

# On the spectrum of Dirac operators

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## 1 Introduction

Let  $\alpha_j$ ,  $j = 1, 2, 3$ ,  $\beta = \alpha_0$  be Hermite (symmetric)  $4 \times 4$  matrices which satisfy the following anti-commuting relations.

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}, \quad 0 \leq \forall j, \forall k \leq 3, \quad (1.1)$$

where  $\delta_{jk}$  denotes Kronecker's delta. The Dirac operator with which we are concerned is defined by

$$H_D u = c \sum_{j=1}^3 \alpha_j D_{x_j} u + mc^2 \beta u + V(x)u, \quad x \in \mathbf{R}^3, \quad (1.2)$$

where  $c$  is the speed of light,  $m$  is a non-negative number and  $V$  is a real-valued function defined on  $\mathbf{R}^3$ . It holds that the free Dirac operator

$$H_0 = c\alpha \cdot D_x + mc^2 \beta,$$

is essentially self-adjoint on  $C_0^\infty(\mathbf{R}^3; \mathbf{C}^4)$  in  $\mathcal{H} = L^2(\mathbf{R}^3; \mathbf{C}^4)$  and

$$\sigma(H_0) = (-\infty, -mc^2] \cup [mc^2, +\infty).$$

in view of the identity  $H_0^2 = (-c^2 \Delta + m^2 c^4) I_4$ .

When the potential decays at infinity, we may expect that the spectrum is almost equal to that of the free operator. In fact when the potentials  $V$  are short range type it holds that

- (1)  $\sigma_{ess}(H_D) = (-\infty, -mc^2] \cup [mc^2, \infty)$ ,
- (2)  $\sigma_{sc}(H_D) = \emptyset$ ,
- (3)  $\sigma_p(H_D) \subset [-mc^2, mc^2]$  is an at most countable set whose elements can only accumulate to the points  $\pm mc^2$  (O.Yamada [12]). This result has been extended to a class of long range type potentials (V. Vogelsang [11]).

On the other hand if we allow the potential to be diverge at infinity, the situation is dramatically changed. In fact it turns out (H. Kalf, T. Okaji and O. Yamada [8]) that the spectrum of  $H_D$  coincides with  $\mathbf{R}$  and there exist no eigenvalues if potentials fulfill the following conditions. There exists a positive number  $\delta$  such that as  $|x| \rightarrow \infty$ ,

$$V(x) - q(|x|)I = \mathcal{O}(|x|^{-1/2-\delta}q^{1/2}(|x|)),$$

$$\partial_r(V(x) - q(|x|)I) = \mathcal{O}(|x|^{-1-\delta}q(|x|)),$$

where  $q(r)$  is a real valued  $C^2([0, \infty))$  function diverging at infinity, which satisfies the following conditions.

- i)  $\inf_{r \in I_a} q(r) > 0$ ,                      ii)  $[q'(r)]_- = \mathcal{O}(r^{-1-\delta}q)$ ,
- iii)  $q'(r) = \mathcal{O}(r^{-1/2-\delta}q^{3/2})$ ,      iv)  $q''(r) = \mathcal{O}(r^{-1-\delta}q^2)$

Our purpose is to investigate the spectrum of Dirac operators with potentials neither decaying nor diverging. In this paper we consider potentials homogeneous of degree zero.

## 2 Main results

We always assume that

(V1)  $V \in C^\infty(\mathbf{R}^3 \setminus \{0\})$  is homogeneous of degree zero.

Let  $V_+ = \max_{\omega \in S^2} V(\omega)$ ,  $V_- = \min_{\omega \in S^2} V(\omega)$ ,  $\Sigma_V = \{\omega \in S^2; \nabla V(\omega) = 0\}$  and define the threshold set

$$\tau(H_D) = (V(\Sigma_V) + mc^2) \cup (V(\Sigma_V) - mc^2).$$

Then our first result for the Dirac operator is as follows.

