

Qualitative analysis of space-time periodic homogenization for nonlinear diffusion equations *

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1 Introduction

In this note, we consider the following nonlinear diffusion equation:

$$(P_\varepsilon) \quad \begin{cases} \partial_t u_\varepsilon = \operatorname{div} (a_\varepsilon \nabla |u_\varepsilon|^{p-1} u_\varepsilon) & \text{in } \Omega \times I, \\ |u_\varepsilon|^{p-1} u_\varepsilon|_{\partial\Omega} = 0, \quad u_\varepsilon|_{t=0} = u_0, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $I = (0, T)$, $0 < p < +\infty$, $\varepsilon > 0$ and $u_0 \in L^{p+1}(\Omega)$. Let $\square = (0, 1)^N$ and $J = (0, 1)$. Let $a = a(y, s) \in [W^{1,1}(\mathbb{R}_+; L^\infty(\mathbb{R}^N))]^{N \times N}$ be an $N \times N$ symmetric matrix field satisfying $(\square \times J)$ -periodicity and the uniform ellipticity, i.e., there exists $\lambda > 0$ such that $\lambda|\xi|^2 \leq a(y, s)\xi \cdot \xi \leq |\xi|^2$ for all $\xi \in \mathbb{R}^N$ and a.e. $(y, s) \in \mathbb{R}^N \times \mathbb{R}_+$. The coefficient matrix field a_ε is given as $a_\varepsilon = a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2})$ for $x \in \Omega$ and $t \in I$.

Homogenization is known as a method of asymptotic analysis for complex structures and systems. Actually, it is often used to replace heterogeneous materials with a large number of microstructures, such as composite materials, with an equivalent homogeneous material; for instance, it is applied to models of heat conduction in composite materials. Such models are often described as linear diffusion equations (LDEs), and then their *space-time homogenization* oscillating both in space and time has been studied in various mathematical fields.

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Space-time homogenization problems for linear diffusion equations were first studied by Bensoussan, Lions and Papanicolaou in [8] based on a method of *asymptotic expansion*, and then various methods have been developed (see, e.g., [14] for *two-scale convergence theory* and [5] for *unfolding method*). Furthermore, homogenization problems for parabolic equations have been studied not only for linear ones but also for nonlinear ones. In [15, 18, 24], doubly-nonlinear parabolic equations are treated, and moreover, as for degenerate p -Laplace parabolic equations, homogenization problems involving scale parameters (e.g., $r > 0$ of $\operatorname{div}[A(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}, \nabla u_\varepsilon)]$) are discussed in [13, 25].

In this note, the critical case of *porous medium types* is treated, and in particular, it is revealed that the difference between degeneracy and singularity of diffusion is deeply related to the representation of the so-called *homogenized matrices*.

2 Main results

We first define *weak solutions* $u_\varepsilon = u_\varepsilon(x, t) : \Omega \times I \rightarrow \mathbb{R}$ of (P_ε) as follows:

Definition 2.1. *A function $u_\varepsilon = u_\varepsilon(x, t) : \Omega \times I \rightarrow \mathbb{R}$ is called a weak solution to (P_ε) , if the following conditions are all satisfied:*

- (i) $u_\varepsilon \in W^{1,2}(I; H^{-1}(\Omega)) \cap L^{p+1}(\Omega \times I)$, $|u_\varepsilon|^{p-1}u_\varepsilon \in L^2(I; H_0^1(\Omega))$ and $u_\varepsilon(t) \rightarrow u_0$ strongly in $H^{-1}(\Omega)$ as $t \rightarrow 0_+$.
- (ii) It holds that

$$\langle \partial_t u_\varepsilon(t), \phi \rangle_{H_0^1(\Omega)} + \int_{\Omega} a_\varepsilon \nabla(|u_\varepsilon|^{p-1}u_\varepsilon)(x, t) \cdot \nabla \phi(x) \, dx = 0$$

for a.e. $t \in I$ and all $\phi \in H_0^1(\Omega)$.

Remark 2.2. For $p \neq 1$, the nonlinear diffusion equation (P_ε) is called a *porous medium equation* (PME) if $1 < p < +\infty$ and a *fast diffusion equation* (FDE) if $0 < p < 1$ (see [21, 22] for details). The well-posedness for (P_ε) can be obtained by [1, 2].

Now, our main results read,

Theorem 2.3. *Let $\varepsilon_n \rightarrow 0_+$ be an arbitrary sequence in $(0, +\infty)$. In addition, suppose that $u_0 \in L^2(\Omega)$ for $p \in (0, 1)$. Let u_{ε_n} be the unique weak solution to (P_{ε_n}) . Then there exist a subsequence of (ε_n) and functions*

$$\begin{aligned} u &\in W^{1,2}(I; H^{-1}(\Omega)) \cap L^{p+1}(\Omega \times I) \cap C_{\text{weak}}(\bar{I}; L^2(\Omega)), \\ z &\in L^2(\Omega \times I; L^2(J; H_{\text{per}}^1(\square)/\mathbb{R})) \end{aligned}$$

such that $|u|^{p-1}u \in L^2(I; H_0^1(\Omega))$,

$$\begin{aligned} |u_{\varepsilon_n}|^{p-1}u_{\varepsilon_n} &\rightarrow |u|^{p-1}u && \text{weakly in } L^2(I; H_0^1(\Omega)), \\ u_{\varepsilon_n} &\rightarrow u && \text{strongly in } L^\rho(I; L^{p+1}(\Omega)) \end{aligned}$$

for any $\rho \in [1, +\infty)$ and

$$\begin{aligned} a_{\varepsilon_n} \nabla |u_{\varepsilon_n}|^{p-1}u_{\varepsilon_n} \\ \rightarrow j_{\text{hom}} := \langle a(\cdot, \cdot) (\nabla |u|^{p-1}u + \nabla_y z) \rangle_{y,s} \quad \text{weakly in } [L^2(\Omega \times I)]^N. \end{aligned}$$

