

ON THE CLOSED IMAGES OF A DEVELOPABLE SPACE

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ABSTRACT

We study the properties of the image of a developable space and an orthocompact developable space under a closed mapping, comparing with Lašnev spaces. Two classes C and C' are defined and their properties are given.

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1. Introduction.

Throughout this paper, all spaces are assumed to be T_1 topological ones and mappings to be continuous and onto. The letter N always denotes natural numbers. The letter Z always denotes a convergent sequence of points of a space such that $Z = \{z_n : n \in N\}$ and $Z \rightarrow p$ implies that Z converges to p as $n \rightarrow \infty$. We denote the topology of X by τ_X . We use the brief expressions HCP and IP in place of "hereditarily closure-preserving" and "interior-preserving", respectively.

As a nice generalization of metric spaces, we have a class of developable spaces, which are defined to be ones X having a sequence $\{U_n : n \in N\}$ of open covers of X such that for each point $p \in X$, $\{S(p, U_n) : n \in N\}$ is a local base at p in X . Untill now, the image of a metric space under a closed mapping, called a Lašnev space, is widely studied. But the study of the image of a developable space, briefly called the closed image of a developable space, has not been published yet. In this paper, we begin on its study, especially using the notion of pair-networks. This is our aim of this paper.

To start with, we give the meanings to the special spaces used later. A space X is called semi-stratifiable if there exists a function $O : \{\text{closed subsets of } X\} \times N \rightarrow \tau_X$, called the semi-stratification of X , satisfying the following conditions:

- (1) For each closed subset F of X ,

$$F = \bigcap \{O(F, n) : n \in N\}$$

and $O(F, n+1) \subset O(F, n)$ for each n .

(2) If F, G are closed subsets of X such that $F \subset G$, then $O(F, n) \subset O(G, n)$ for each n .

2. The closed image of a developable space.

DEFINITION [2]. Let $P = \{(F_\alpha, V_\alpha) : \alpha \in A\}$ be a collection of ordered pairs of subsets of a space X .

P is called a pair-network for X if whenever $p \in U \in \tau_X$, there exists $\alpha \in A$ such that $p \in F_\alpha \subset V_\alpha \subset U$. P is called discrete (resp. HCP) if the family $\{F_\alpha : \alpha \in A\}$ is discrete (resp. HCP) in X . P is called σ -discrete (resp. σ -HCP) in X if $P = \bigcup \{P_n : n \in \mathbb{N}\}$ with each P_n discrete (resp. HCP) in X . The other terms for P are similar.

In this paper, we assume that every F_α is closed in X , but every V_α is not necessarily open in X . Unless otherwise is stated explicitly, we assume that P has the members $\{(F_\alpha, V_\alpha) : \alpha \in A\}$ or $\{(F_\alpha, V_\alpha) : \alpha \in A_n, n \in \mathbb{N}\}$.

THEOREM 1. For a Fréchet space X , the following are equivalent:

(1) X has a σ -HCP pair-network P such that if $Z \rightarrow p \in U \in \tau_X$, then there exists $\alpha \in A$ such that $p \in F_\alpha \subset V_\alpha \subset U$ and Z is cofinal in V_α , where Z is cofinal in V_α means $z_n \in V_\alpha$ for infinitely many n .

(2) X has a σ -HCP pair-network P such that if $Z \rightarrow p \in U \in \tau_X$, then there exists $\alpha \in A$ such that $p \in F_\alpha \subset V_\alpha \subset U$ and Z is residual in V_α , where Z is residual in V_α means $\{z_n : n \geq m\} \subset V_\alpha$ for some $m \in \mathbb{N}$.

(3) X has a σ -HCP pair-network P such that if $Z \rightarrow p \in U \in \tau_X$ and $Z \subset X - \{p\}$, then there exists $\alpha \in A$ such that $p \in F_\alpha \subset V_\alpha \subset U$, $F_\alpha - \{p\} \subset \text{Int } V_\alpha$ and Z is residual in $\text{Int } V_\alpha$.

