf{C}^{n+}$ is that A is a non-singular positive definite diagonal matrix. We obtain the following facts:

- (1) In the case $\mathbf{C}^{n+}$, if $A \in M_n^+$ and $A \succeq O$, then there exists a real number $s \geq 1$ such that $A^\lambda \succeq O$ for all $\lambda \in [s, +\infty)$.
- (2) In the case $\mathbf{C}^{n+}$, if $A \in M_n^+$, $A \succeq O$, $\det A \neq 0$ and $A\mathbf{C}^{n+} \subsetneq \mathbf{C}^{n+}$, then there exists a real number $s' < 0$ such that $A^\lambda \not\succeq O$ for all $\lambda \in (-\infty, s']$.

Indeed, let $A \in M_n$ be entrywise positive and positive semi-definite. We may assume $\|A\| = 1$. Let $1, a_1, \dots, a_m, 0 \leq m \leq n-1$, be distinct eigenvalues of A . Since A can be diagonalized by a real orthogonal matrix, each entry of A^λ is written in the form

$$f(\lambda) = \alpha_0 + \alpha_1 a_1^\lambda + \dots + \alpha_m a_m^\lambda$$

for some real numbers α_k . Then α_0 must be positive, since $A^n \succeq O$ for all $n \in \mathbf{N}$ by (2.2) (1) and $0 \leq a_k < 1, 1 \leq k \leq m$. From the continuity of the function we can find a number $s \geq 1$ such that $f(\lambda) > 0$ for all $\lambda \geq s$. So (1) holds. Suppose, in addition, that A is non-singular and $A\mathbf{C}^{n+} \subsetneq \mathbf{C}^{n+}$. If $A^{-\lambda_0} \succeq O$ for some $\lambda_0 > 0$, then $A^{-\ell\lambda_0} \succeq O$ for all $\ell \in \mathbf{N}$. From (1), $A^{\ell\lambda_0} \succeq O$ for a large $\ell \in \mathbf{N}$. This implies that $A^{\ell\lambda_0}$ is diagonal, and so is A , a contradiction. Therefore, (2) holds.

(2.5). For a matrix ordered standard form $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$, suppose that $A \in L(\mathcal{H})$, and $B \in \mathcal{M}$ is an injective operator with a dense range. Then, $O \preceq A \preceq \hat{B}$ if and only if there exists an element $C \in Z(\mathcal{M})$ with $O \leq C \leq I$ such that $A = C\hat{B}$. In particular, if $\mathcal{M}$ is a factor, then one can choose a scalar λ with $0 \leq \lambda \leq 1$ such that $A = \lambda\hat{B}$.

Proof. Consider the polar decomposition $B = U|B|$ of B . By assumption U is a unitary element of $\mathcal{M}$, and so $\hat{U} \geq O$ and $\hat{U}^* \geq O$ by (2.2). Hence we may assume B to be positive semi-definite. Let $B = \int_0^{\|B\|} \lambda dE_\lambda$ be a spectral decomposition of B . Put $P_n = \int_{\frac{1}{n}}^{\|B\|} dE_\lambda$ for $n \in \mathbb{N}$. Then one sees that $\hat{P}_n \nearrow I$ and $\hat{P}_n A \hat{P}_n \leq \hat{P}_n \hat{B} \hat{P}_n$ by (2.2). Since $\hat{P}_n \hat{B} \hat{P}_n$ is invertible on $\hat{P}_n \mathcal{H}$, where the inverse shall be denoted by $(\hat{P}_n \hat{B} \hat{P}_n)^{-1}$, we have

$$O \leq \hat{P}_n A \hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \leq \hat{P}_n.$$

There then exists an element c_n in an order ideal $Z_{\hat{P}_n \mathcal{H}^+}$ of a selfdual cone $\hat{P} \mathcal{H}^+$ with $\|c_n\| \leq 1$ such that $\hat{P}_n A \hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \xi = c_n \xi$ for all $\xi \in \hat{P}_n \mathcal{H}$. By [I, Theorem VI.1,2 3)] we obtain that $c_n \in Z(\hat{P}_n \mathcal{M}|_{\hat{P}_n \mathcal{H}})^+$. Since $\hat{P}_n Z(\mathcal{M}) \hat{P}_n = Z(\hat{P}_n \mathcal{M} \hat{P}_n)$, we can find an element $C_n \in Z(\mathcal{M})$ such that $c_n \xi = \hat{P}_n C_n \hat{P}_n \xi$ for all $\xi \in \hat{P}_n \mathcal{H}$. Since $P_n B = B P_n, n \in \mathbb{N}$, we have