Here and henceforth, $H_{\text{per}}^1(\square)/\mathbb{R} := \{g \in H_{\text{loc}}^1(\mathbb{R}^N) : g \text{ is } \square\text{-periodic and } \langle g \rangle_y := \int_{\square} g(y) dy = 0\}$, ∇_y denotes the gradient operator with respect to y and $\langle \cdot \rangle_{y,s}$ denotes the mean over $\square \times J$, that is,

$$\langle g \rangle_{y,s} = \int_0^1 \int_{\square} g(y, s) dy ds \quad \text{for } g \in L^1(\square \times J).$$

Moreover, the limit u solves the weak form of the homogenized equation,

$$(P_0) \quad \begin{cases} \langle \partial_t u(t), \phi \rangle_{H_0^1(\Omega)} + \int_{\Omega} j_{\text{hom}}(x, t) \cdot \nabla \phi(x) dx = 0 & \text{for } \phi \in H_0^1(\Omega), \\ u(\cdot, 0) = u_0 & \text{in } \Omega \end{cases}$$

for a.e. $t \in I$.

Furthermore, the homogenized diffusion flux j_{hom} is characterized as follows:

Theorem 2.4. *In addition to all the assumptions of Theorem 2.3, suppose that $u_0 \geq 0$ for $p \geq 2$,*

$$u_0 \in L^{3-p}(\Omega) \text{ if } p \in (0, 1]; \log u_0 \in L_{\text{loc}}^1(\Omega) \text{ if } p = 3; u_0^{3-p} \in L_{\text{loc}}^1(\Omega) \text{ if } p \in (3, +\infty).$$

Let u be a limit of weak solutions (u_{ε_n}) to (P_{ε_n}) as a sequence $\varepsilon_n \rightarrow 0_+$ and let u be a weak solution of the homogenized equation (P_0) . Then $z = z(x, t, y, s)$ appeared in Theorem 2.3 is represented as

$$z(x, t, y, s) = \sum_{k=1}^N (\partial_{x_k} v(x, t)) \Phi_k(x, t, y, s),$$

where

$$v = \begin{cases} |u|^{p-1}u & \text{if } p \in (0, 2), \\ u^p & \text{if } p \in [2, +\infty) \end{cases}$$

and $\Phi_k \in L^\infty(\Omega \times I; L^2(J; H_{\text{per}}^1(\square)/\mathbb{R}))$ is the corrector characterized as follows:

(i) In case $p \in (0, 1]$ (i.e., FDE and LDE), $\Phi_k = \Phi_k(x, t, y, s)$ solves the cell problem,

$$\begin{cases} \frac{1}{p} |u(x, t)|^{1-p} \partial_s \Phi_k(x, t, y, s) = \operatorname{div}_y (a(y, s) [\nabla_y \Phi_k(x, t, y, s) + e_k]) & \text{in } \square \times J, \\ \Phi_k(x, t, y, 0) = \Phi_k(x, t, y, 1) & \text{in } \square \end{cases}$$

for each $(x, t) \in \Omega \times I$. Here $\{e_k\} = \{[\delta_{jk}]_{j=1,2,\dots,N}\}$ stands for a canonical basis of $\mathbb{R}^N$.

(ii) In case $p \in (1, +\infty)$ (i.e., PME), Φ_k is given by

$$\Phi_k(x, t, y, s) = \begin{cases} p |u(x, t)|^{p-1} \Psi_k(x, t, y, s) & \text{if } u(x, t) \neq 0, \\ 0 & \text{if } u(x, t) = 0, \end{cases}$$

where $\Psi_k = \Psi_k(x, t, y, s)$ solves the cell problem,

$$\begin{cases} \partial_s \Psi_k(x, t, y, s) = \operatorname{div}_y (a(y, s) [p |u(x, t)|^{p-1} \nabla_y \Psi_k(x, t, y, s) + e_k]) & \text{in } \square \times J, \\ \Psi_k(x, t, y, 0) = \Psi_k(x, t, y, 1) & \text{in } \square \end{cases}$$

for each $(x, t) \in [u \neq 0] := \{(x, t) \in \Omega \times I : u(x, t) \neq 0\}$.

Moreover, the homogenized flux $j_{\text{hom}}(x, t)$ can be written as

$$j_{\text{hom}}(x, t) = a_{\text{hom}}(x, t) \nabla v(x, t),$$

where a_{hom} is the homogenized matrix given by

$$(1) \quad a_{\text{hom}}(x, t) e_k = \int_0^1 \int_{\square} a(y, s) [\nabla_y \Phi_k(x, t, y, s) + e_k] \, dy ds.$$

Remark 2.5. The homogenized matrix (1) is described in terms of solutions to cell problems. For the nonlinear diffusion case $p \neq 1$, the cell problem involves the limit $u(x, t)$ of solutions, which is a function of (x, t) . Thus Φ_k also depends on (x, t) , and hence, so does a_{hom} . On the other hand, for the linear diffusion case $p = 1$, Φ_k is independent of (x, t) . Thus a_{hom} is a constant $N \times N$ matrix. In particular, it is noteworthy that the representation of a_{hom} depends on $p \in (0, +\infty)$ since it is determined by Φ_k .

