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# On Regular Surfaces of General Type II.

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1. Introduction. In this paper a surface shall mean a compact complex manifold of dimension 2. We denote by $|mK_S|$ ( $m \in N$ ) a pluricanonical system on a surface S and by $\Phi_{mK_S}$ the associated rational map (the pluricanonical map), assuming that $|mK_S|$ is not empty. A surface S is called of general type if $\Phi_{mK_S}(S)$ in the projective space $P^N$ ($N = \dim mK_S$ ) for a large number m is a variety of dimension 2. If S is a surface of general type the following results are well-known.

Theorem 1 (Mumford [ ]). If m is sufficiently large, $\Phi_{mK_S}$ is a birational morphism and $\Phi_{mK_S}(S) \cong X = \mathrm{Proj} \bigoplus_r H^0(S, \underline{O}(rK_S))$. X is a normal variety with only a finite number of rational double points as singularities. If S is a minimal surface, then S is the minimal resolution of X.

Theorem 2 (Mumford [ ]). Assume that S is minimal. Then we have $H^1(S, \underline{O}(mK_S)) = 0$, for $m \neq 0, 1$, $m \in Z$.

Theorem 3 (Riemann-Roch Theorem for pluricanonical systems). Letting $\bar{c}_1^2$ be the self intersection number for the canonical divisor of the minimal model of S, we have

$$\dim H^0(S,\underline{O}(mK_S)) = \chi(\underline{O}) + (\bar{c}_1^{\,2}/2)\, m(m-1),$$

where $\chi(\underline{O})$ denotes the Euler characteristic of the structure sheaf $\underline{O}_S$ of $S$.

Theorem 4 (Iitaka [ ]). The m-genus $P_m(S) = \dim H^0(S,\underline{O}_S(mK_S))$ is deformation-invariant.

As an immediate corollary to Theorems 3 and 4, we obtain the following

Theorem 5 (Deformation Invariance of the Minimality). If $S$ is minimal, then any deformation of $S$ is also a minimal surface of general type.

From now on, we denote by $S$ a minimal surface of general type with the following numerical conditions:

$$* \quad \begin{cases} p_g(S) = \dim H^0(S,\underline{O}(K_S)) = 0, \\ q(S) = \dim H^1(S,\underline{O}) = 0, \\ K_S^{\,2} = 2. \end{cases}$$

A surface of this type shall be called a numerical Campedelli surface.

In section 2, we study the property of the tricanonical system $|3K_S|$ on a numerical Campedelli surface. In spite of Bombieri's comprehensive work [ ] on pluricanonical maps, the tricanonical system on $S$ was not completely surveyed. ~~But~~ And there remains still an open problem: Is the tricanonical map of $S$ is a birational morphism?

It is an interesting but, in general, a very difficult

problem to determine the complex structures on a given underlying differentiable manifold. In our case the problem is rather easy under some conditions. In section 3, we shall determine the structure of $S$ under the condition that the fundamental group of $S$ is a direct sum of three copies of the cyclic group of order 2.

2. Regularity of the tricanonical maps.

Let $S$ be a numerical Campedelli surface. Then we have the following

Theorem 5 (Regularity of tricanonical maps). The tricanonical system $|3K_S|$ on $S$ is free from base points and fixed components.

For the proof we need some results .

Definition. An effective divisor $D$ on a surface $F$ is called 1-connected if

$$D_1 \cdot D_2 > 0,$$

for any non-trivial decomposition $D = D_1 + D_2$, $D_i > 0$.

Theorem 6 (Ramanujam vanishing theorem [ ]). If an effective divisor $D$ on a regular surface (i.e. $q(F) = 0$ ) is 1-connected , then $H^1(F, \underline{O}(-D)) = 0$.

Theorem 7 (Bombieri [ ]). Let $F$ be a minimal surface of general type and $P$ a point on $F$. Let $p: \tilde{F} \to F$ denote a quadric transformation at $P$ and $E$ the exceptional curve over $P$. If an effective divisor $D$ is numerically equivalent to $2p^*K_F - 2E$, then $D$ is 1-connected except in the case where $K_F^2 = 1$.