PROOF. (3) $\rightarrow$ (2) $\rightarrow$ (1) is trivial. (1) $\rightarrow$ (3): Let P be a σ -HCP pair-network satisfying the condition of (1). Without loss of generality we can assume $A_n \subset A_{n+1}$ for each $n \in \mathbb{N}$. For each $\delta \subset A_n$, $n \in \mathbb{N}$, let

$$F(\delta) = \bigcap \{F_\alpha : \alpha \in \delta\}, \quad V(\delta) = \bigcup \{V_\alpha : \alpha \in \delta\}.$$

Since the family of all intersections of members of a HCP family is also HCP in a Fréchet space [5, Remark 3.7], the pair-collection

$$P' = \{(F(\delta), V(\delta)) : \delta \subset A_n, n \in \mathbb{N}\}$$

is a σ -HCP pair-network for X . We show that P' has the required properties in (3). Let $Z \rightarrow p \in U \in \tau_X$ and $Z \subset X - \{p\}$. Set for each n

$$\delta_n = \{\alpha \in A_n : p \in F_\alpha \subset V_\alpha \subset U\}.$$

Then $p \in F(\delta_n) \subset V(\delta_n) \subset U$ for each n .

Claim 1: $F(\delta_n) - \{p\} \subset \text{Int } V(\delta_n)$ for some n .

Assume not. Take a sequence $\{p_n : n \in \mathbb{N}\}$ of points such that $p_n \in F(\delta_n) - \{p\} - \text{Int } V(\delta_n)$ for each n . Since X is Fréchet, for each n there exists a convergent sequence $Z(n)$ of points of $X - V(\delta_n)$ such that $Z(n) \rightarrow p_n$. Note that $\{F(\delta_n) : n \in \mathbb{N}\}$ forms a decreasing local network at p in X . Then $p_n \rightarrow p$ as $n \rightarrow \infty$, implying

$$p \in \overline{\bigcup \{Z(n) : n \in \mathbb{N}\}}.$$

Using Fréchet-ness of X , we can take a convergent sequence Z of points of $\bigcup \{Z(n) : n \in \mathbb{N}\}$ such that $Z \rightarrow p$. Because

$p_n \neq p$, $n \in \mathbb{N}$, $Z \cap Z(n) \neq \emptyset$ for infinitely many n . We can take a convergent subsequence $Z' = \{z_{n(k)} : k \in \mathbb{N}\}$

of Z such that $z_{n(k)} \in Z(n_k)$ and $k \leq n(k) < n(k+1)$, $k \in \mathbb{N}$.

By the property of P stated in (1), there exists $\alpha \in A_n$,

$n \in \mathbb{N}$, such that $p \in F_\alpha \subset V_\alpha \subset U$ and Z' is cofinal in V_α . But this is a contradiction because $V_\alpha \subset V(\delta_k)$ for every $k \geq n$. Hence Claim 1 is established.

Claim 2: Z is residual in $\text{Int } V(\delta_m)$ for some m .

Assume the contrary, i.e., Z is cofinal in $X - \text{Int } V(\delta_n)$ for every n . Then there exists a subset $\{k(n) : n \in \mathbb{N}\}$ of $\mathbb{N}$ such that $z_{k(n)} \in X - \text{Int } V(\delta_n)$ and $k \leq k(n) < k(n+1)$, $n \in \mathbb{N}$. Using Fréchet-ness of X , for each n we can take a sequence $Z(n)$ of points of $X - V(\delta_n)$ such that $Z(n) \rightarrow z_{k(n)}$. Since $z_{k(n)} \notin p$ for each n , we can use the same argument as above to get a contradiction, which implies the validity of Claim 2. Now, let k be the maximum of n and m in Claims 1 and 2, respectively. Since $F(\delta_s) \subset F(\delta_t)$ and $V(\delta_t) \subset V(\delta_s)$ for every s, t with $t \leq s$, and this k satisfies both claims. This completes the proof (1) $\rightarrow$ (3).

In the sequel, we denote by $\mathcal{C}$ the class of all Fréchet spaces satisfying one and hence all of (1) to (3) in Theorem 1. With respect to the properties of $\mathcal{C}$, the following hold:

THEOREM 2. $\mathcal{C}$ has the following properties:

- (1) $\mathcal{C}$ is closed under closed mappings.
- (2) $\mathcal{C}$ is closed under subspaces.
- (3) $\{\text{closed images of a developable space}\} \subset \mathcal{C}$.
- (4) $\mathcal{C}$ is not finitely productive.