**Theorem 2.1** *Suppose that (V-1) holds,  $m$  is nonnegative and  $\Sigma_V$  is at most countable. Then*

- 1)  $\sigma(H_D) = (-\infty, V_+ - mc^2] \cup [V_- + mc^2, +\infty)$
- 2)  $\sigma_p(H_D)$  is an at most countable set whose elements can only accumulate to  $\tau(H_D)$ ,
- 3)  $\sigma_{sc}(H_D) = \emptyset$ .

We can improve the above result when the light speed  $c$  is large enough.

**Theorem 2.2** *Suppose that (V1) holds and  $m$  is positive. Then there exists a positive constant  $c_0$  such that the spectrum of  $H_D$  is purely absolutely continuous if  $c \geq c_0$ .*

For Schrödinger operators

$$H_S = -\frac{1}{2}\Delta + V(x) \quad \text{in } L^2(\mathbf{R}^d), \quad d \geq 3$$

it is well known that the same conclusion is true for more general potentials. In fact Lavine [7] proved that if  $(x \cdot D)V \leq 0$  (repulsive), then the spectrum of  $H_D$  is purely absolutely continuous. In connection to the study of asymptotic behavior of solutions Herbst [3] has proved a uniform limiting absorption principle for homogeneous potentials of degree zero by use of complex dilation method. Later Agmon, Cruz and Herbst [1] applied Mourre theory to generalize it and Hassel, Melrose and Vasy [2] investigated more general operator from the view point of propagation of singularities.

### 3 Idea of proof of Theorem 2.1

Define a unitary operator  $(Uf)(x) = (h/c)^{-3/2}f(hx/c)$ . Then

$$U^{-1}cD_xU = hD_x := p \text{ and } U^{-1}VU = V.$$

Let  $H = U^{-1}H_DU = \alpha \cdot p + mc^2\beta + V$  and define a selfadjoint operator  $A_1$ , called conjugate operator

$$A_1 = \frac{1}{2h} (H_0^{-1}px + xpH_0^{-1}) - \frac{\gamma}{2h}(G + G^*)$$

where  $H_0 = \alpha \cdot p + mc^2\beta$ ,  $\gamma > 0$  is a small parameter and  $G = E_h^{-2}p \cdot \nabla(\tilde{V}(x))$ . Here

$$E_h = \begin{cases} \sqrt{|hD|^2 + m^2c^4}, & \text{if } m > 0 \\ \sqrt{|hD|^2 + 1}, & \text{if } m = 0. \end{cases}$$

and  $\tilde{V}(x) = (1 - \chi_\varepsilon(x))|x|^2V(x)$  with  $\chi_\varepsilon \in C_0^\infty(\mathbf{R}^3)$  satisfying

$$\chi_\varepsilon(x) = \begin{cases} 0 & |x| \geq 2\varepsilon \\ 1 & |x| \leq \varepsilon \end{cases}, \quad \varepsilon > 0.$$

If  $m > 0$ , then  $\frac{hD}{H_0} = hDH_0/E_h^2$  is called the classical velocity operator ([10]).

Theorem 2.1 is a simple consequence of Mourre estimates [9] outside the threshold  $\tau(H)$ .

**Theorem 3.1** *Let  $I \subset \mathbf{R} \setminus \tau(H)$ . Then there exist positive constant  $\delta > 0$  and a compact operator  $K$  in  $L^2(\mathbf{R}^3; \mathbf{C}^4)$  such that*

$$E_H(I)[iH, A_1]E_H(I) \geq \delta E_H(I) + K,$$

where  $E_H$  is the spectral projection of  $H$ .

The most essential step to prove Mourre estimate is the following result.

**Lemma 3.2**

- (i)  $[iH_0, A_1] = -h^2 \Delta E_h^{-2}$ ,
- (ii) if  $\varepsilon$ ,  $\gamma$  and  $h$  are small enough, then there exists a positive constant  $C$  such that

$$[iH, A_1] \geq \frac{1}{2} E_h^{-1} \{ |hD|^2 + W(x, D) \} E_h^{-1},$$

$$W(x, D) = \gamma|x|^2|\nabla V(x)|^2 - C(\gamma + h)|hD|,$$

- (iii)  $[[H, iA_1], iA_1] \leq C$

Once the Mourre estimate is verified, it holds that the (local) limiting absorption principle.