Now we proceed to the proof of Theorem 1. Let $p: \tilde{S} \to S$ be the quadric transformation at a point $P$ and $E$ the associated exceptional curve. Let us consider the following natural exact sequence of sheaves:

$$0 \longrightarrow \underline{O}_{\tilde{S}}(3p^*K_S - E) \longrightarrow \underline{O}_{\tilde{S}}(3p^*K_S) \longrightarrow \underline{O}_E \longrightarrow 0.$$

Then it is obvious that $|3K_S|$ is free from base point at $P$ if and only if $H^1(\tilde{S},\underline{O}(3p^*(K_S - E)) = 0$. By the Serre duality we have

$$\dim H^1(\tilde{S},\underline{O}(3p^*K_S - E)) = \dim H^1(\tilde{S},\underline{O}(2E - 2p^*K_S)).$$

Hence Theorem 7 yields the vanishing of the cohomology group under the condition that $|2p^*K_S - 2E| \neq \emptyset$. Now assume that $|2p^*K_S - 2E| = \emptyset$. Since $\dim H^0(S,\underline{O}(2K_S)) = 3$, this implies that the rational map $\Phi_{2K_S}$ associated with the bicanonical system $|2K_S|$ is a local isomorphism at $P$. Therefore there exists an effective divisor $D \in |2p^*K_S - E|$ such that $D$ is irreducible in a neighbourhood of $E$ and that the unique irreducible component $D_0$ which simply intersects $E$ satisfies $D_0^2 \geq 0$. Now we shall take the following exact sequence of cohomology groups:

$$0 \longrightarrow H^0(\tilde{S},\underline{O}(2E - 2p^*K_S)) \longrightarrow H^0(\tilde{S},\underline{O}(E)) \longrightarrow H^0(D, \underline{O}_D(E))$$
$$\longrightarrow H^1(\tilde{S},\underline{O}(2E - 2p^*K_S)) \longrightarrow H^1(\tilde{S},\underline{O}(E)).$$

Note that $H^0(\tilde{S},\underline{O}(2E - 2p^*K_S)) = 0$ and that $\dim H^1(\tilde{S},\underline{O}(E)) = \dim H^1(\tilde{S},\underline{O}(p^*K_S)) = \dim H^1(S,\underline{O}(K_S))$ $= q(S) = 0$. Hence, for the proof of Theorem 5, it is sufficient to show the equality

$$\dim H^0(D,\underline{O}(E)) = \dim H^0(\tilde{S},\underline{O}(E)) = 1.$$

On the other hand we have the following natural commutative diagram

$$\begin{array}{ccccccc} 0 & \to & H^0(D,\underline{O}) & \longrightarrow & H^0(D,\underline{O}(E)) & \xrightarrow{r} & H^0(D\cdot E,\underline{O}) \\ & & \downarrow & & \downarrow & & \downarrow \text{ identity} \\ 0 & \to & H^0(D_0,\underline{O}) & \longrightarrow & H^0(D_0,\underline{O}(E)) & \xrightarrow{r} & H^0(D.E,\underline{O}) \end{array}$$

of which the rows are exact. But it is obvious that the virtual genus of $D_0$ is not 0. Since the degree of the divisor E on $D_0$ is 1, the restriction map r is the zero-map. This implies that

$$\dim H^0(D,\underline{O}(E)) = \dim H^0(D,\underline{O}).$$

Moreover we have $\dim H^0(D,\underline{O}) = 1$. In fact, there exists the following natural exact sequence

$$0 \longrightarrow H^0(\tilde{S},\underline{O}(E - 2p^*K_S)) \longrightarrow H^0(\tilde{S},\underline{O}) \longrightarrow H^0(D,\underline{O}) \longrightarrow H^1(\tilde{S},\underline{O}(E - 2p^*K_S)),$$

where $\dim H^1(\tilde{S},\underline{O}(E - 2p^*K_S)) = \dim H^1(\tilde{S},\underline{O}(3p^*K_S)) = \dim H^1(S,\underline{O}(3K_S)) = 0$. Thus $\dim H^0(D,\underline{O}) = \dim H^0(S,\underline{O}) = 1$ and the assertion is proved.