All except (4) are easily seen from Theorem 1. (4) is a direct consequence of Theorem 9, stated later.

We give a characterization of developable spaces in terms of pair-networks somewhat different from the results of Burke [2, Theorem 2.1].

THEOREM 3. For a space X , the following are equivalent:

- (1) X is a developable space.
- (2) X is first countable and $X \in \mathcal{C}$.
- (3) X is a strongly Fréchet space having a σ -locally finite pair-network $\mathcal{P}$ satisfying the same condition as in Theorem 1, (1).

(4) X has a σ -locally finite pair-network $\mathcal{P}$ such that each V_α is open in X .

PROOF. As well-known, a space X is developable if and only if X has a σ -discrete pair-network $\mathcal{P}$ such that each V_α is open in X , [4]. So, (1) $\rightarrow$ (4) $\rightarrow$ (2) and (4) $\rightarrow$ (3) are obvious. (2) $\rightarrow$ (1): We shall show that

X has a σ -discrete pair-network $\mathcal{P}$ such that all V_α are open in X . Let $\mathcal{P}$ be a σ -HCP pair-network for X satisfying the same condition as in Theorem 1, (1). For each $n \in \mathbb{N}$, let

$$X_n = \{p \in X : \text{ord}(p, F_n) \geq \aleph_0\},$$

where $F_n = \{F_\alpha : \alpha \in A_n\}$. Since X is Fréchet and each F_n is HCP in X , each X_n is a discrete closed subset of X . Let

$$X_{0n} = \{p \in X : F_\alpha - \text{Int } V_\alpha = \{p\} \text{ for some } \alpha \in A_n, n \in \mathbb{N}\}.$$

Then obviously $\bigcup \{X_{0n} : n \in \mathbb{N}\}$ is a σ -discrete closed subset of X . For each n , by the method of [10] we can construct a σ -discrete family H_n of closed subsets of X from

$$B_n = F_n \cup \{\{x\} : x \in X_{0n} \cup X_n\}$$

such that H_n satisfying the following: For each subfamily

$B_0 \subset B_n$, if $p \in B_0 - \bigcup (B_n - B_0)$, then $p \in H \subset B_0 - \bigcup (B_n - B_0)$ for some $H \in H_n$. For each $H \in H_n$, $n \in \mathbb{N}$, with $H \cap (X_{0n} \cup X_n) = \emptyset$, choose an open subset $V(H)$ of X such that

$$H \subset V(H) \subset \bigcap \{\text{Int } V_\alpha : \alpha \in \delta\},$$

where δ is a finite subset of A_n such that

$$H \subset \bigcap \{F_\alpha : \alpha \in \delta\} - \bigcup \{F_\alpha : \alpha \notin A_n - \delta\}.$$

For each point $p \in X$, let $\{O_n(p) : n \in \mathbb{N}\}$ be a local base at p in X . Construct the pair-collection

$$\begin{aligned} P' = & \{(\{p\}, O_n(p)) : p \in X_k, k, n \in \mathbb{N}\} \\ & \cup \{(\{p\}, O_n(p)) : p \in X_{0n}, k, n \in \mathbb{N}\} \\ & \cup \{(H, V(H)) : H \in H_n', n \in \mathbb{N}\}, \end{aligned}$$

where

$$H_n' = \{H \in H_n : H \cap (X_{0n} \cup X_n) = \emptyset\}, n \in \mathbb{N}.$$

Then it is easy to see that P' is a σ -discrete pair-network for X such that the second subset of each pair of P' is open in X , proving that X is developable.

(3) $\rightarrow$ (2): It suffices to show that X is first countable.