$$\begin{aligned} \hat{P}_{n+1} C_{n+1} \hat{P}_{n+1} \xi &= \hat{P}_{n+1} A \hat{P}_{n+1} (\hat{P}_{n+1} \hat{B} \hat{P}_{n+1})^{-1} \hat{P}_n \xi \\ &= \hat{P}_{n+1} A \hat{P}_n (\hat{P}_n \hat{B} \hat{P}_n)^{-1} \xi = \hat{P}_n C_n \hat{P}_n \xi \end{aligned}$$

for all $\xi \in \hat{P}_n \mathcal{H}$. Since $\{\hat{P}_n C_n \hat{P}_n\}$ is a bounded sequence, one can define

$$C\xi = \lim_{n \rightarrow \infty} \hat{P}_n C_n \hat{P}_n \xi, \quad \xi \in \mathcal{H}.$$

Thus $C \in Z(\mathcal{M}), O \leq C \leq I$ and we get

$$\begin{aligned} A &= s\text{-}\lim_{n \rightarrow \infty} \hat{P}_n A \hat{P}_n \\ &= s\text{-}\lim_{n \rightarrow \infty} \hat{P}_n C_n \hat{P}_n A \hat{P}_n \\ &= C\hat{B}. \end{aligned}$$

The converse implication is immediate. Indeed, if $C \in Z(\mathcal{M})$ with $O \leq C \leq I$, then $I - C \geq O$, and so $I - C \geq O$. Hence $\hat{B} - C\hat{B} = (I - C)\hat{B} \geq O$. This completes the proof. $\square$

§3. COMPLETE ORDER OF OPERATORS

Consider two matrix ordered standard forms $(\mathcal{M}^{(1)}, \mathcal{H}^{(1)}, \mathcal{H}_n^{(1)+})$ and $(\mathcal{M}^{(2)}, \mathcal{H}^{(2)}, \mathcal{H}_n^{(2)+})$ with respective canonical involutions $J^{(1)}$ and $J^{(2)}$. For an arbitrary element $\xi \in \mathcal{H}^{(1)}$, let R_ξ be a right slice map of $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ into $\mathcal{H}^{(2)}$ such that

$$R_\xi(\xi' \otimes \eta') = (\xi', \xi)\eta', \xi' \in \mathcal{H}^{(1)}, \eta' \in \mathcal{H}^{(2)}.$$

For any element $x \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$, we put

$$r(x)\xi = R_{J^{(1)}\xi}(x), \xi \in \mathcal{H}^{(1)}.$$

Then $r(x)$ is a map of Hilbert-Schmidt class of $\mathcal{H}^{(1)}$ to $\mathcal{H}^{(2)}$. A set of all maps of Hilbert-Schmidt class of $\mathcal{H}^{(1)}$ to $\mathcal{H}^{(2)}$ is denoted by $HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$. A set of all completely positive maps of $(\mathcal{H}^{(1)}, \mathcal{H}_n^{(1)+'})$ to $(\mathcal{H}^{(2)}, \mathcal{H}_n^{(2)+})$ in $HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$ is denoted by $CPHS(\mathcal{H}^{(1)+'}, \mathcal{H}^{(2)+})$. Here $\mathcal{H}_n^{(1)+'}, n \in \mathbf{N}$, means a family of the self-dual cones associated with $\mathcal{M}^{(1)'}$, that is $\mathcal{H}_n^{(1)+'} = \{^t[\xi_{ij}]_{i,j=1}^n \mid [\xi_{ij}]_{i,j=1}^n \in \mathcal{H}_n^{(1)+}\}$. We shall write $\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+}$ for a selfdual cone associated with a von Neumann tensor product $\mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)}$. It was shown in [MT, SW1] that

$$\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+} = \{x \in \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \mid r(x) \in CPHS(\mathcal{H}^{(1)+'}, \mathcal{H}^{(2)+})\}.$$

Thus

$$r : \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \rightarrow HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$$

is an isometry mapping $\mathcal{H}^{(1)+} \otimes \mathcal{H}^{(2)+}$ onto $CPHS(\mathcal{H}^{(1)+'}, \mathcal{H}^{(2)+})$.