3. The structure of Campedelli surfaces.

In this section we shall study numerical Campedelli surfaces of special type.

Definition (cf. Campedelli [ ]). A numerical Campedelli surface is called a Campedelli surface if its fundamental group is isomorphic to $Z/(2) + Z/(2) + Z/(2)$.

If $S$ is a Campedelli surface, the universal covering $\bar{S}$ of $S$ has the following numerical characters:

$$\begin{cases} \chi(\bar{S},\underline{O}_{\bar{S}}) = 8\chi(S,\underline{O}_S) = 8, \\ q(\bar{S}) = 0, \\ p_g(\bar{S}) = \chi(\bar{S},\underline{O}_{\bar{S}}) - q(\bar{S}) - 1 = 7, \\ K_{\bar{S}}^2 = 8\,K_S^2 = 16. \end{cases}$$

The fundamental group $G$ of $S$ acts on $\bar{S}$ as the covering transformation group of the unramified covering $e: \bar{S} \longrightarrow S$, and $G$ naturally operates on the vector space $H^0(\bar{S},\underline{O}(K_{\bar{S}}))$ as linear transformations. Hence we obtain a canonical representation $k: G \longrightarrow GL(7,C)$ and the induced representation $k': G \longrightarrow PGL(6,C)$.

Lemma 1. $k'$ is a faithful representation.

Proof. Let $g \in G$ be an element of $\ker k'$. Since $g^2 = id$, $k(g) = \pm id$. Hence $p_g(\bar{S}/\langle g\rangle) = 7$ or $0$. But $p_g(\bar{S}/\langle g\rangle) = 3$, if $g$ is of order 2. Hence $g = id$.

Let $V$ denotes the image of $\bar{S}$ by the canonical map $\Phi_{K_{\bar{S}}}$ associated with the canonical system $|K_{\bar{S}}|$.

Then $k'(g)$ $(g \in G)$ induces an automorphism of $V$. Thus we obtain a natural homomorphism $a: G \longrightarrow \mathrm{Aut}(V)$, where $\mathrm{Aut}(V)$ denotes the automorphism group of $V$.

Lemma 2. $a$ is injective.

Proof. A trivial consequence of Lemma 1. ~~Assume that $g \in G$ induces the identity on $V$. Then $V$ is contained in an eigenspace of $k'(g)$. Since $V$ is not contained in any proper linear subspace of $P^6$, this implies that $k'(g) = id$. Lemma 1 yields the equality $g = id$.~~

Lemma 3. The canonical system $K_{\bar{S}}$ of $\bar{S}$ is not composed of a pencil.

Proof. Assume that $V$ is a curve. Since $q(\bar{S}) = 0$, $V$ must be a (possibly singular) rational curve. An automorphism of $V$ induces a unique automorphism of the non-singular model $P^1$ of $V$. Hence, in virtue of the above lemma, we infer that there exists a faithful representation $a': G \longrightarrow \mathrm{PGL}(1,C)$. On the other hand, it is obvious that $\mathrm{PGL}(1,C)$ does not contain a subgroup isomorphic to $(Z/(2))^3$. This is a contradiction.

Since $G$ is a commutative group, we may assume that $k(G)$ is contained in the diagonal subgroup of $GL(7,C)$. Let $w_1, \ldots, w_7$ be a basis of $H^0(\bar{S}, \underline{O}(K_{\bar{S}}))$ such that $g^*(w_j) = \pm w_j$ for any $g \in G$.

Lemma 4. The linear subspace $W$ of $H^0(\bar{S}, \underline{O}(2K_{\bar{S}}))$ spanned by $w_1^2, w_2^2, \ldots, w_7^2$ is 3-dimensional.