**Theorem 3.3** *If  $J \subset \subset \mathbf{R} \setminus \{\tau(H) \cup \sigma_p(H)\}$ , then for any  $s > 1/2$ ,*

$$\sup_{\lambda \in J, \varepsilon > 0} \| \langle x \rangle^{-s} (H - \lambda \mp i\varepsilon)^{-1} \langle x \rangle^{-s} \|_{\mathbf{B}(L^2)} < \infty.$$

## 4 Uniform limiting absorption principle

To prove Theorem 2.2, we shall establish a uniform limiting absorption principle which is derived from the one for relativistic Schrödinger operators  $H_R$ .

Define a unitary operator  $(Uf)(x) = (mc)^{-3/2} f(mc x)$ . Then

$$U^{-1} c D_x U = mc^2 D_x \text{ and } U^{-1} V U = V.$$

Then

$$U^{-1}H_DU = mc^2 \left\{ \alpha \cdot D_x + \beta + \frac{1}{mc^2}V \right\}$$

Let  $p = D_x$  and

$$H = \alpha \cdot p + \beta + \frac{1}{mc^2}V, \quad H_R = \sqrt{|p|^2 + 1} + \frac{1}{mc^2}V.$$

It is known that there exists a unitary operator  $T$ , called Foldy-Wouthuysen-Tani transform defined later explicitly such that

$$T(\alpha \cdot p + \beta)T^{-1} = \begin{pmatrix} EI_2 & 0 \\ 0 & -EI_2 \end{pmatrix},$$

where  $E = \sqrt{|p|^2 + 1}$  is called the relativistic Schrödinger operators. Consider two unitary operator  $T_{\pm}$  on  $L^2(\mathbf{R}^3; \mathbf{C}^4)$

$$T_{\pm} = \sqrt{\frac{E+1}{2E}}I_4 \pm \sqrt{\frac{E-1}{2E}}\beta \frac{\alpha \cdot D}{|D|} = \frac{1}{\sqrt{2E}} \left( \sqrt{E+1}I_4 \pm \beta \frac{\alpha \cdot D}{\sqrt{E+1}} \right).$$

Then it holds that

$$T_+^* = T_-, \quad T_+T_- = I_4.$$

$T = T_+$  is called Foldy-Wouthuysen-Tani transform  $T$  ([10]).

The result in Theorem 2.1 can be improved if we consider  $H$  as a perturbation of a pair of relativistic Schrödinger operators because

$$H_R = \sqrt{|D|^2 + 1} + \frac{1}{mc^2}V(x)$$

has a nice property as follows.

**Theorem 4.1** *Let  $m > 0$  and  $H_R = \sqrt{|D|^2 + 1} + \frac{1}{mc^2}V$ . Suppose that (V1). Then there exist positive constants  $L$  and  $c_0$  such that*

$$\sup_{\lambda \in \mathbf{R}, \varepsilon > 0} \|\langle x \rangle^{-1}(H_R - \lambda \mp i\varepsilon)^{-1}\langle x \rangle^{-1}\|_{\mathbf{B}(L^2)} < L, \text{ for all } c > c_0$$

**Corollary 4.2** *If  $c$  is large enough, then the spectrum of  $H_R$  is purely absolutely continuous.*

We can show that the conclusion of Corollary 4.2 is true without limitation on  $c$  by using the Mourre theory and absence of eigenvalues of  $H_R$ . In applying a perturbation argument, however, we need a uniform estimate as in Theorem 4.1

**Theorem 4.3** *Suppose that (V1) . Then there exists a positive constant  $c_0$  such that if  $c > c_0$ , then it holds that*

$$\sup_{\lambda \in \mathbf{R}, \varepsilon > 0} \|\langle x \rangle^{-1} (THT^{-1} - \lambda \mp i\varepsilon)^{-1} \langle x \rangle^{-1}\|_{\mathbf{B}(L^2)} < \infty.$$