Let $P = \bigcup \{P_n : n \in \mathbb{N}\}$ be a pair-network for X satisfying the same condition as in Theorem 1, (1), where each $P_n = \{(F_\alpha, V_\alpha) : \alpha \in A_n\}$ is locally finite in X . Without loss of generality we can assume $A_n \subset A_{n+1}$, $n \in \mathbb{N}$. For each point p , $A_n(p) = \{\alpha \in A_n : p \in F_\alpha\}$, $n \in \mathbb{N}$, is finite. For each n , set

$$\Delta_n = \{\delta \subset A_n(p) : p \in \text{Int } V(\delta)\},$$

where

$$V(\delta) = \bigcup \{V_\alpha : \alpha \in \delta\}, \delta \in \Delta_n.$$

We show that

$$\{\text{Int } V(\delta) : \delta \in \bigcup \{\Delta_n : n \in \mathbb{N}\}\}$$

is a local base at p in X . Let $p \in U \in \tau_X$. For each n , we take $\delta_n \subset A_n(p)$ such that

$$\delta_n = \{\alpha \in A_n(p) : V_\alpha \subset U\}.$$

Assume $p \notin \text{Int } V(\delta_n)$ for each n . Since X is strongly Fréchet, there exists a sequence $\{p_n : n \in \mathbb{N}\}$ of points of X such that $p_n \rightarrow p$ and $p_n \notin V(\delta_n)$, $n \in \mathbb{N}$. By the property of P , there

exists $\alpha \in A_n$, $n \in \mathbb{N}$, such that $p \in F_\alpha \subset V_\alpha \subset U$ and $\{p_n\}$ is cofinal in V_α . But this is a contradiction. Hence we have $p \in \text{Int } V(\delta_n) \subset U$ for some m .

As the corollaries, we have two: The former is already known [9, Cor. to Proposition 4] and the latter is known for the case when X is an Moore space [3, Corollary 1.1]. The proof of the latter is the same as that of (2) $\rightarrow$ (1).

COROLLARY 1. If a closed image of a developable space is first countable, then it is developable.

COROLLARY 2. If X is a closed image of a developable space, then $X = X_0 \cup X_1$, where X_0 is a σ -discrete closed subset and X_1 is a developable space.

The proof of (3) $\rightarrow$ (2) above assures the following theorem:

THEOREM 4. If X is a strongly Fréchet space and $X \in \mathcal{C}$, then X has a σ -HCP pair-network such that all V_α are open in X .

But we do not know whether such a space is developable.

QUESTION 1. If X is a strongly Fréchet space and $X \in \mathcal{C}$, then is X developable ?

The following gives another characterization of the class $\mathcal{C}$, which is similar to that of Lašnev spaces in terms of σ -HCP k -networks by Foged.

THEOREM 5. A space X belongs to $\mathcal{C}$ if and only if X is a Fréchet space which has a σ -HCP pair-network $\mathcal{P}$ such that if $K \subset U \in \tau_X$ with K compact in X , then there exists a finite subcollection $\{(F_\alpha, V_\alpha) : \alpha \in \delta\}$ of $\mathcal{P}$ such that

$$K \subset \bigcup \{V_\alpha : \alpha \in \delta\} \subset U$$

and $K \cap F_\alpha \neq \emptyset$ for each $\alpha \in \delta$.

PROOF. If part is trivial. Only if part: Let $\mathcal{P}$ be a σ -HCP pair-network for X satisfying the condition of Theorem 1, (1). Assume $A_n \subset A_{n+1}$ for each n . For each $\delta \subset A_n$, $n \in \mathbb{N}$, set

$$F(\delta) = \bigcap \{F_\alpha : \alpha \in \delta\}, \quad V(\delta) = \bigcup \{V_\alpha : \alpha \in \delta\}$$

and

$$\mathcal{Q} = \{(F(\delta), V(\delta)) : \delta \subset A_n, n \in \mathbb{N}\}.$$

Then $\mathcal{Q}$ is a σ -HCP pair-network for X . We shall show that $\mathcal{Q}$ has the required property. Let $K \subset U \in \tau_X$ with K compact in X . For each $n \in \mathbb{N}$, let

$$A_{0n} = \{\alpha \in A_n : F_\alpha \cap K \neq \emptyset \text{ and } V_\alpha \subset U\}.$$

