

Hyperbolicity of Localizations

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1. INTRODUCTION

Let $P(x, D)$ be a differential operator of order m in an open set $\Omega \subset \mathbf{R}^{n+1}$ with coordinates $x = (x_0, x') = (x_0, x_1, \dots, x_n)$, hence a sum of differential polynomials $P_j(x, D)$ of order j ($j \leq m$) with symbols $P_j(x, \xi)$. In [7] Ivrii-Petkov has proved a necessary condition for the Cauchy problem to $P(x, D)$ is correctly posed which asserts that $P_{m-j}(z)$ must vanish of order $r - 2j$ at z if $P_m(z)$ vanishes of order r at z with $z = (x, \xi) \in T^*\Omega \setminus 0$. This enables us to define the localization $P_{z_0}(z)$ at a multiple characteristic z_0 (of $P_m(z)$), which is a polynomial on $T_{z_0}(T^*\Omega)$, following Helffer [4].

In this note we show that $P_{z_0}(z)$ is hyperbolic, that is verifies Gårding's condition if the Cauchy problem to $P(x, D)$ is correctly posed. The proof is based on the arguments of Svensson [9] and Nishitani [8].

Since $P_{z_0}(z)$ is hyperbolic, following Atiyah-Bott-Gårding [1], one can define the localizations $P_{(z_0, z_1, \dots, z_s)}(z)$ successively as the localization of $P_{(z_0, z_1, \dots, z_{s-1})}(z)$ at z_s which are hyperbolic polynomials on $T_{z_0}(T^*\Omega) \cong \dots \cong T_{z_s}(T^*\Omega)$ (see also Hörmander [5, II]). It may occur the case that the lineality $\Lambda_{(z_0, z_1, \dots, z_s)}(P_m)$ of $P_{m(z_0, z_1, \dots, z_s)}(z)$ (see (2.8) below) is an involutive subspace with respect to the canonical symplectic structure on $T_{z_0}(T^*\Omega)$. In this case we prove that for the Cauchy problem to be correctly posed it is necessary that

$$P_{(z_0, z_1, \dots, z_s)}(z) = P_{m(z_0, z_1, \dots, z_s)}(z),$$

that is, no lower order terms of $P_{(z_0, \dots, z_s)}(z)$ occur. This argument was also used in Bernardi-Bove-Nishitani [2] with $s = 1$.

2. LOCALIZATION IS HYPERBOLIC

We denote by $L_{z_0}^{m,r}$ the set of pseudodifferential operators P near z_0 with symbol $P(x, \xi)$ verifying

$$P(x, \xi) \sim \sum_{j=0}^{\infty} P_{m-j}(x, \xi)$$

in every homogeneous symplectic coordinates around z_0 where $P_{m-j}(x, \xi)$ are positively homogeneous of degree $m - j$ in ξ and vanish of order at least $r - 2j$ and $P_m(x, \xi)$ vanishes exactly to the order r at z_0 . Note that we may replace in the definition "every" by "some".

Lemma 2.1 (Helffer [4]). *Let $P \in L_{z_0}^{m,r}$. Then*

$$(2.1) \quad Q(x, \xi) = \exp\left\{\frac{i}{2} \sum_{j=0}^n \frac{\partial^2}{\partial x_j \partial \xi_j}\right\} P(x, \xi)$$

is invariantly defined in $L_{z_0}^{m,r}/L_{z_0}^{m,r+1}$: Let χ be a homogeneous symplectic coordinates around z_0 and let F be a Fourier integral operator associated with χ and $\hat{P} = FPF^{-1}$.

Then we have

$$\hat{Q}(\chi(x, \xi)) = Q(x, \xi)$$

in $L_{z_0}^{m,r}/L_{z_0}^{m,r+1}$ where $\hat{Q}$ is associated with $\hat{P}$ by (2.1).

Definition 2.1. We define the localization $P_{z_0}(x, \xi)$ of $P \in L_{z_0}^{m,r}$ at $z_0 = (x_0, \xi_0)$ as the lowest order term of the Taylor expansion of

$$\mu^{2m} Q(x_0 + \mu x, \mu^{-2} \xi_0 + \mu^{-1} \xi)$$

as $\mu \rightarrow 0$ which is invariantly defined as a polynomial on $T_{z_0}(T^*\Omega)$: If y are local coordinates around the origin and $\hat{P}(y, \eta)$ is the full symbol of P for the coordinates $(y, \eta dy)$ then we have

$$\hat{P}_{w_0}(y'(x_0), {}^t y'(x_0)^{-1} \xi + {}^t (y' \xi_0)'(x_0) x) = P_{z_0}(x, \xi), \quad w_0 = (y(x_0), {}^t y'(x_0)^{-1} \xi_0).$$