As for the qualitative properties of a_{hom} , we have

Propositon 2.6 (cf. [2, Proposition 1.8]). *Under the same assumptions as in Theorem 2.4, let a_{hom} and $\{\Phi_k\}_{k=1,2,\dots,N}$ be defined as in Theorem 2.4. Then the following (i) and (ii) hold true:*

(i) (*Improved uniform ellipticity*) It holds that

$$\begin{aligned} & \lambda \sum_{k=1}^N \left(1 + \int_0^1 \|\Phi_k(x, t, \cdot, s)\|_{L^2(\square)}^2 ds \right) |\xi_k|^2 \\ & \leq a_{\text{hom}}(x, t) \xi \cdot \xi \leq \sum_{k=1}^N \left(1 + \int_0^1 \|\Phi_k(x, t, \cdot, s)\|_{L^2(\square)}^2 ds \right) |\xi_k|^2 \end{aligned}$$

for any $\xi = [\xi_k]_{k=1,2,\dots,N} \in \mathbb{R}^N$ and a.e. $(x, t) \in \Omega \times I$.

(ii) (*Symmetry and asymmetry*) The homogenized matrix $a_{\text{hom}}(x, t)$ is not symmetric (respectively, symmetric) when $u(x, t) \neq 0$ (respectively, $u(x, t) = 0$).

As mentioned in [2, 6, 12, 14, 18], the gradient ∇v_{ε_n} does not converge to ∇v strongly in $[L^2(\Omega \times I)]^N$ in general. Indeed, one can prove that

$$(2) \quad \nabla v_{\varepsilon_n} - \nabla v - \sum_{k=1}^N (\partial_{x_k} v) \nabla_y \Phi_k(x, t, \frac{x}{\varepsilon_n}, \frac{t}{\varepsilon_n^2}) \rightarrow 0 \text{ strongly in } [L^2(\Omega \times I)]^N,$$

and hence, due to the oscillation of Φ_k , the breaking of strong compactness in $L^2(I; H_0^1(\Omega))$ is obtained. However, to guarantee strong convergence (2), we shall require regularities: $\nabla v \in [L^\sigma(\Omega \times I)]^N$ and $\nabla_y \Phi_k \in [L^\rho(\Omega \times I)]^N$ along with $\frac{1}{\sigma} + \frac{1}{\rho} = \frac{1}{2}$. Hence additional assumptions for the coefficient $a(y, s)$ and given data will also be required. This note provides a corrector result (introduced by [10]) without assumptions for the smoothness of $a(y, s)$.

Theorem 2.7. *Let u be a limit of weak solutions (u_{ε_n}) to (P_{ε_n}) as a sequence $\varepsilon_n \rightarrow 0_+$ such that u is a weak solution to (P_0) and let Φ_k be the corrector given by Theorem 2.4. Set*

$$v_{\varepsilon_n} := \begin{cases} |u_{\varepsilon_n}|^{p-1} u_{\varepsilon_n} & \text{if } p \in (0, 2), \\ u_{\varepsilon_n}^p & \text{if } p \in [2, +\infty), \end{cases} \quad v := \begin{cases} |u|^{p-1} u & \text{if } p \in (0, 2), \\ u^p & \text{if } p \in [2, +\infty). \end{cases}$$

Then it holds that

$$\lim_{\varepsilon_n \rightarrow 0_+} \left\| \nabla v_{\varepsilon_n} - \nabla v - \sum_{k=1}^N \mathcal{U}_{\varepsilon_n}(\partial_{x_k} v) \mathcal{U}_{\varepsilon_n}(\nabla_y \Phi_k) \right\|_{L^2(\Omega \times I)} = 0,$$

where $\mathcal{U}_{\varepsilon_n}$ is the averaging operator (see Definition 4.4 below).

Remark 2.8. Theorem 2.7 also implies the breaking of strong compactness in $L^2(I; H_0^1(\Omega))$ for the pressure $v_{\varepsilon_n} \in L^2(I; H_0^1(\Omega))$ since the oscillating corrector terms do not vanish as $\varepsilon_n \rightarrow 0_+$.

3 Uniform estimates and convergence

In this section, we shall derive uniform estimates for (v_ε) and $(v_\varepsilon^{1/p})$ and discuss their convergence to prove Theorems 2.3 and 2.4.

Lemma 3.1. *Let $0 < p < +\infty$. For each $\varepsilon > 0$ let $u_\varepsilon \in L^2(I; H_0^1(\Omega))$ be the unique weak solution of (P_ε) and set $v_\varepsilon = |u_\varepsilon|^{p-1}u_\varepsilon$. Then the following (i)–(iv) hold true:*

- (i) (v_ε) and $(v_\varepsilon^{1/p})$ are bounded in $L^2(I; H_0^1(\Omega)) \cap L^\infty(I; L^{(p+1)/p}(\Omega))$ and $L^\infty(I; L^{p+1}(\Omega))$, respectively.
- (ii) $(\partial_t v_\varepsilon^{1/p})$ is bounded in $L^2(I; H^{-1}(\Omega))$.
- (iii) $(v_\varepsilon^{1/p})$ is bounded in $L^\infty(I; L^2(\Omega))$.
- (iv) $(v_\varepsilon^{1/p})$ is bounded in $L^\infty(I; L^{3-p}(\Omega)) \cap L^2(I; H_0^1(\Omega))$, provided that $p \in (0, 2)$ and $u_0 \in L^{3-p}(\Omega)$.