Proof. Lemma 3 implies that the transcendental degree over $C$ of the field $C(w_2/w_1,\ldots,w_7/w_1)$ is 2. Hence the transcendental degree of $C(w_2^2/w_1^2,\ldots,w_7^2/w_1^2)$ is also 2. This yields the inequality

$$\dim W \geqq 3.$$

On the other hand, since $w_j^2$ is G-invariant, $W$ can be regarded as a subspace of $H^0(S,\underline{O}(2K_S))$. But the Riemann-Roch theorem geves an equality $\dim H^0(S,\underline{O}(2K_S)) = 3$. This completes the proof.

Lemma 5. Let $K$ be an extension of the rational function field $C(x_1,\ldots,x_n)$ defined by

$$K_r = C(x_1,\ldots,x_n,\sqrt{Q_1},\ldots,\sqrt{Q_r}),$$

where $Q_j$ is a quadric polynomial in $x_i$. Assume that $K_r : C(x_1,\ldots,x_n) = 2^r$. Then the integral closure of $C[x_1,\ldots,x_n]$ in $K_r$ is $R_r = C[x_1,\ldots,x_n, Q_1,\ldots, Q_r]$.

Proof. Trivial.

Corollary. Let $K$ be as above. Let $Q_{r+1}$ be another quadric polynomial in $x_i$. Assume that $K_{r+1} = K_r$. Then $\sqrt{Q_{r+1}}$ is a linear combination cf $x_1,\ldots, x_n, \sqrt{Q_1},\ldots,\sqrt{Q_r}$.

Let $w_1^2$, $w_2^2$, $w_3^2$ be a basis of $W$. From Lemma 4, we infer that there are quadric relations

$$w_j^2 = a_j w_1^2 + b_j w_2^2 + c_j w_3^2, \qquad j = 4, 5, 6, 7.$$

The above corollary asserts that, if the complete intersection defined by the above quadrics is reducible

then its any irreducible component is contained in a hyperplane in $P^6$. Since the image $V$ of $\bar{S}$ is contained in the complete intersection $V'$ defined by the above 4 equations and $V$ is not contained in any hyperplane, $V' = V$ is a irreducible surface. Thus we obtain the following

Corollary. $V$ is a complete intersection of type $(2,2,2,2)$ in $P^6$.

As an immediate consequence of this corollary, we have

Theorem 8. The canonical homomorphism

$$\otimes^m H^0(\bar{S},\underline{O}(K_{\bar{S}})) \longrightarrow H^0(\bar{S},\underline{O}(mK_{\bar{S}}))$$

is surjective.

Proof. Let $\underline{O}_V(m)$ denote the sheaf of the hypersurface section of degree $m$. Since $V$ is a complete intersection of type $(2,2,2,2)$, we have

$$\dim H^0(V,\underline{O}_V(m)) \geq 8 + 8\,m(m-1) = \dim H^0(\bar{S},\underline{O}_{\bar{S}}(mK_{\bar{S}}))$$

Moreover $H^0(V,\underline{O}_V(1))$ generates $H^0(V,\underline{O}_V(m))$. This proves the theorem.

Now the following theorem is trivial.

Theorem 9. The canonical model $\bar{X}$ of $\bar{S}$ is isomorphic to a complete intersection of type $(2,2,2,2)$ in $P^6$. The canonical model $X$ of $S$ is the quotient of $\bar{X}$ by the action of following subgroup $G$ of $PGL(6,C)$.

$$G = \left\langle \begin{pmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ 0 & & & & & -1 \end{pmatrix}, \begin{pmatrix} 1 & & & & & \\ & -1 & & & 0 & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & & & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & & & & \\ & -1 & & & 0 & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ 0 & & & & & -1 \end{pmatrix} \right\rangle$$

The following theorem is a corollary of Theorem 9 and the forms of the defining equations.

Theorem 10. The moduli space of Campedelli surfaces is a ~~normal~~ unirational variety of dimension 6.

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