Writing $Q(x, \xi)$ as the sum of homogeneous parts $Q_{m-j}(x, \xi)$, it is clear that

$$(2.2) \quad \begin{aligned} P_{z_0}(x, \xi) &= \sum_{r-2j \geq 0} Q_{m-j, z_0}(x, \xi), \\ Q_{m-j, z_0}(z) &= P_{m-j, z_0}(z) + \sum_{i < j, |\alpha| = j-i} c_\alpha P_{m-i, z_0(\alpha)}^{(\alpha)}(z) \end{aligned}$$

with some constants c_α where $Q_{m-j, z_0}(x, \xi)$ and $P_{m-j, z_0}(x, \xi)$ are defined by

$$P_{m-j, z_0}(z) = \lim_{\mu \rightarrow 0} \mu^{-(r-2j)} P_{m-j}(z_0 + \mu z).$$

Let $P(x, D) = \sum_{j=0}^m P_j(x, D)$ be a differential operator of order m on Ω containing the origin where $P_j(x, D)$ is the homogeneous part of degree j with symbol $P_j(x, \xi)$. Assume that the plane $x_0 = 0$ is non characteristic and we are concerned with the Cauchy problem with respect to $x_0 = \text{const.}$. Let $z_0 \in T^*\Omega \setminus 0$ be a characteristic of P_m of order r ;

$$d^j P_m(z_0) = 0 \quad \text{for } j < r, \quad d^r P_m(z_0) \neq 0.$$

By the necessary condition of Ivrii-Petkov [7] stated in Introduction we conclude that $P \in L_{z_0}^{m,r}$ provided that the Cauchy problem for P is correctly posed. Then we have from Lemma 2.1 that

Proposition 2.2 (cf. Ivrii and Petkov [7]). Assume that the Cauchy problem for $P(x, D)$ is correctly posed near the origin and let $z_0 \in T^*\Omega \setminus 0$ be a multiple characteristic of P_m . Then the localization $P_{z_0}(z)$ is an invariantly defined polynomial on $T_{z_0}(T^*\Omega)$.

Let us denote by $\tilde{P}_{z_0}(x, \xi)$ the lowest order term of the Taylor expansion of

$$\mu^{2m} P(x_0 + \mu x, \mu^{-2} \xi_0 + \mu^{-1} \xi)$$

as $\mu \rightarrow 0$. Note that $\tilde{P}_{z_0}(x, \xi)$ is not coordinates free but we have

Lemma 2.3. *The following two conditions are equivalent.*

- (i) $\tilde{P}_{z_0}(z)$ is hyperbolic with respect to $\theta = (0, e_0)$,
- (ii) $P_{z_0}(z)$ is hyperbolic with respect to θ .

Proof. Recall that $\tilde{P}_{z_0}(z) = \sum_{r-2j \geq 0} P_{m-j, z_0}(z)$. Since $\tilde{P}_{z_0}(z)$ is hyperbolic if and only if $P_{m-j, z_0}(z)$ are weaker than $P_{m, z_0}(z) = Q_{m, z_0}(z)$ (see Hörmander [5, II], Svensson [9]) the proof is immediate by (2.2).

Now our aim is to prove

Theorem 2.4. *Assume that the Cauchy problem for $P(x, D)$ is correctly posed near the origin and let $z_0 \in T^*\Omega \setminus 0$ be a multiple characteristic of P_m . Then the localization $P_{z_0}(z)$ is a hyperbolic polynomial with respect to $\theta = (0, e_0)$.*

Let z_0 be a characteristic of order r_0 of $P_m(z)$ so that $P_{z_0}(z)$ is a polynomial of degree r_0 . We denote by $P_{(z_0, z_1)}(z)$ the localization of $P_{z_0}(z)$ at z_1 , that is the first coefficient of $\mu^{r_0} P_{z_0}(\mu^{-1} z_1 + z)$ that does not vanish identically in z :

$$\mu^{r_0} P_{z_0}(\mu^{-1} z_1 + z) = \mu^{r_1} (P_{(z_0, z_1)}(z) + O(\mu)), \quad \mu \rightarrow 0$$