Proof. See [2, Lemma 4.1]. □

As for $p \geq 2$, we have the following local uniform estimates:

Lemma 3.2. *Under the same assumptions as in Theorem 2.4, for any $\omega \Subset \Omega$, there exists a constant $C_\omega \geq 0$ such that the following holds true:*

- (i) In case $2 \leq p < 3$,

$$\int_0^T \|\nabla v_\varepsilon^{1/p}(t)\|_{L^2(\omega)}^2 dt \leq C_\omega.$$

- (ii) In case $p = 3$,

$$\sup_{t \in \bar{I}} \left(\int_{[v_\varepsilon^{1/p}(\cdot, t) \leq 1] \cap \omega} [-\log v_\varepsilon^{1/p}(\cdot, t)] dx \right) + \int_0^T \|\nabla v_\varepsilon^{1/p}(t)\|_{L^2(\omega)}^2 dt \leq C_\omega,$$

provided that $\log u_0 \in L_{\text{loc}}^1(\Omega)$.

- (iii) In case $p > 3$,

$$\sup_{t \in \bar{I}} \left(\int_\omega v_\varepsilon^{(3-p)/p}(\cdot, t) dx \right) + \int_0^T \|\nabla v_\varepsilon^{1/p}(t)\|_{L^2(\omega)}^2 dt \leq C_\omega,$$

provided that $u_0^{3-p} \in L_{\text{loc}}^1(\Omega)$.

Proof. See [3, Lemma 3.1]. □

By Lemmas 3.1 and 3.2, we have

Propositon 3.3. *Under the same assumptions as in Theorem 2.3, there exist a subsequence (ε_n) of (ε) and $v \in L^2(I; H_0^1(\Omega)) \cap L^\infty(I; L^{(p+1)/p}(\Omega))$ such that*

$$\begin{aligned} v_{\varepsilon_n} &\rightarrow v && \text{weakly in } L^2(I; H_0^1(\Omega)), \\ v_{\varepsilon_n}^{1/p} &\rightarrow v^{1/p} && \text{strongly in } C(\bar{I}; H^{-1}(\Omega)), \\ \partial_t v_{\varepsilon_n}^{1/p} &\rightarrow \partial_t v^{1/p} && \text{weakly in } L^2(I; H^{-1}(\Omega)), \\ v_{\varepsilon_n} &\rightarrow v && \text{strongly in } L^\rho(I; L^{(p+1)/p}(\Omega)), \\ v_{\varepsilon_n}^{1/p} &\rightarrow v^{1/p} && \text{strongly in } L^\rho(I; L^{p+1}(\Omega)), \\ \nabla v_{\varepsilon_n}^{1/p} &\rightarrow \nabla v^{1/p} && \text{weakly in } [L^2(I; L^2(\omega))]^N \end{aligned}$$

for any $\rho \in [1, +\infty)$ and $\omega \Subset \Omega$.

Proof. See [2, Lemma 4.3]. □

4 Space-time unfolding method

In this section, we briefly review the space-time unfolding method to characterize the limit of $a_{\varepsilon_n} \nabla v_{\varepsilon_n}$ as $\varepsilon_n \rightarrow 0_+$. The unfolding method was first introduced in [9] (see [7, 10, 11, 16] for more details), and then its space-time version was developed in [5, 20]. This method is also known as the intermediate notion between weak convergence and strong convergence, and weak and strong convergences for unfolded sequences are equivalent to weak and strong *two-scale convergence* (see, e.g., [4, 14, 17, 19, 23, 26] for more details).

Throughout this section, let $1 < q < +\infty$, when no confusion can arise. Moreover, q' denotes the Hölder conjugate of q , i.e., $1/q + 1/q' = 1$.

Definition 4.1 (cf. [5, Definition 2.1]). *For $\varepsilon > 0$, define the sets $\hat{\Omega}_\varepsilon \subset \Omega$ and $\hat{I}_\varepsilon \subset I$ by*

$$\begin{aligned} \hat{\Omega}_\varepsilon &:= \text{interior} \left(\bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{\square}) \right), \quad \Xi_\varepsilon := \{\xi \in \mathbb{Z}^N : \varepsilon(\xi + \square) \subset \Omega\}, \\ \hat{I}_\varepsilon &:= \{t \in I : \varepsilon^2 (\lfloor \frac{t}{\varepsilon^2} \rfloor + 1) \leq T\}, \end{aligned}$$

respectively. Here $\varepsilon(\xi + \overline{\square})$ denotes the closed ε -cell $[0, \varepsilon]^N$ with the origin at $\varepsilon\xi \in \varepsilon\mathbb{Z}^N$ and $\lfloor \cdot \rfloor$ denotes the floor function (i.e., $\lfloor \cdot \rfloor$ denotes the integer part of $\cdot$). Set $\Lambda_\varepsilon := (\Omega \times I) \setminus (\hat{\Omega}_\varepsilon \times \hat{I}_\varepsilon)$. For $\varepsilon > 0$, the space-time unfolding operator $\mathcal{T}_\varepsilon : \mathcal{M}(\Omega \times I) \rightarrow$