Then HCP-ness of $\{F_\alpha : \alpha \in A_n\}$ implies

$$\{F_\alpha : \alpha \in A_{0n}\} \upharpoonright K = \{F_1, F_2, \dots, F_{k(n)}\}$$

with some $k(n) \in \mathbb{N}$, [7, Proposition 3.7]. For each i with $1 \leq i \leq k(n)$, choose $(F(\delta_{ni}), V(\delta_{ni})) \in \mathcal{Q}$ such that

$$\delta_{ni} = \{\alpha \in A_{0n} : F_\alpha \cap K = F_i\}.$$

Obviously $\bigcup \{V(\delta_{ni}) : 1 \leq i \leq k(n)\} \subset U$. Assume

$$K \not\subset \bigcup \{V(\delta_{ni}) : 1 \leq i \leq k(n)\}$$

for each n . Choose a sequence $\{p_n : n \in \mathbb{N}\}$ of points of X such that

$$p_n \in K - \bigcup \{V(\delta_{ni}) : 1 \leq i \leq k(n)\}, n \in \mathbb{N}.$$

Since K is metrizable, $\{p_n\}$ has a convergent subsequence Z to some point $p \in K$ in X . By the property of $\mathcal{P}$, there exists $\alpha_0 \in A_m$, $m \in \mathbb{N}$, such that $p \in F_{\alpha_0} \subset V_{\alpha_0} \subset U$ and

Z is cofinal in V_{α_0} . But this is a contradiction because

$$V_{\alpha_0} \subset \bigcup \{V(\delta_{m_i}) : 1 \leq i \leq k(m)\}.$$

This completes the proof.

Viewing Theorem 1, (1), we can easily observe that a space X belongs to C if and only if X is a Fréchet space having a σ -HCP pair-network P such that the following conditions:

(C1) For each $\alpha \in A$, there exists an open subset W_α of X such that $V_\alpha = F_\alpha \cup W_\alpha$.

(C2) If $Z \rightarrow p \in U \in \tau_X$ and $Z \subset X - \{p\}$, then there exists $\alpha \in A$ such that $p \in F_\alpha \subset V_\alpha \subset U$, $F_\alpha - \{p\} \subset W_\alpha$ and Z is residual in W_α .

By setting one more additional condition to P , we define a class C' of spaces as follows: A space X belongs to C' if and only if X is a Fréchet space having a σ -HCP pair-network P satisfying the following additional condition (IP) besides (C1) and (C2):

(IP) For each n , $W_n = \{W_\alpha : \alpha \in A_n\}$ is an IP family of open subsets of X .

With respect to the properties of C' , the following holds and that corresponds to Theorem 2 for C .

THEOREM 6. C' has the following properties:

- (1) C' is closed under closed mappings.
- (2) C' is closed under subspaces.
- (3) A closed image of an orthocompact developable space belongs to C' .
- (4) C' is not finitely productive.

PROOF. (2) is obvious and (4) is a direct consequence of Theorem 9. So, we state the proofs of (1) and (3) only. First, we show (1). Let $f : X \rightarrow Y$ be a closed mapping of X onto a space Y and let $X \in \mathcal{C}'$. Let P be a σ -HCP pair-network for X assured by the definition of $X \in \mathcal{C}'$. Assume $A_n \subset A_{n+1}$, $n \in \mathbb{N}$. For each $\delta \subset A_n$, $n \in \mathbb{N}$, set

$$\begin{aligned} F(\delta) &= \bigcap \{f(F_\alpha) : \alpha \in \delta\}, \\ W(\delta) &= Y - f(X - \bigcup \{W_\alpha : \alpha \in \delta\}), \\ V(\delta) &= F(\delta) \cup W(\delta). \end{aligned}$$

Obviously $\{F(\delta) : \delta \subset A_n\}$ is a HCP family of closed subsets and $\{W(\delta) : \delta \subset A_n\}$ is an IP family of open subsets of Y . Thus, the pair-collection

$$P' = \{(F(\delta), V(\delta)) : \delta \subset A_n, n \in \mathbb{N}\}$$

is a σ -HCP pair-network for Y satisfying (C1) and (IP).

We show that P' satisfies the condition (C2) in Y . Let

$Z \rightarrow y \in U \in \tau_Y$ and $Z \subset Y - \{y\}$. For each n , let

$$\delta_n = \{\alpha \subset A_n : F_\alpha \cap f^{-1}(y) \neq \emptyset \text{ and } V_\alpha \subset f^{-1}(U)\}.$$

Then obviously, without loss of generality we can assume

$y \in F(\delta_n) \subset V(\delta_n) \subset U$ for each $n \in \mathbb{N}$.