Indeed, r is isometric. Suppose that $HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)})$ has an inner product

$$\langle A, B \rangle = \sum_{k=1}^{\infty} (Ae_k, Be_k),$$

where $\{e_k\}$ is a complete orthogonal basis of $\mathcal{H}^{(1)}$. Noticing that $\{J^{(1)}e_k\}$ is a complete orthogonal basis of $\mathcal{H}^{(1)}$, we obtain for a complete orthogonal basis $\{f_k\}$

$$\begin{aligned}
& \langle r(J^{(1)}e_i \otimes f_j), r(J^{(1)}e_{i'} \otimes f_{j'}) \rangle \\
&= \sum_{k=1}^{\infty} (r(J^{(1)}e_i \otimes f_j)(e_k), r(J^{(1)}e_{i'} \otimes f_{j'})(e_k)) \\
&= \sum_{k=1}^{\infty} (R_{J^{(1)}e_k}(J^{(1)}e_i \otimes f_j), R_{J^{(1)}e_k}(J^{(1)}e_{i'} \otimes f_{j'})) \\
&= \sum_{k=1}^{\infty} ((J^{(1)}e_i, J^{(1)}e_k)f_j, (J^{(1)}e_{i'}, J^{(1)}e_k)f_{j'}) \\
&= \sum_{k=1}^{\infty} ((e_k, e_i)f_j, (e_k, e_{i'})f_{j'}) \\
&= \delta_{ii'}\delta_{jj'}
\end{aligned}$$

for $i, j, i', j' = 1, 2, \dots$.

Therefore, $(r(\mathcal{M}^{(1)} \otimes \mathcal{M}^{(2)})r^{-1}, HS(\mathcal{H}^{(1)}, \mathcal{H}^{(2)}), r(J^{(1)} \otimes J^{(2)})r^{-1}, CPHS(\mathcal{H}^{(1)+}, \mathcal{H}^{(2)+}))$ is a standard form. Using the Radon-Nikodym theorem for L^2 -spaces [S, Theorem 1.2], we obtain the following theorem:

(3.1). *Let $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ be a matrix ordered standard form. Then $(r(\mathcal{M}' \otimes \mathcal{M})r^{-1}, HS(\mathcal{H}, \mathcal{H}), r(J \otimes J)r^{-1}, CPHS(\mathcal{H}^+, \mathcal{H}^+))$ is a standard form which is isomorphic to $(\mathcal{M}' \otimes \mathcal{M}, \mathcal{H} \otimes \mathcal{H}, J \otimes J, \mathcal{H}^+ \otimes \mathcal{H}^+)$ by the identification $r : \mathcal{H} \otimes \mathcal{H} \rightarrow HS(\mathcal{H}, \mathcal{H})$ defined as above. If $A, B \in HS(\mathcal{H}, \mathcal{H})$ satisfies $0 \leq_{cp} A \leq_{cp} B$, then there exists an element $C \in \mathcal{M}' \otimes \mathcal{M}$ with $0 \leq C \leq I$ such that $A = rCr^{-1}B$.*

(3.2). *If in (3.1) $\mathcal{M}$ is an injective factor (or semi-finite injective von Neumann algebra) on a separable Hilbert space $\mathcal{H}$, then the above statement is valid for $A \in L(\mathcal{H})$ instead of $A \in HS(\mathcal{H}, \mathcal{H})$.*

Proof. Suppose that $\mathcal{M}$ is the von Neumann algebra in the statement. There then exists an increasing net $\{E_i\}$ of completely positive projections of finite rank on $\mathcal{H}$ which converges strongly to 1 by [M1, Theorem 1.4]. It follows that $0 \leq_{cp} E_i A \leq_{cp} E_i B$. Hence

$$\text{Tr}(A^* E_i A) \leq \text{Tr}(B^* E_i B) \leq \text{Tr}(B^* B).$$

Considering a limit with respect to i , we have $\text{Tr}(A^* A) < +\infty$. Using (3.1) we obtain the desired result. $\square$

(3.3). For a matrix ordered standard form $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$, any element $A \in HS(\mathcal{H})$ can be uniquely decomposed into the following:

$$A = A_1 - A_2 + i(A_3 - A_4)$$

where $A_1 \perp A_2, A_3 \perp A_4, A_i \in CPHS(\mathcal{H}^+)$.

