

Improved error estimates for the Davenport–Heilbronn theorems

神戸大学・大学院理学研究科 谷口 隆

Takashi Taniguchi

Department of Mathematics, Faculty of Sciences,
Kobe University

Abstract

This is a résumé of the preprint [BTT] of Manjul Bhargava, Frank Thorne and the author, based on the author’s talk at RIMS conference¹.

1 Introduction

The purpose of this article is to give an outline of the proof the following theorem, obtained by Bhargava, Thorne and the author [BTT, Theorem 1.1]:

Theorem 1 Let $N_3^\pm(X)$ denote the number of isomorphism classes of cubic fields F satisfying $0 < \pm \text{Disc}(F) < X$. Then

$$N_3^\pm(X) = C^\pm \frac{1}{12\zeta(3)} X + K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} X^{5/6} + O(X^{2/3+\epsilon})$$

where $C^+ = 1$, $C^- = 3$, $K^+ = 1$, and $K^- = \sqrt{3}$.

We briefly recall the history of this counting problem: The first main term is due to Davenport and Heilbronn [DH71], while the second main term was conjectured by Datskovsky and Wright [DW88, p. 125] and Roberts [Rob01] and proven in [BST13] and [TT13b]. The latter works in turn built on the successively improved error terms obtained in [DH71], [Bel97], and [BBP10].

We also obtain a variation of Theorem 1 that counts isomorphism classes of cubic fields satisfying certain specified sets of *local conditions*. For details, see [BTT, Theorems 1.4, 1.5].

2 Binary cubic forms and cubic rings

A *cubic ring* is a unitary commutative ring that is free of rank three as a $\mathbb{Z}$ -module. Its *discriminant* is the determinant of the trace form $\langle x, y \rangle = \text{Tr}(xy)$. The lattice of *integral binary cubic forms* is defined by $V(\mathbb{Z}) := \{au^3 + bu^2v + cuv^2 + dv^3 \mid a, b, c, d \in \mathbb{Z}\}$, and the *discriminant* of $f(u, v) = au^3 + bu^2v + cuv^2 + dv^3 \in V(\mathbb{Z})$ is defined by $\text{Disc}(f) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$. The group $\text{GL}_2(\mathbb{Z})$ acts on $V(\mathbb{Z})$ by $(\gamma \cdot f)(u, v) = f((u, v) \cdot \gamma) / (\det \gamma)$.

The correspondence of Levi [Lev14] and Delone–Faddeev [DF64], as further extended by Gan, Gross, and Savin [GGS02] to include the degenerate case, is as follows:

Theorem 2 ([Lev14, DF64, GGS02]) There is a canonical, discriminant-preserving bijection between the set of $\text{GL}_2(\mathbb{Z})$ -orbits on $V(\mathbb{Z})$ and the set of isomorphism classes of cubic rings.

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We want to count cubic fields, which is equivalent to count maximal cubic domains. Let $N_{\max}^{\pm}(X)$ counts the number of maximal cubic rings R satisfying $0 < \pm \text{Disc}(R) < X$. For a squarefree integer q , let $N^{\pm}(X; q)$ denotes the number of cubic rings R satisfying $0 < \pm \text{Disc}(R) < X$ which are non maximal at all prime divisors of q . Then by inclusion-exclusion, we have

$$N_{\max}^{\pm}(X) = \sum_q N^{\pm}(X; q) = \sum_{q < Q} N^{\pm}(X; q) + \sum_{q \geq Q} N^{\pm}(X; q). \quad (1)$$

The latter sum is $O(X/Q^{1-\epsilon})$, since $N^{\pm}(X; q) = O(X/q^{2-\epsilon})$.

We use the correspondence of Theorem 2 to analyze $N^{\pm}(X; q)$ for small q . Let $\Psi_q: V(\mathbb{Z}) \rightarrow \{0, 1\}$ be the indicator functions of cubic rings that are non maximal at all prime divisors of q . It was proved in [DH71] that Ψ_q factors through the reduction map $V(\mathbb{Z}) \rightarrow V(\mathbb{Z}/q^2\mathbb{Z})$.