**Corollary 4.4** *If  $c$  is large enough, then the spectrum of  $H_D$  is purely absolutely continuous.*

## 5 Proof of Theorem 4.3

Approximate  $H$  by a pair of  $\pm H_R$  via FWT transform.

$$\begin{aligned} & T_+(\alpha \cdot D + \beta + V)T_- \\ &= \begin{pmatrix} \sqrt{|D|^2 + 1} + \frac{1}{mc^2}V & 0 \\ 0 & -(\sqrt{|D|^2 + 1} - \frac{1}{mc^2}V) \end{pmatrix} + W, \quad (5.1) \end{aligned}$$

where

$$W = \frac{1}{mc^2} \{T_+VT_- - V\}.$$

Let  $\tilde{H} = T_+(\alpha \cdot D + \beta)T_- + (mc^2)^{-1}V$ . Then  $T_+HT_- = \tilde{H} + W$ . We shall use the following results to handle the remainder term  $W$ .

**Lemma 5.1** *Suppose*

$$\sup_{\lambda \in J, \varepsilon > 0} \|\langle x \rangle^{-s} (\tilde{H} - \lambda \mp i\varepsilon)^{-1} \langle x \rangle^{-s}\| = M < \infty.$$

*If  $\|\langle x \rangle^s W(x) \langle x \rangle^s\| < \frac{1}{M}$ , then the same conclusion is valid for  $H$  replaced by  $H + W$ .*

**Lemma 5.2 (Remainder estimate):** Let  $\tilde{V}(x) = (1 - \chi_\varepsilon(x))V(x)$ . Then

$$T_+ \tilde{V} T_- - \tilde{V} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}.$$

where

$$\langle x \rangle W_{jj} \langle x \rangle \in \mathcal{L}(\mathcal{H}), \quad \langle x \rangle^{1/2} W_{jk} \langle x \rangle^{1/2} \in \mathcal{L}(\mathcal{H}), \quad j \neq k.$$

**Proof:** Let

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \text{with } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then it holds that

$$T_\pm = a + b, \quad a = \begin{pmatrix} A_+ & 0 \\ 0 & A_+ \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \pm A_- \\ \mp A_- & 0 \end{pmatrix}.$$

Here

$$A_+ = \frac{1}{\sqrt{2E}} \sqrt{E+1}, \quad A_- = \frac{\sigma \cdot D}{\sqrt{2E} \sqrt{E+1}}.$$

Note that  $a^* = a$ ,  $b^* = -b$  and

$$\begin{aligned} T_+ V T_- - V &= \frac{1}{2} [[a, V], a] - \frac{1}{2} [[b, V], b] \\ &\quad + \frac{1}{2} ([a, V]b + b[a, V]) - \frac{1}{2} ([b, V]a + a[b, V]). \end{aligned} \quad (5.2)$$

To derive the conclusion we use a calculus of  $\Psi$ DO. Let  $g = (|x|^2 + 1)^{-1} dx^2 + (|\xi|^2 + 1)^{-1} d\xi^2$  be a metric on  $\mathbf{R}^{2d}$ . A smooth function  $a(x, \xi)$  defined on  $\mathbf{R}^d \times \mathbf{R}^d$  belongs to a class of symbols  $S_\ell^m(g)$  if

$$\forall \alpha, \beta, \quad |\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C \langle x \rangle^{\ell - |\beta|} \langle \xi \rangle^{m - |\alpha|}, \quad (x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d$$

Define the pseudo-differential operator  $\text{OP}(a)$  with symbol  $a$  by

$$\text{OP}(a)u(x) = (2\pi)^{-d} \int e^{i(x-y) \cdot \xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi, \quad u \in C_0^\infty(\mathbf{R}^d)$$