The proof of the above proposition is immediate from a decomposition theorem of vectors in the ordered Hilbert space.

§4. DECOMPOSITION OF POSITIVE MAPS

The purpose of this section is to show that any order isomorphism between non-commutative L^2 -spaces associated with von Neumann algebras is decomposed into a sum of a completely positive and a completely co-positive maps. The result is an L^2 version of a theorem of Kadison [K] for a Jordan isomorphism on operator algebras.

We first generalize a theorem of A. Connes [C] for the polar decomposition of an order isomorphism, to the case where a von Neumann algebra is non- σ -finite.

(4.1). Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \tilde{J}, \tilde{\mathcal{H}}^+)$ be standard forms, and A be a linear bijection of $\mathcal{H}$ onto $\tilde{\mathcal{H}}$ satisfying $A\mathcal{H}^+ = \tilde{\mathcal{H}}^+$. Then for a polar decomposition $A = U|A|$ of A we obtain the following properties:

- (1) There exists a unique invertible operator B in $\mathcal{M}^+$ such that $|A| = BJB$.
(cf. [I, Corollary II.3.2])
- (2) There exists a unique Jordan $*$ -isomorphism α of $\mathcal{M}$ onto $\tilde{\mathcal{M}}$ such that

$$(\alpha(X)\xi, \xi) = (XU^{-1}\xi, U^{-1}\xi)$$

for all $X \in \mathcal{M}, \xi \in \tilde{\mathcal{H}}^+$.

Proof. (1) Let $\mathcal{M}$ be non- σ -finite. Choose an increasing net $\{p_i\}_{i \in I}$ of σ -finite projections in $\mathcal{M}$ converging strongly to 1. Put $q_i = p_i J p_i J$. By [[C, Theorem 4.2] $q_i \mathcal{H}^+$ is a closed face of $\tilde{\mathcal{H}}^+$. Since A is an order isomorphism, $A(q_i \mathcal{H}^+)$ is a closed face of $\tilde{\mathcal{H}}^+$. There then exists a σ -finite projection $p'_i \in \tilde{\mathcal{M}}$ such that $A(q_i \mathcal{H}^+) = q'_i \tilde{\mathcal{H}}^+$ where q'_i denotes $p'_i J p'_i J$. Hence $q'_i A q_i$ is an order isomorphism

of $q_i\mathcal{H}^+$ onto $q'_i\tilde{\mathcal{H}}^+$. These cones appear respectively in the reduced standard forms $(q_i\mathcal{M}q_i, q_i\mathcal{H}, q_iJq_i, q_i\mathcal{H}^+)$ and $(q'_i\tilde{\mathcal{M}}q'_i, q'_i\tilde{\mathcal{H}}, q'_iJq'_i, q'_i\tilde{\mathcal{H}}^+)$. Put $A_i = (q'_iAq_i)^*q'_iAq_i$. Then $A_i \in q_i\mathcal{M}^+q_i$ is an order automorphism on $q_i\mathcal{H}^+$. By [C, Theorem 3.3] there exists a unique invertible operator $B_i \in q_i\mathcal{M}^+q_i$ such that $A_i = B_iJ_iB_iJ_i$, where J_i denotes q_iJq_i . Taking a logarithm of both sides, we have $\log A_i = \log B_i + J_i(\log B_i)J_i$. Since $\{A_i\}$ is a bounded net, $\{\log B_i\}$ is bounded. Indeed, we have in a standard form that a map

$$X \mapsto \delta_X = \frac{1}{2}(X + JXJ)$$

is a Jordan isomorphism of a selfadjoint part of $\mathcal{M}$ into a selfadjoint part of a set of all order derivations $D(\mathcal{H}^+)$ by [I, Corollary VI.2.3]. It is known that any isomorphism of a JB-algebra into another JB-algebra is isometry (see [HS, Proposition 3,4.3]). Hence

$$\|\delta_X\| = \|X\|, \quad X \in \mathcal{M}_{\text{s.a.}}$$

Thus $\{\log B_i\}$ is bounded. It follows that $\{p_i(\log B_i)p_i\}$ is bounded because $p_i\mathcal{M}p_i$ and $q_i\mathcal{M}q_i$ are $*$ -isomorphic. Therefore, one can find a subnet of $\{p_i \log B_i p_i\}$ which converges to some element $C \in \mathcal{M}^+$ in the σ -weak topology. We may index the subnet as the same $i \in \mathbf{I}$. We then have for $\xi, \eta \in \mathcal{H}$