$\mathcal{M}(\Omega \times I \times \square \times J)$ is defined by

$$\mathcal{T}_\varepsilon(w)(x, t, y, s) = \begin{cases} w(\varepsilon \lfloor \frac{x}{\varepsilon} \rfloor + \varepsilon y, \varepsilon^2 \lfloor \frac{t}{\varepsilon^2} \rfloor + \varepsilon^2 s) & \text{for a.e. } (x, t, y, s) \in \hat{\Omega}_\varepsilon \times \hat{I}_\varepsilon \times \square \times J, \\ 0 & \text{for a.e. } (x, t, y, s) \in \Lambda_\varepsilon \times \square \times J, \end{cases}$$

for $w \in \mathcal{M}(\Omega \times I)$. Moreover, the unfolding operator (still denoted by $\mathcal{T}_\varepsilon$) can be defined analogously for $W \in [\mathcal{M}(\Omega \times I)]^N = \mathcal{M}(\Omega \times I; \mathbb{R}^N)$. Here $\mathcal{M}(A)$ stands for the set of Lebesgue measurable functions on $A \subset \mathbb{R}^{N+1}$.

As for the weak compactness of space-time unfolded sequences, we have

Propositon 4.2. *For any bounded sequence (w_ε) in $L^q(\Omega \times I)$, there exist a sequence $\varepsilon_n \rightarrow 0_+$ and a function $w \in L^q(\Omega \times I \times \square \times J)$ such that*

$$\mathcal{T}_{\varepsilon_n}(w_{\varepsilon_n}) \rightarrow w \quad \text{weakly in } L^q(\Omega \times I \times \square \times J).$$

In addition, assume that (w_ε) is bounded in $L^q(I; W^{1,q}(\Omega))$ and $w_{\varepsilon_n} \rightarrow w$ strongly in $L^q(\Omega \times I)$ for a limit $w \in L^q(I; W^{1,q}(\Omega))$. Then there exist a (not relabeled) subsequence of (ε_n) and a function $w_1 \in L^q(\Omega \times I; L^q(J; W_{\text{per}}^{1,q}(\square)/\mathbb{R}))$ such that

$$\mathcal{T}_{\varepsilon_n}(\nabla w_{\varepsilon_n}) \rightarrow \nabla w + \nabla_y w_1 \quad \text{weakly in } [L^q(\Omega \times I \times \square \times J)]^N.$$

Proof. See [20, Proposition 2.9]. □

Remark 4.3. We note that $\nabla_y w_1$ vanishes in the sense of weak convergence due to the periodicity in $\square$ of $w_1 \in L^2(\Omega \times I; L^2(J; H_{\text{per}}^1(\square)/\mathbb{R}))$. Thus the weak convergence of the unfolded sequence for the gradient plays a crucial role in characterizing the limit of the diffusion flux $a_\varepsilon \nabla v_\varepsilon$.

We next introduce the space-time averaging operator.

Definition 4.4. *Under the same assumption as in Definition 4.1, the space-time averaging operator $\mathcal{U}_\varepsilon: L^q(\Omega \times I \times \square \times J) \rightarrow L^q(\Omega \times I)$ is defined as follows:*

$$\mathcal{U}_\varepsilon(\Psi)(x, t) = \begin{cases} \int_0^1 \int_\square \Psi(\varepsilon \lfloor \frac{x}{\varepsilon} \rfloor + \varepsilon \sigma, \varepsilon^2 \lfloor \frac{t}{\varepsilon^2} \rfloor + \varepsilon^2 \rho, \{\frac{x}{\varepsilon}\}, \{\frac{t}{\varepsilon^2}\}) d\sigma d\rho & \text{for a.e. } (x, t) \in \hat{\Omega}_\varepsilon \times \hat{I}_\varepsilon, \\ 0 & \text{for a.e. } (x, t) \in \Lambda_\varepsilon, \end{cases}$$

for $\Psi \in L^q(\Omega \times I \times \square \times J)$. Here $\{\cdot\}$ denotes the fraction part of $\cdot$ (i.e., $\{\cdot\} := \cdot - \lfloor \cdot \rfloor$).

As for the strong convergence of unfolded sequences, we have

Propositon 4.5. *Let (w_ε) be bounded in $L^q(\Omega \times I)$ and let $w \in L^q(\Omega \times I \times \square \times J)$. Then the following (i)–(iii) are equivalent:*

- (i) $\mathcal{T}_\varepsilon(w_\varepsilon) \rightarrow w$ strongly in $L^q(\Omega \times I \times \square \times J)$ and $\lim_{\varepsilon \rightarrow 0+} \iint_{\Lambda_\varepsilon} |w_\varepsilon(x, t)|^q dx dt = 0$.
- (ii) $w_\varepsilon - \mathcal{U}_\varepsilon(w) \rightarrow 0$ strongly in $L^q(\Omega \times I)$.
- (iii) $\lim_{\varepsilon \rightarrow 0+} \|w_\varepsilon\|_{L^q(\Omega \times I)} = \|w\|_{L^q(\Omega \times I \times \square \times J)}$.