3 Shintani zeta fucntions and Landau's method

Our proof will apply the theory of *Shintani zeta functions* [Shi72] associated with the space of binary cubic forms. Shintani introduced the following Dirichelt series:

$$\xi^{\pm}(s) := \sum_{\substack{x \in \text{GL}_2(\mathbb{Z}) \backslash V(\mathbb{Z}) \\ \pm \text{Disc}(x) > 0}} \frac{|\text{Stab}(x)|^{-1}}{|\text{Disc}(x)|^s}.$$

More generally, we consider

$$\xi_q^{\pm}(s) := \sum_{\substack{x \in \text{GL}_2(\mathbb{Z}) \backslash V(\mathbb{Z}) \\ \pm \text{Disc}(x) > 0}} \Psi_q(x) \frac{|\text{Stab}(x)|^{-1}}{|\text{Disc}(x)|^s} =: \sum_{n \geq 1} \frac{a_q^{\pm}(n)}{n^s}.$$

Then $N^{\pm}(X; q) = \sum_{n < X} a_q^{\pm}(n)$. By applying Perron's formula,

$$N^{\pm}(X; q) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \xi_q^{\pm}(s) \frac{X^s}{s} ds.$$

The Shintani zeta function $\xi_q^{\pm}(s)$ enjoys an analytic continuation and a functional equation. It has simple poles at $s = 1$ and $5/6$, and the explicit formulas of the respective residues $r_1^{\pm}(q)$ and $r_{5/6}^{\pm}(q)$ are obtained by Datskovsky and Wright [DW86, Proposition 5.3 and Theorem 6.2]. Let

$$E^{\pm}(X; q) := N^{\pm}(X; q) - r_1^{\pm}(q)X - r_{5/6}^{\pm}(q) \frac{X^{5/6}}{5/6}.$$

If we formally shift the contour to the left, we get these two main terms by the residue theorem and also get an integral expression of $E^{\pm}(X; q)$. There is a convergence problem, so we consider the Riesz mean. Then we come back the original count by the finite differencing. This was established in the classical work of Landau [Lan12, Lan15], and we have $E^{\pm}(X; q) = O_q(X^{3/5})$ where the implied constant depends on q . For our purpose we need an estimate of $E^{\pm}(X; q)$ uniform in q , which we now discuss.

4 Uniform Landau and estimating the error term

Let

$$\hat{d}(\Psi_q) := q^8 \cdot \sup_N \frac{1}{N} \sum_{\substack{y \in \text{GL}_2(\mathbb{Z}) \backslash V^*(\mathbb{Z}) \\ 0 < |\text{Disc}^*(y)| < N}} |\widehat{\Psi_q}(y)|.$$

Here V^* is the dual space and Disc^* is an invariant polynomial on V^* . The following estimate follows from [LDTT22] by Lowry-Duda, Thorne and the author:

Theorem 3 ([LDTT22]) We have

$$\sum_{q \in [Q, 2Q]} |E(X; q)| \ll X^{3/5} \left(\sum_{q \in [Q, 2Q]} r_1(q) \right)^{3/5} \left(\sum_{q \in [Q, 2Q]} \hat{d}(\Psi_q) \right)^{2/5},$$

provided that

$$\left(\sum_{q \in [Q, 2Q]} \hat{d}(\Psi_q) \right)^{2/5} \ll X \left(\sum_{q \in [Q, 2Q]} r_1(q) \right)^{3/5}.$$

We know $r_1(q) \asymp q^{-2}$, so the problem is reduced to establish an estimate of

$$\frac{1}{N} \sum_{q \in [Q, 2Q]} \sum_{\substack{y \in \text{GL}_2(\mathbb{Z}) \backslash V^*(\mathbb{Z}) \\ 0 < |\text{Disc}^*(y)| < N}} |\widehat{\Psi}_q(y)|. \quad (2)$$

By the Chinese remainder theorem, $\widehat{\Psi}_q = \prod_{p|q} \widehat{\Psi}_p$. Thorne and the author [TT13a] gave an explicit formula of $\widehat{\Psi}_p$, and in particular obtained the following upper bound:

Propositoin 4 ([TT13a]) We have

$$\widehat{\Psi}_p(y) \ll \begin{cases} p^{-2} & p^2 \mid y, \\ p^{-3} & p^4 \mid \text{Disc}^*(y), \\ p^{-4} & p^3 \mid \text{Disc}^*(y), \\ p^{-5} & p^2 \mid \text{Disc}^*(y). \end{cases}$$

Moreover, $\widehat{\Psi}_p(y) = 0$ if $p^2 \nmid \text{Disc}^*(y)$.

This proposition shows that the function $\widehat{\Psi}_p$ takes mostly quite small values, and has a thin support. Thus by switching the sum in (2), we can estimate the quantity rather effectively. As a consequence we have $\ll Q^{-6+\epsilon}$ for (2). Thus the total error in the squarefree sieve (1) is $\ll X/Q^{1-\epsilon} + X^{3/5}Q^{1/5+\epsilon}$. Choosing $Q = X^{1/3-\epsilon'}$, we have the desired bound.

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