$$\begin{aligned} ((C + J C J)q_j\xi, q_j\eta) &= \lim_i ((p_i(\log B_i)p_i + Jp_i(\log B_i)p_iJ)q_j\xi, q_j\eta) \\ &= ((\log B_j + J_j(\log B_j)J_j)q_j\xi, q_j\eta) \\ &= \lim_i (\log A_i q_j\xi, q_j\eta) \\ &= (\log A^* A q_j\xi, q_j\eta), \end{aligned}$$

using the facts that $q_iXq_iJq_iXq_iJq_i = p_iXp_iJp_iXp_iJq_i$ for all $X \in \mathcal{M}$, and under the strong topology $\{A_i\}$ converges to A^*A ; hence $\{q_i(\log A_i)q_i\}$ converges to $\log A^*A$. Since $\bigcup_{i \in \mathbf{I}} q_i\mathcal{H}$ is dense in $\mathcal{H}$, we obtain the equality $C + J C J = \log A^*A$. Therefore, $e^C J e^C J = A^*A$. Thus there exists an element $B \in \mathcal{M}^+$ such that $|A| = B J B J$. Since, in addition, $q_iBq_iJq_iBq_iJq_i = q_i|A|q_i$, one easily sees the invertibility and the unicity of B using the same properties as in the σ -finite case.

(2) From (1) we have $U = AB^{-1}JB^{-1}J$. It follows that U is an isometry satisfying $U\mathcal{H}^+ = \tilde{\mathcal{H}}^+$. Let p_i and q_i be as in (1). There then exists a σ -finite projection $p'_i \in \tilde{\mathcal{M}}$ such that $U(q_i\mathcal{H}^+) = q'_i\tilde{\mathcal{H}}^+$ with $q'_i = p'_i\tilde{J}p'_i\tilde{J}$. Using also [C, Theorem 3.3], one can find a unique Jordan $*$ -isomorphism α_i of $q_i\mathcal{M}q_i$ onto $q'_i\tilde{\mathcal{M}}q'_i$ such that

$$(\alpha_i(q_iXq_i)\xi, \xi) = (q_iXq_iU^{-1}\xi, U^{-1}\xi)$$

for all $X \in \mathcal{M}, \xi \in q'_i\tilde{\mathcal{H}}^+$. Fixed now $X \in \mathcal{M}_{s.a.}$. Since $p'_i\tilde{\mathcal{M}}p'_i$ and $q'_i\tilde{\mathcal{M}}q'_i$ are $*$ -isomorphic, there exists a unique operator $Y_i \in p'_i\tilde{\mathcal{M}}_{s.a.}p'_i$ such that $Y_i|_{q'_i\tilde{\mathcal{H}}} = \alpha_i(q_iXq_i)$. Using an isometry between the Jordan algebras, one sees that $\{\alpha_i(q_iXq_i)\}$ is a bounded net, because $\|\alpha_i(q_iXq_i)\| = \|q_iXq_i\| \leq \|X\|, i \in \mathbf{I}$. Thus $\{Y_i\}$ is bounded. We may then say that $\{Y_i\}$ converges to some operator $Y \in \tilde{\mathcal{M}}_{s.a.}$ in the σ -weak topology. We then have for $\xi \in \tilde{\mathcal{H}}^+$

$$\begin{aligned} (Yq'_j\xi, q'_j\xi) &= \lim_i (Y_iq'_j\xi, q'_j\xi) = \lim_i (\alpha_i(q_iXq_i)q'_j\xi, q'_j\xi) \\ &= \lim_i (q_iXq_iU^{-1}q'_j\xi, U^{-1}q'_j\xi) \\ &= (XU^{-1}q'_j\xi, U^{-1}q'_j\xi). \end{aligned}$$