Proof. See [20, Proposition 2.13]. □

5 Sketch of proof for Theorem 2.3

By employing Propositions 3.3 and 4.2, one can prove Theorem 2.3; indeed, there exist a subsequence and $z \in L^2(\Omega \times I; L^2(J; H_{\text{per}}^1(\square)/\mathbb{R}))$ such that

$$\mathcal{T}_{\varepsilon_n}(\nabla v_{\varepsilon_n}) \rightarrow \nabla v + \nabla_y z \quad \text{weakly in } [L^2(\Omega \times I \times \square \times J)]^N.$$

For any $\phi \in C_c^\infty(\Omega)$ and $\psi \in C_c^\infty(I)$, let $\varepsilon > 0$ be small enough such that $\phi\psi = 0$ on Λ_ε . Then we see by [11, Propositions 1.5 and 1.8] that

$$\begin{aligned} & \int_0^T \left\langle \partial_t v(t)^{1/p}, \phi \right\rangle_{H_0^1(\Omega)} \psi(t) dt \\ &= \lim_{\varepsilon_n \rightarrow 0+} \int_0^T \left\langle \partial_t v_{\varepsilon_n}(t)^{1/p}, \phi \right\rangle_{H_0^1(\Omega)} \psi(t) dt \\ &= - \lim_{\varepsilon_n \rightarrow 0+} \int_0^T \int_\Omega a\left(\frac{x}{\varepsilon_n}, \frac{t}{\varepsilon_n^2}\right) \nabla v_{\varepsilon_n}(x, t) \cdot \nabla \phi(x) \psi(t) dx dt \\ &= - \lim_{\varepsilon_n \rightarrow 0+} \int_0^T \int_\Omega \int_0^1 \int_\square a(y, s) \mathcal{T}_{\varepsilon_n}(\nabla v_{\varepsilon_n})(x, t, y, s) \cdot \mathcal{T}_{\varepsilon_n}(\nabla \phi \psi)(x, t, y, s) dy ds dx dt \\ &= - \int_0^T \int_\Omega \left\langle a(y, s) (\nabla v(x, t) + \nabla_y z(x, t, y, s)) \right\rangle_{y, s} \cdot \nabla \phi(x) \psi(t) dx dt, \end{aligned}$$

which completes the proof.

6 Sketch of proof for Theorem 2.4

We only consider the case where $0 < p < 1$ for simplicity. We first note that, for any $b \in C_{\text{per}}^\infty(\square)/\mathbb{R}$, there exists a unique solution $w \in C_{\text{per}}^\infty(\square)/\mathbb{R}$ to

$$(3) \quad \Delta_y w(y) = b(y) \quad \text{in } \square.$$

Set $B := \nabla_y w \in [C_{\text{per}}^\infty(\square)/\mathbb{R}]^N$ (i.e., $\text{div}_y B(y) = b(y)$), $\Psi(x, t, y, s) = \phi(x)\psi(t)b(y)c(s)$ and $\Psi_\varepsilon(x, t) = \phi(x)\psi(t)b(\frac{x}{\varepsilon})c(\frac{t}{\varepsilon^2})$ for any $\phi \in C_c^\infty(\Omega)$, $\psi \in C_c^\infty(I)$, $b \in C_{\text{per}}^\infty(\square)/\mathbb{R}$ and $c \in C_{\text{per}}^\infty(J)$. Let $\varepsilon_n > 0$ be small enough such that $\Psi_{\varepsilon_n} = 0$ on Λ_{ε_n} . Then we observe

that

$$\begin{aligned}
& \lim_{\varepsilon_n \rightarrow 0_+} \int_0^T \int_{\Omega} v_{\varepsilon_n}^{1/p}(x, t) \partial_t(\varepsilon_n \Phi_{\varepsilon_n}(x, t)) \, dx dt \\
&= \lim_{\varepsilon_n \rightarrow 0_+} \varepsilon_n \int_0^T \int_{\Omega} v_{\varepsilon_n}^{1/p}(x, t) \phi(x) \partial_t \psi(t) b\left(\frac{x}{\varepsilon_n}\right) c\left(\frac{t}{\varepsilon_n^2}\right) \, dx dt \\
&\quad + \lim_{\varepsilon_n \rightarrow 0_+} \int_0^T \int_{\Omega} v_{\varepsilon_n}^{1/p}(x, t) \phi(x) \psi(t) \nabla \cdot B\left(\frac{x}{\varepsilon_n}\right) \partial_s c\left(\frac{t}{\varepsilon_n^2}\right) \, dx dt \\
&= - \lim_{\varepsilon_n \rightarrow 0_+} \int_0^T \int_{\Omega} \nabla v_{\varepsilon_n}^{1/p}(x, t) \phi(x) \psi(t) \cdot B\left(\frac{x}{\varepsilon_n}\right) \partial_s c\left(\frac{t}{\varepsilon_n^2}\right) \, dx dt \\
&= - \lim_{\varepsilon_n \rightarrow 0_+} \int_0^T \int_{\Omega} \int_0^1 \int_{\square} \mathcal{T}_{\varepsilon_n}(\nabla v_{\varepsilon_n}^{1/p})(x, t, y, s) \mathcal{T}_{\varepsilon_n}(\phi \psi)(x, t, y, s) \cdot B(y) \partial_s c(s) \, dy ds dx dt.
\end{aligned}$$