(see Hörmander [5, II] and Atiyah-Bott-Gårding [1]). We call r_1 the order of z_1 . From Lemma 3.4.2 in Atiyah-Bott-Gårding [1] it follows that $P_{(z_0, z_1)}(z)$ is again hyperbolic with respect to θ . Furthermore z_1 is a characteristic of P_{m, z_0} of order r_1 and $P_{m(z_0, z_1)}(z)$ is the principal part of $P_{(z_0, z_1)}(z)$. On the other hand Corollary 12.4.9 in Hörmander [5, II] shows that

$$d^\nu Q_{m-j, z_0}(z_1) = 0, \quad \nu < r_1 - 2j$$

where $d^\nu Q(z)$ denotes the ν -th differential of Q with respect to z . Since $Q_{m-j, z_0}(z)$ are homogeneous of degree $r_0 - 2j$ it is clear that

$$P_{(z_0, z_1)}(z) = \sum_{r_1 - 2j \geq 0} Q_{m-2j(z_0, z_1)}(z)$$

where

$$Q_{m-j(z_0, z_1)}(z) = \lim_{\mu \rightarrow 0} \mu^{-(r_1 - 2j)} Q_{m-j, z_0}(z_1 + \mu z)$$

which is homogeneous of degree $r_1 - 2j$ in z . Repeating the same arguments we get

Lemma 2.5. *Let $P_{(z_0, \dots, z_k)}(z)$ be the localization of $P_{(z_0, \dots, z_{k-1})}(z)$ at z_k of which order is r_k (≥ 2);*

$$P_{(z_0, \dots, z_k)}(z) = (P_{(z_0, \dots, z_{k-1})})_{z_k}(z).$$

Then we have for every j with $r_k - 2j > 0$

$$d^\nu Q_{m-j(z_0, \dots, z_{k-1})}(z_k) = 0, \quad \nu < r_k - 2j$$

and hence

$$Q_{m-j(z_0, \dots, z_k)}(z) = \lim_{\mu \rightarrow 0} \mu^{-(r_k - 2j)} Q_{m-j(z_0, \dots, z_{k-1})}(z_k + \mu z)$$

exists. Moreover $P_{(z_0, \dots, z_k)}(z)$ is equal to

$$\sum_{r_k - 2j \geq 0} Q_{m-j(z_0, \dots, z_k)}(z)$$

and hyperbolic with respect to θ .

Corollary 2.6. Let z_k be a characteristic of $P_{m(z_0, \dots, z_{k-1})}(z)$ of order $r_k (\geq 2)$. Then we have

$$(2.3) \quad d^\nu P_{m-j(z_0, \dots, z_{k-1})}(z_k) = 0, \quad \nu < r_k - 2j$$

and then

$$(2.4) \quad P_{m-j(z_0, \dots, z_k)}(z) = \lim_{\mu \rightarrow 0} \mu^{-(r_k - 2j)} P_{m-j(z_0, \dots, z_{k-1})}(z_k + \mu z)$$

exists.

Proof. Assume that (2.3) and

$$(2.5) \quad \begin{aligned} Q_{m-j(z_0, \dots, z_{k-1})}(z) &= P_{m-j(z_0, \dots, z_{k-1})}(z) \\ &+ \sum_{i < j, |\alpha| = j-i} c_\alpha P_{m-i(z_0, \dots, z_{k-1})}^{(\alpha)}(z) \end{aligned}$$

hold with $k = p$ where c_α are constants. Then it is easy to see that (2.5) with $k = p+1$ holds. Thus (2.3) with $k = p+1$ follows from Lemma 2.5. By induction on k we get the desired conclusion.

Here we give another formula which defines $P_{(z_0, \dots, z_s)}(z)$ directly. Let $0 < \mu_0 < \mu_1 < \dots < \mu_s$ be a sequence of parameters with

$$(2.6) \quad \mu_j = O(\mu_{j+1}^{m+1}) \quad \text{as} \quad \mu_{j+1} \rightarrow 0.$$

Then we have

$$(2.7) \quad \begin{aligned} &(\mu_0 \cdots \mu_s)^{2m} Q(x_0 + \mu_0 x_1 + \cdots + \mu_0 \cdots \mu_{s-1} x_s + \mu_0 \cdots \mu_s x, \\ &(\mu_0 \cdots \mu_s)^{-2} (\xi_0 + \mu_0 \xi_1 + \cdots + \mu_0 \cdots \mu_{s-1} \xi_s + \mu_0 \cdots \mu_s \xi) \\ &= \mu_0^{r_0} \cdots \mu_s^{r_s} (P_{(z_0, \dots, z_s)}(z) + O(\mu_s)) \end{aligned}$$

where $z_j = (x_j, \xi_j)$ and r_j is the order of z_j .