Taking a limit with respect to j , we obtain

$$(Y\xi, \xi) = (XU^{-1}\xi, U^{-1}\xi)$$

for all $\xi \in \tilde{\mathcal{H}}^+$. It is known that any normal state on the von Neumann algebra $\tilde{\mathcal{M}}$ is represented by a vector state with respect to an element of $\tilde{\mathcal{H}}^+$ (see [H, Lemma 2.10 (1)]). Therefore, the above element Y is uniquely determined. Moreover, we have $q'_iYq'_i = \alpha_i(q_iXq_i)$. It follows that $\{\alpha_i(q_iXq_i)\}$ converges to Y in the strong topology. Hence one can define $\alpha(X) = Y$ for all $X \in \mathcal{M}$. It is now immediate that $\alpha(X^2) = \alpha(X)^2$ for all $X \in \mathcal{M}_{s.a.}$. Considering the inverse order isomorphism U^{-1} , we have $\alpha(\mathcal{M}) = \tilde{\mathcal{M}}$. This completes the proof. $\square$

In the following proposition we deal with a reduced matrix ordered standard form by a completely positive projection.

(4.2). *With $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ a matrix ordered standard form, let E be a completely positive projection on $\mathcal{H}$. Then $(EME, E\mathcal{H}, E_n\mathcal{H}_n^+)$ is a matrix ordered standard*

Proof. The statement was shown in [MN, Lemma 3] where $\mathcal{M}$ is σ -finite. In the case where $\mathcal{M}$ is not σ -finite, since E is a completely positive projection, there exists a von Neumann algebra $\mathcal{N}$ such that $(\mathcal{N}, E\mathcal{H}, E_n\mathcal{H}_n^+)$ is a matrix ordered standard form by [M2, Lemma 3]. Hence $E\mathcal{M}|_{E\mathcal{H}} = \mathcal{N}$ and $(E\mathcal{M}E, E\mathcal{H}, E_n\mathcal{H}_n^+)$ is a matrix ordered standard form by using the same discussion as in the proof in [M3]. $\square$

Now, we shall state the decomposition theorem for an order isomorphism between non-commutative L^2 -spaces.

(4.3). *Let $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \tilde{\mathcal{H}}_n^+)$ be matrix ordered standard forms. Suppose that A is a 1-positive map of $\mathcal{H}$ into $\tilde{\mathcal{H}}$ such that $A\mathcal{H}^+$ is a selfdual cone in the closed range of A . If both the support projection E and the range projection F of A are completely positive, then there exists a central projection P of $E\mathcal{M}E$ such that AP is completely positive and $A(E - P)$ is completely co-positive.*

In particular, if A is an order isomorphism of $\mathcal{H}$ onto $\tilde{\mathcal{H}}$, then there exists a central projection P of $\mathcal{M}$ such that AP is completely positive and $A(1 - P)$ is completely co-positive.

Proof. We first consider the case where A is an order isomorphism. Let U, B and α be as in (4.1). It follows from a theorem of Kadison [K] that there exists a central projection P of $\mathcal{M}$ satisfying

$$\alpha : \mathcal{M}_P \rightarrow \tilde{\mathcal{M}}_{\alpha(P)}, \text{ onto } *- \text{isomorphism}$$

and

$$\alpha : \mathcal{M}_{1-P} \rightarrow \tilde{\mathcal{M}}_{\alpha(1-P)}, \text{ onto } *- \text{anti-isomorphism.}$$

Indeed, $\alpha(P)$ is a central projection of $\tilde{\mathcal{M}}$. Since α preserves a $*$ -operation and power, $\alpha(P)$ is a projection. Suppose that Q is an arbitrary projection in $\mathcal{M}$. Since α is order preserving, we have $\alpha(QP) \leq \alpha(P)$ and $\alpha(Q(1 - P)) \leq \alpha(1 - P)$. It follows that two projections $\alpha(P)$ and $\alpha(QP)$ are commutative, and so are $\alpha(1 - P)$ and $\alpha(Q(1 - P))$. Hence $\alpha(Q) = \alpha(QP + Q(1 - P))$ and $\alpha(P)$ commute. Since α is bijective, a set $\alpha(Q)$ generates a von Neumann algebra $\tilde{\mathcal{M}}$. Therefore, $\alpha(P)$ belongs to a center of $\tilde{\mathcal{M}}$. Now, there then exists a unique completely positive

isometry $u : P\mathcal{H} \rightarrow \alpha(P)\tilde{\mathcal{H}}$ such that

$$u(P\mathcal{H}^+) = \alpha(P)\tilde{\mathcal{H}}^+ \quad \text{and} \quad \alpha(x) = u x u^{-1}, \quad x \in \mathcal{M}_P$$