Combining Lemmas 3.1 and 3.2 with Proposition 4.2, we see that there exists $\tilde{z} \in L^2_{\text{loc}}(\Omega \times I; L^2(J; H^1_{\text{per}}(\square)/\mathbb{R}))$ such that

$$\mathcal{T}_{\varepsilon_n}(\nabla v_{\varepsilon_n}^{1/p}) \rightarrow \nabla v + \nabla_y \tilde{z} \quad \text{weakly in } [L^2(\omega \times I \times \square \times J)]^N$$

for any $\omega \Subset \Omega$, and hence,

$$\begin{aligned}
& \int_0^T \int_{\Omega} \int_0^1 \int_{\square} \tilde{z}(x, t, y, s) \partial_s \Psi(x, t, y, s) \, dy ds dx dt \\
&= \lim_{\varepsilon_n \rightarrow 0_+} \int_0^T \int_{\Omega} a_{\varepsilon_n} \nabla v_{\varepsilon_n}(x, t) \cdot \nabla(\varepsilon_n \Psi_{\varepsilon_n}(x, t)) \, dx dt \\
&= \int_0^T \int_{\Omega} \int_0^1 \int_{\square} a(y, s) (\nabla v(x, t) + \nabla_y z(x, t, y, s)) \cdot \nabla_y \Psi(x, t, y, s) \, dy ds dx dt.
\end{aligned}$$

Since one can prove $\tilde{z} = \frac{1}{p} |v|^{(1-p)/p} z$ as in [2, Lemma 5.4], by setting $z = \sum_{k=1}^N (\partial_{x_k} v) \Phi_k$, we obtain the cell problem and (1) (see [2] for details).

7 Sketch of proof for Theorem 2.7

We first observe that

$$\begin{aligned}
& \left\| \nabla v_{\varepsilon_n} - \nabla v - \sum_{k=1}^N \mathcal{U}_{\varepsilon_n}(\partial_{x_k} v) \mathcal{U}_{\varepsilon_n}(\nabla_y \Phi_k) \right\|_{L^2(\Omega \times I)} \\
&\leq \left\| \nabla v_{\varepsilon_n} - \mathcal{U}_{\varepsilon_n}(\nabla v) - \mathcal{U}_{\varepsilon_n}(\nabla_y z) \right\|_{L^2(\Omega \times I)} \\
&\quad + \left\| \mathcal{U}_{\varepsilon_n}(\nabla v) - \nabla v \right\|_{L^2(\Omega \times I)} + \left\| \mathcal{U}_{\varepsilon_n}(\nabla_y z) - \sum_{k=1}^N \mathcal{U}_{\varepsilon_n}(\partial_{x_k} v) \mathcal{U}_{\varepsilon_n}(\nabla_y \Phi_k) \right\|_{L^2(\Omega \times I)} \\
&=: I_1^{\varepsilon_n} + I_2^{\varepsilon_n} + I_3^{\varepsilon_n}.
\end{aligned}$$

Then we shall estimate the terms $I_1^{\varepsilon_n}$, $I_2^{\varepsilon_n}$ and $I_3^{\varepsilon_n}$ below.

To prove

$$(4) \quad I_1^{\varepsilon_n} \rightarrow 0 \quad \text{as } \varepsilon_n \rightarrow 0_+,$$

we claim that

$$(5) \quad \lim_{\varepsilon_n \rightarrow 0_+} \|\nabla v_{\varepsilon_n}\|_{L^2(\Omega \times I)}^2 = \|\nabla v_0 + \nabla_y z\|_{L^2(\Omega \times I \times \square \times J)}^2.$$

Indeed, noting by [2, Lemma 6.1] that

$$v_{\varepsilon_n}(t)^{1/p} \rightarrow v_0(t)^{1/p} \quad \text{weakly in } L^{p+1}(\Omega) \quad \text{for all } t \in \bar{I},$$

we see by the weak form that

$$\limsup_{\varepsilon_n \rightarrow 0_+} \int_0^T \int_{\Omega} a_{\varepsilon_n} \nabla v_{\varepsilon_n} \cdot \nabla v_{\varepsilon_n} \, dx dt \leq \int_0^T \int_{\Omega} a_{\text{hom}}(x, t) \nabla v \cdot \nabla v \, dx dt.$$

On the other hand, by the J -periodicity of Φ_k , we have

$$\begin{aligned} & \liminf_{\varepsilon_n \rightarrow 0_+} \int_0^T \int_{\Omega} a_{\varepsilon_n} \nabla v_{\varepsilon_n} \cdot \nabla v_{\varepsilon_n} \, dx dt \\ & \geq \int_0^T \int_{\Omega} \int_0^1 \int_{\square} a(y, s) (\nabla v + \nabla_y z) \cdot (\nabla v + \nabla_y z) \, dy ds dx dt \\ & = \int_0^T \int_{\Omega} a_{\text{hom}}(x, t) \nabla v \cdot \nabla v \, dx dt. \end{aligned}$$

Thus it follows that

$$(6) \quad \lim_{\varepsilon_n \rightarrow 0_+} \int_0^T \int_{\Omega} a_{\varepsilon_n} \nabla v_{\varepsilon_n} \cdot \nabla v_{\varepsilon_n} \, dx dt = \int_0^T \int_{\Omega} a_{\text{hom}}(x, t) \nabla v \cdot \nabla v \, dx dt.$$