It is easily verified that  $W_{jk}$  is a  $2 \times 2$  matrix-valued pseudo-differential operator with symbol

$$W_{jj}(x, \xi) \in S_{-2}^{-2}(g), \quad W_{jk}(x, \xi) \in S_{-1}^{-1}(g), \quad j \neq k.$$

Let  $\hat{H} = \begin{bmatrix} H_+ & W_{12} \\ W_{21} & H_- \end{bmatrix}$  where

$$H_+ = E + \frac{1}{mc^2}V, \quad H_- = -(E - \frac{1}{mc^2}V)$$

Then  $(U^{-1}H_D U - z)u = f$  is equivalent to

$$(\hat{H} - \frac{1}{mc^2}z) \begin{bmatrix} u_+ \\ u_- \end{bmatrix} = \frac{1}{mc^2} \begin{bmatrix} f_+ \\ f_- \end{bmatrix},$$

which means that if  $\zeta = z(mc^2)^{-1}$ ,

$$(H_+ - \zeta)u_+ - W_{12}(H_- - \zeta)^{-1}W_{21}u_+ = \frac{1}{mc^2} [f_+ - W_{12}(H_- - \zeta)^{-1}f_-],$$

$$(H_- - \zeta)u_- - W_{21}(H_+ - \zeta)^{-1}W_{12}u_- = \frac{1}{mc^2} [f_- - W_{21}(H_+ - \zeta)^{-1}f_+]$$

In virtue of

$$\sigma(H_+) \cap \sigma(H_-) = \emptyset, \quad \text{if } \frac{1}{mc^2} (V_+ - V_-) < 2$$

it follows from Lemma 5.1 with  $s = 0$  and  $s = 1$  that

$$\|\langle x \rangle^{-1}u_{\pm}\| \leq C (\|\langle x \rangle f_+\| + \|\langle x \rangle f_-\|).$$

## 6 Proof of Theorem 4.1

We shall apply weakly conjugate operator method to  $H_R$  (a weak version of Mourre estimates).

This method is applied for many cases. One of them treats the free Dirac operator with positive mass  $\alpha \cdot D + m\beta$  (Iftimovici and Măntoiu [6].)

In our case we consider relativistic Schrödinger operators with homogeneous potential.

$$A_2 = \frac{1}{2} (E^{-1}D_x \cdot x + x \cdot D_x E^{-1}), \quad E = (|D|^2 + 1)^{1/2}$$

**Lemma 6.1** *There exist positive numbers  $c_0$  and  $\delta$  such that*

$$\begin{aligned} \langle [iH_R, A_2]u, u \rangle &\geq \delta \|B_0^{1/2}u\|^2 \\ B_0^{-1/2}[[H_R, iA_2], iA_2]B_0^{-1/2} &\in \mathbf{B}(L^2(\mathbf{R}^3)), \end{aligned}$$

where  $B_0 = |D|^2(|D|^2 + 1)^{-1}$ .

**Proof:** A simple computation gives

$$[iE, A_2] = |D|^2(|D|^2 + 1)^{-1} = B_0 > 0.$$

Moreover

$$\begin{aligned} 2[iV, A_2] &= E^{-1}D \cdot x iV + D \cdot x E^{-1}iV - iVx \cdot DE^{-1} - iVE^{-1}x \cdot D \\ &\quad + 3E^{-1}V - 3VE^{-1} \\ &= 2E^{-1}D \cdot ixV - 2iVx \cdot DE^{-1} + [D \cdot x, E^{-1}]iV - iV[E^{-1}, x \cdot D] \\ &\quad + 3E^{-1}V - 3VE^{-1} \end{aligned}$$

where  $[E^{-1}, ix \cdot D] = [E^{-1}, D \cdot ix] = E^{-1}[ix \cdot D, E]E^{-1} = B_0E^{-1}$ . Let  $V_1(x) = \chi(x)V(x)$  and  $V_2(x) = (1 - \chi(x))V(x)$  with  $\chi \in C_0^\infty(\mathbf{R}^3)$ . Note that

$$\langle E^{-1}D_j u, x_j V_1 u \rangle = \left\langle \frac{D_j}{|D|} B_0^{1/2} u, x_j \langle x \rangle V_1 \langle x \rangle^{-1} B_0^{-1/2} B_0^{1/2} u \right\rangle.$$