Let $\Lambda_{(z_0, \dots, z_s)}(P_m)$ be the lineality of $P_{m(z_0, \dots, z_s)}$ which is a linear subspace defined by

$$(2.8) \quad \{z | P_{m(z_0, \dots, z_s)}(w + tz) = P_{m(z_0, \dots, z_s)}(w), \forall t \in \mathbf{R}, \forall w \in T_{z_0}(T^*\Omega)\}$$

and let $\sigma = \sum_{j=0}^n d\xi_j \wedge dx_j$ be the canonical symplectic two form on $T^*\Omega$. For $S \subset T_{z_0}(T^*\Omega)$ we denote by S^σ the annihilator of S with respect to σ :

$$S^\sigma = \{z \in T_{z_0}(T^*\Omega) | \sigma(z, w) = 0, \forall w \in S\}.$$

Theorem 2.7. Assume that the Cauchy problem for $P(x, D)$ is correctly posed near the origin and

$$\Lambda_{(z_0, \dots, z_s)}(P_m)^\sigma \subset \Lambda_{(z_0, \dots, z_s)}(P_m).$$

Then we have

$$P_{(z_0, \dots, z_s)}(z) = P_{m(z_0, \dots, z_s)}(z),$$

that is, no lower order terms occur in $P_{(z_0, \dots, z_s)}(z)$.

Example 2.1. Let

$$P(x, \xi) = (\xi_0^2 - x_1^2 \xi_n^2 - \xi_1^2)(\xi_0^2 - x_1^2 \xi_n^2 - 2\xi_1^2) + p_2(\xi_0, x_1 \xi_n, \xi_1) \xi_n$$

where p_2 is a homogeneous polynomial of degree 2. With $z_0 = (0, e_n)$ it is clear that

$$P_{4, z_0} = (\xi_0^2 - x_1^2 - \xi_1^2)(\xi_0^2 - x_1^2 - 2\xi_1^2), \quad Q_{3, z_0} = 6ix_1 \xi_1 + p_2(\xi_0, x_1, \xi_1).$$

Let z_1 be $\xi_0 = x_1 = a$, $a \in \mathbf{R}$, $\xi_1 = 0$ so that

$$P_{4(z_0, z_1)} = 4a^2(\xi_0 - x_1)^2, \quad Q_{3(z_0, z_1)} = p_2(a, a, 0).$$

Since $\Lambda_{(z_0, z_1)}(P_4)^\sigma \subset \Lambda_{(z_0, z_1)}(P_4)$ it follows from Theorem 2.7 that $p_2(a, a, 0) = 0$. Similarly choosing z_1 to be $\xi_0 = a$, $x_1 = -a$, $\xi_1 = 0$ we get $p_2(a, -a, 0) = 0$. Thus

$$p_2(\xi_0, x_1, \xi_1) = c(\xi_0^2 - x_1^2) + \xi_1 p_1(\xi_0, x_1, \xi_1)$$

where p_1 is linear. Finally one can write

$$P(x, \xi) = (\xi_0^2 - x_1^2 \xi_n^2 - \xi_1^2 + c\xi_n)(\xi_0^2 - x_1^2 \xi_n^2 - 2\xi_1^2) + \xi_1 L(\xi_0, x_1 \xi_n, \xi_1) \xi_n$$

with a linear function L .

Example 2.2. Let

$$P(x, \xi) = (\xi_0 - x_0 \xi_n)^2 (\xi_0 + x_0 \xi_n) + \alpha(\xi_0 - x_0 \xi_n) \xi_n + \beta(\xi_0 + x_0 \xi_n) \xi_n$$

where $\alpha, \beta \in \mathbf{C}$. With $z_0 = (0, e_n)$ we have

$$P_{3, z_0} = (\xi_0 - x_0)^2 (\xi_0 + x_0), \quad Q_{2, z_0} = \alpha(\xi_0 - x_0) + (\beta - i)(\xi_0 + x_0).$$

Taking z_1 to be $\xi_0 = 1$, $x_0 = 1$ it follows that

$$P_{3(z_0, z_1)} = 2(\xi_0 - x_0)^2, \quad Q_{2(z_0, z_1)} = 2(\beta - i).$$

Since $\Lambda_{(z_0, z_1)}(P_3)^\sigma \subset \Lambda_{(z_0, z_1)}(P_3)$ we have $\beta = i$ by Theorem 2.7. Set

$$p_1(x, \xi) = \xi_0 - x_0 \xi_n, \quad p_2(x, \xi) = (\xi_0 - x_0 \xi_n)(\xi_0 + x_0 \xi_n) + (\alpha + i) \xi_n$$

then $\beta = i$ implies that

$$P(x, D) = p_1^w(x, D) p_2^w(x, D)$$

where $p_j^w(x, D)$ are Weyl realizations of $p_j(x, \xi)$, see Hörmander [5, III].

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