Claim 1: $F(\delta_n) - \{y\} \subset W(\delta_n)$ for some m .

To see it, assume the contrary. Then we can choose a point $p_n \in F(\delta_n) - \{y\} - W(\delta_n)$ for each n . Since $\{F(\delta_n) : n \in \mathbb{N}\}$ forms a decreasing local network at y in Y , $p_n \rightarrow y$ as $n \rightarrow \infty$ in Y . Using the closedness of f and Fréchet-ness of X , we can choose a sequence $\{q_{n(k)} : k \in \mathbb{N}\}$ of points of $X - f^{-1}(y)$ such that $\{q_{n(k)}\}$ converges to some point of $f^{-1}(y)$, $f(q_{n(k)}) = p_{n(k)}$

and

$$q_{n(k)} \notin \bigcup \{W_\alpha : \alpha \in \delta_{n(k)}\} \text{ for each } k$$

where $k \leq n(k) < n(k+1)$, $k \in \mathbb{N}$. By (C2) of C' , there exists

$\alpha \in A_n$, $n \in \mathbb{N}$, such that $\{q_{n(k)}\}$ is residual in W_α and

$\alpha \in \delta_n$. But this is a contradiction. Hence Claim 1 is established.

By the same argument as above, we can show that Z is residual in $W(\delta_m)$ for some m . This completes the proof of (1).

Since an orthocompact developable space X has a σ -discrete pair-network P such that for each n $\{V_\alpha : \alpha \in A_n\}$ is an IP family of open subsets of X , obviously $X \in C'$, which combined with (1) implies (3).

We give two lemmas used in the proof of Theorem 7.

LEMMA 1. Let $X \in C'$. Then for each discrete family $\{F_\lambda : \lambda \in \Lambda\}$ of closed subsets of X there exist families $\{\omega_\lambda : \lambda \in \Lambda\}$ of open subsets of X satisfying the following:

(1) For each λ , ω_λ is an outer base of F_λ in X .

(2) $\bigcup \{\omega_\lambda \mid (X - F_\lambda) : \lambda \in \Lambda\}$ is IP in X .

PROOF. For each $\lambda \in \Lambda$, there exists a sequence $\{O(\lambda, n) : n \in \mathbb{N}\}$ of open subsets of X such that

$$F_\lambda = \bigcap \{O(\lambda, n) : n \in \mathbb{N}\},$$

$$O(\lambda, n+1) \subset O(\lambda, n) \subset O(F_\lambda, n) \cap (X - \bigcup \{F_\mu : \mu \neq \lambda\}).$$

Let P be a σ -HCP pair-network for X assured by $X \in C'$. Let

$\lambda \in \Lambda$ be fixed for a while. Set

$$\omega_n = \{W_\alpha \cap O(\lambda, n) : \alpha \in A_n\}, n \in \mathbb{N}.$$

Let $\{\omega(\delta) : \delta \in \Delta(\lambda)\}$ be the totality of subfamilies of

$\bigcup \{w_n : n \in \mathbb{N}\}$ such that

$$W(\delta) = F_\lambda \cup \left(\bigcup w(\delta) \right)$$

is an open neighborhood of F_λ in X . We show that $\{W(\delta) :$

$\delta \in \Delta(\lambda)\}$ is an outer base of F_λ in X . Let $F_\lambda \subset O \in \tau_X$. Let

$$w_n' = \{W \in w_n : W \subset O\}, n \in \mathbb{N},$$

$$w(\delta) = \bigcup \{w_n' : n \in \mathbb{N}\}.$$

Then $F_\lambda \subset W(\delta) \subset O$. To see that $W(\delta)$ is open in X , assume the contrary. Take a point $p \in F_\lambda - \text{Int } W(\delta)$. Since X is Fréchet, there exists a sequence Z of points of $X - W(\delta)$ such that $Z \rightarrow p$ in X . By the property of P , we can choose $\alpha \in A_n$, $n \in \mathbb{N}$, such that

$$p \in F_\alpha \subset V_\alpha \subset O, F_\alpha - \{p\} \subset W_\alpha$$

and Z is residual in W_α . This implies also that Z is residual in $W_\alpha \cap O(\lambda, n)$. But this is a contradiction. From the property (IP) of P , we can easily see that (2) is satisfied for thus constructed

$$w_\lambda = \{W(\delta) : \delta \in \Delta(\lambda)\}, \lambda \in \Lambda.$$

This completes the proof.