By virtue of

$$B_0^{-1/2} = \sqrt{1 + |D|^{-2}} \leq 1 + |D|^{-1},$$

and Hardy's inequality

$$\| |x|^{-1} u \|_{L^2(\mathbf{R}^d)} \leq C_d \| |D| u \|_{L^2(\mathbf{R}^d)} \text{ with } C_d = \frac{2}{d-2},$$

it holds that

$$\| \langle x \rangle^{-1} B_0^{-1/2} v \| \leq (1 + C_3) \| v \|,$$

Since

$$\sup |x_j \langle x \rangle V_1(x)| \leq \max_{\text{supp } \chi} \{ |x|(1 + |x|) \} \|V\|_\infty,$$

if we take  $\text{supp } \chi \subset \{|x| \leq 1/2\}$ , we obtain

$$|\langle E^{-1}D \cdot x iV_1 u, u \rangle| \leq \frac{3}{4}(1 + C_0)\|V\|_\infty \|B_0^{1/2}u\|^2.$$

Similarly

$$|\langle [D \cdot x, E^{-1}]iV_1 u, u \rangle| \leq \frac{3}{2}(1 + C_0)\|V\|_\infty \|B_0^{1/2}u\|^2,$$

$$|\langle E^{-1}V_1 u, u \rangle| \leq \frac{9}{4}(1 + C_0)^2\|V\|_\infty \|B_0^{1/2}u\|^2.$$

To deal with  $V_2$  we now use the identities

$$\begin{aligned} -[iV_2, A_2] &= \frac{1}{2} ([E^{-1}, iV_2]D \cdot x + x \cdot D[E^{-1}, iV_2]) \\ &\quad + \frac{1}{2} (E^{-1}[D \cdot x, V_2] - [iV_2, x \cdot D]), \\ [E^{-1}, iV_2] &= E^{-1}[iV_2, E]E^{-1}, \\ [E, iV_2] &= E^{-1}D \cdot \nabla V + K(x, D), \end{aligned}$$

where  $K$  is a  $\Psi$ DO with symbol satisfying

$$\begin{aligned} K(x, \xi) &= (\xi^2 + m^2 c^2)^{-3/2} \xi^2 \Delta V + (\xi^2 + m^2 c^2)^{-1/2} \Delta V + \dots, \\ \forall \alpha, \beta, |\partial_x^\beta \partial_\xi^\alpha K(x, \xi)| &\leq C \langle x \rangle^{-2-|\beta|} \langle \xi \rangle^{-|\alpha|}, \quad (x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d. \end{aligned}$$

Thus, it holds that

$$|\langle E^{-1}K(x, D)E^{-1}u, u \rangle| \leq C\|B_0^{1/2}u\|^2.$$

Therefore if we take  $c$  to be large, then

$$\langle [iH_R, A_2]u, u \rangle \geq (1 - C_1(mc^2)^{-1}\|V\|_\infty)\|B_0^{1/2}u\|^2 \geq \delta\|B_0^{1/2}u\|^2.$$

Q.E.D.

Let

$$F_\varepsilon = \langle u, (H_R - \lambda \mp i\varepsilon B)^{-1}u \rangle$$

where  $B = [H, iA_2]$ . Then it holds that

$$\left| \frac{dF_\varepsilon}{d\varepsilon} \right| \leq C\varepsilon^{-1/2} |F_\varepsilon|^{1/2} \|\langle x \rangle u\|_{L^2}.$$

Integrating it on  $[\varepsilon, \varepsilon_0] \subset (0, 1)$  with aid of a Gronwall-type lemma and taking the limit

$$\lim_{\varepsilon \rightarrow +0} F_\varepsilon = F_0,$$

we can conclude that

$$|F_0| \leq C \|\langle x \rangle u\|_{L^2}^2.$$

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