Now, let $\mathbb{I} \in \mathbb{R}^{N \times N}$ be a unit matrix and let $\gamma > 0$ be such that $(a(y, s) - \gamma \mathbb{I}) \xi \cdot \xi \geq \tilde{\lambda} |\xi|^2$ for all $\xi \in \mathbb{R}^N$ and some $\tilde{\lambda} > 0$. Then we infer by the J -periodicity of Φ_k that

$$\begin{aligned} & \liminf_{\varepsilon_n \rightarrow 0_+} \int_0^T \int_{\Omega} (a_{\varepsilon_n} - \gamma \mathbb{I}) \nabla v_{\varepsilon_n} \cdot \nabla v_{\varepsilon_n} \, dx dt \\ & \geq \int_0^T \int_{\Omega} \int_0^1 \int_{\square} (a(y, s) - \gamma \mathbb{I}) (\nabla v + \nabla_y z) \cdot (\nabla v + \nabla_y z) \, dy ds dx dt \\ & = \int_0^T \int_{\Omega} a_{\text{hom}}(x, t) \nabla v \cdot \nabla v \, dx dt \\ & \quad - \int_0^T \int_{\Omega} \int_0^1 \int_{\square} \gamma \mathbb{I} (\nabla v + \nabla_y z) \cdot (\nabla v + \nabla_y z) \, dy ds dx dt, \end{aligned}$$

and hence, (6) ensures that

$$\begin{aligned} & \limsup_{\varepsilon_n \rightarrow 0_+} \int_0^T \int_{\Omega} \gamma \mathbb{I} \nabla v_{\varepsilon_n} \cdot \nabla v_{\varepsilon_n} \, dx dt \\ & \leq \int_0^T \int_{\Omega} \int_0^1 \int_{\square} \gamma \mathbb{I} (\nabla v + \nabla_y z) \cdot (\nabla v + \nabla_y z) \, dy ds dx dt, \end{aligned}$$

which together with the lower semi continuity yields (5). Thus (4) follows from the implication (iii) $\Rightarrow$ (ii) of Proposition 4.5.

We next claim that

$$(7) \quad I_2^{\varepsilon_n} \rightarrow 0 \quad \text{as } \varepsilon_n \rightarrow 0_+.$$

This also follows from the implication (iii) $\Rightarrow$ (ii) of Proposition 4.5.

We finally show that

$$(8) \quad I_3^{\varepsilon_n} \rightarrow 0 \quad \text{as } \varepsilon_n \rightarrow 0_+.$$

It suffices to prove that

$$(9) \quad \mathcal{U}_{\varepsilon_n}((\partial_{x_k} v) \nabla_y \Phi_k) - \mathcal{U}_{\varepsilon_n}(\partial_{x_k} v) \mathcal{U}_{\varepsilon_n}(\nabla_y \Phi_k) \rightarrow 0 \quad \text{strongly in } [L^2(\Omega \times I)]^N.$$

To this end, we shall use the following fact:

$$(10) \quad \nabla_y \Phi_k \in [L^\infty(\Omega \times I; L^2(\square \times J))]^N$$

(see [2, Appendix] for the proof). Since $\partial_{x_k} v$ is independent of $(y, s) \in \square \times J$, noting that, for any $(\xi, \zeta) \in \Xi_{\varepsilon_n} \times \Theta_{\varepsilon_n} := \{\zeta \in \mathbb{N} \cup \{0\} : \varepsilon^2(\zeta + J) \subset I\}$, $\mathcal{U}_{\varepsilon_n}(\partial_{x_k} v)$ can be regarded as a constant in $\varepsilon_n(\xi + \square) \times \varepsilon_n^2(\zeta + J)$, we derive that

$$\begin{aligned} & \|\mathcal{U}_{\varepsilon_n}((\partial_{x_k} v) \nabla_y \Phi_k) - \mathcal{U}_{\varepsilon_n}(\partial_{x_k} v) \mathcal{U}_{\varepsilon_n}(\nabla_y \Phi_k)\|_{L^2(\Omega \times I)}^2 \\ & = \sum_{\zeta \in \Theta_{\varepsilon_n}} \sum_{\xi \in \Xi_{\varepsilon_n}} \int_{\varepsilon^2(\zeta + J)} \int_{\varepsilon(\xi + \square)} |\mathcal{U}_{\varepsilon_n}((\partial_{x_k} v - \mathcal{U}_{\varepsilon_n}(\partial_{x_k} v)) \nabla_y \Phi_k)|^2 \, dx dt \\ & \leq \|\nabla_y \Phi_k\|_{L^\infty(\Omega \times I; L^2(\square \times J))}^2 \|\partial_{x_k} v - \mathcal{U}_{\varepsilon_n}(\partial_{x_k} v)\|_{L^2(\Omega \times I)}^2 \rightarrow 0 \quad \text{as } \varepsilon_n \rightarrow 0_+. \end{aligned}$$

Here we used the facts (7) and (10) in the last line (see [20] for details). Thus we have (9). Combining (4), (7) and (8), we obtain

$$\left\| \nabla v_{\varepsilon_n} - \nabla v - \sum_{k=1}^N \mathcal{U}_{\varepsilon_n}(\partial_{x_k} v) \mathcal{U}_{\varepsilon_n}(\nabla_y \Phi_k) \right\|_{L^2(\Omega \times I)} \leq I_1^{\varepsilon_n} + I_2^{\varepsilon_n} + I_3^{\varepsilon_n} \rightarrow 0 \quad \text{as } \varepsilon_n \rightarrow 0_+,$$

which completes the proof.

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