A space X is called d-IP-expandable [6] if for each discrete family $\{F_\lambda : \lambda \in \Lambda\}$ of closed subsets and each family $\{U_\lambda : \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset U_\lambda$, $\lambda \in \Lambda$, there exists an IP family $\{V_\lambda : \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset V_\lambda \subset U_\lambda$, $\lambda \in \Lambda$.

LEMMA 2. If $X \in \mathcal{C}'$, then X is orthocompact.

PROOF. By the lemma above, X is d-IP-expandable.

Since a submetacompact, d-IP-expandable space is orthocompact, [6, Theorem 2.5], X is orthocompact.

From Lemmas 2 and 3, we have a characterization of orthocompact developable spaces in terms of pair-networks as follows:

THEOREM 7. For a space X , the following are equivalent:

- (1) X is an orthocompact developable space.
- (2) X is a first countable space and $X \in \mathcal{C}'$.

A space X is called d-paracompact [1] if for each open cover $\mathcal{U}$ of X , there exists a $\mathcal{U}$ -mapping of X onto a developable space. A space X is called subdevelopable if τ_X contains a developable subtopology. With respect to the notions, we have the following:

THEOREM 8. If $X \in \mathcal{C}'$, then X is both d-paracompact and subdevelopable.

PROOF. If $X \in \mathcal{C}'$, then by Lemma 1 X is D-expandable and hence is d-paracompact [1, Theorem 1]. Since a d-paracompact space with a G_δ -diagonal is subdevelopable [8, Theorem 4], X is subdevelopable.

But we do not know whether the above holds for the class $\mathcal{C}$.

QUESTION 2. If $X \in \mathcal{C}$, then is X d-paracompact or subdevelopable?

It is well-known as Heyman's result that for any non-discrete spaces X, Y , the product space $X \times Y$ being Lašnev means both X, Y are metrizable. This is true for the class $\mathcal{C}'$.

we state it more generally.

THEOREM 9. Let X, Y be non-discrete spaces. If $X \times Y \in \mathcal{C}'$, then $X \times Y$ is an orthocompact developable space.

PROOF. By the virtue of Theorem 7, it suffices to show that both X, Y are first countable. Let P be a σ -HCP pair-network for $X \times Y$ defining $X \times Y \in \mathcal{C}'$. Let Z be a sequence of points of X such that $Z \rightarrow x$ and $Z \subset X - \{x\}$. Let y be an arbitrary point of Y . We show that y has a countable local base in Y . Obviously

$$Z' = \{(z_k, y) : k \in \mathbb{N}\} \rightarrow (x, y)$$

in $X \times Y$. Since $\{W_\alpha : \alpha \in A_n\}$, $n \in \mathbb{N}$, is IP in $X \times Y$ by (IP), for each pair $(m, n) \in \mathbb{N}^2$ with

$$(z_m, y) \in W_\alpha \quad \text{for some } \alpha \in A_n,$$

there exists an open subset $O(m, n)$ of X such that

$$(z_m, y) \in O(m, n) \subset \bigcap \{W_\alpha : \alpha \in A_n, (z_m, y) \in W_\alpha\}.$$

Let N_0 be the totality of such pairs (m, n) . Let $p : X \times Y \rightarrow Y$ be the projection. By the property of P , it is easily seen that $\{p(O(m, n)) : (m, n) \in N_0\}$ is a local base at y in Y . This completes the proof.

COROLLARY. Let X, Y be non-discrete spaces. If $X \times Y$ is the closed image of an orthocompact developable space, then $X \times Y$ is an orthocompact developable space.

But, we do not know whether Theorem 9 holds for the class $\mathcal{C}$:

QUESTION 3. For non-discrete spaces X , Y , does $X \times Y$ imply that $X \times Y$ is developable ?

Finally, we pose the following question about the characterization of a closed image of a developable space:

QUESTION 4. If a space X belongs to $\mathcal{C}$, then is X a closed image of a developable space ?

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