by [M3, Proposition 2.4] which is also valid for the non- σ -finite case. Hence $(UxU^{-1}\xi, \xi) = (uxu^{-1}\xi, \xi), x \in \mathcal{M}_P, \xi \in \alpha(P)\tilde{\mathcal{H}}^+$. We have from the unicity of a completely positive isometry $UP = u$. Note that $\alpha(P)UP = UP$. Indeed, we have for $\xi \in \alpha(1-P)\tilde{\mathcal{H}}^+$ the equality

$$\|PU^{-1}\xi\|^2 = (UPU^{-1}\xi, \xi) = (\alpha(P)\xi, \xi) = 0.$$

This yields $PU^{-1}\alpha(1-P) = 0$, and so $PU^{-1} = PU^{-1}\alpha(P)$. Therefore, we obtain that $AP = UBJBJP = uBJBJP$ and AP is completely positive.

We next consider a $*$ -isomorphism $\alpha' : \mathcal{M}_{1-P} \rightarrow \tilde{\mathcal{M}}'_{1-\alpha(P)}$ defined by $\alpha'(X) = \tilde{J}\alpha(X)^*\tilde{J}, X \in \mathcal{M}_{1-P}$. There then exists a unique completely positive isometry $v : (1-P)\mathcal{H} \rightarrow \alpha(1-P)\tilde{\mathcal{H}}$ such that

$$v(1-P)\mathcal{H}^+ = (1-\alpha(P))\tilde{\mathcal{H}}^+ \quad \text{and} \quad \alpha'(x) = v x v^{-1}, \quad x \in \mathcal{M}_{1-P}.$$

Then we have $\alpha(x) = \tilde{J}v x^* v^{-1} \tilde{J}, x \in \mathcal{M}_{1-P}$. Note that the complete positivity above means $v_n(1-P)_n \mathcal{H}_n^+ = (1-\alpha(P))_n \tilde{\mathcal{H}}_n^{+'}$, where $\tilde{\mathcal{H}}_n^{+'}$ denotes the selfdual cones associated with $\tilde{\mathcal{M}}'$. Hence v is a completely co-positive map under the setting $(\mathcal{M}, \mathcal{H}, \mathcal{H}_n^+)$ and $(\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \tilde{\mathcal{H}}_n^+)$. Hence

$$\begin{aligned} (UxU^{-1}\xi, \xi) &= (\tilde{J}v x^* v^{-1} \tilde{J}\xi, \xi) \\ &= (\tilde{J}\xi, v x^* v^{-1} \tilde{J}\xi) \\ &= (v x v^{-1}\xi, \xi) \end{aligned}$$

for all $x \in \mathcal{M}_{1-P}, \xi \in (1-P)\mathcal{H}^+$. It follows that $U(1-P) = v$. We conclude by the equality $A(1-P) = vBJBJ(1-P)$ that $A(1-P)$ is completely co-positive.

We now consider a general A . Since $A\mathcal{H}^+ \subset \tilde{\mathcal{H}}^+$, we have $A\mathcal{H}^+ \subset F\tilde{\mathcal{H}}^+$. Since F is a projection, $F\tilde{\mathcal{H}}^+$ is a selfdual cone in $F\tilde{\mathcal{H}}$. It follows from the selfduality of $A\mathcal{H}^+$ that $A\mathcal{H}^+ = F\tilde{\mathcal{H}}^+$. This yields from (4.2) that FAE is an order isomorphism of $E\mathcal{H}$ onto $F\tilde{\mathcal{H}}$ in the sense of matrix ordered standard forms $(EME, E\mathcal{H}, E_n\mathcal{H}_n^+)$

and $(F\tilde{M}F, F\tilde{\mathcal{H}}, F_n\tilde{\mathcal{H}}_n^+)$. Using the first part of the proof, we obtain the desired result. Indeed, there exists a central projection $P \in EME$ such that FAP is completely positive and $FA(E - P)$ is completely co-positive under the reduced matrix ordered standard forms. We obtain the inclusion

$${}^t(A_n(E_n - P_n)\mathcal{H}_n^+) = {}^t(F_n A_n(E_n - P_n)\mathcal{H}_n^+) \subset F_n\tilde{\mathcal{H}}_n^+ \subset \tilde{\mathcal{H}}_n^+.$$

This completes the proof. $\square$

Finally, the author wishes to express his sincere gratitude to Professor Y. Katayama for having pointed out the problem of Section 4 to him.

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