

A NOTE ON GAMMA FUNCTIONS

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1. Introduction

Let X be the Gamma distributed random variable,

$$P\{X \leq x\} = \int_0^x \frac{\beta^\xi}{\Gamma(\xi)} t^{\xi-1} e^{-\beta t} dt,$$

for $x > 0$, where $\beta > 0$, $\xi > 0$. Let us denote $Y = \alpha \log X$ for $\alpha \neq 0$. We obtain

$$P\{Y \leq x\} = \int_{-\infty}^x \frac{\beta^\xi}{|\alpha| \Gamma(\xi)} e^{(\xi/\alpha)t} e^{-\beta e^{t/\alpha}} dt, \quad x \in \mathbb{R}, \quad (1)$$

and then the characteristic function of distribution function of the random variable Y is

$$Ee^{izY} = \frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}}, \quad z \in \mathbb{R}. \quad (2)$$

The author will discuss about the Lévy representation,

$$\begin{aligned} & \frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}} \\ &= \exp \left[iz \left\{ \alpha \frac{\Gamma'(\xi)}{\Gamma(\xi)} - \alpha \log \beta + \alpha^3 \int_{-\infty}^{-0} \frac{x^2}{1 + \alpha^2 x^2} \frac{e^{\xi x}}{(1 - e^x)} dx \right\} \right. \\ & \quad \left. + \int_{-\infty}^{-0} \left(e^{izx} - 1 - \frac{izx}{1 + x^2} \right) \frac{e^{(\xi/\alpha)x}}{(1 - e^{x/\alpha})|x|} dx \right], \end{aligned} \quad (3)$$

for $\alpha > 0$ and also discuss on Thorin's representation. The Lévy representation and Thorin's representation were found by B. Grigelionis in the paper [1]. In this note, as an application of the Lévy representation it is shown that Gauss's multiplication formula or the duplication formula of Legendre can be obtained from a property of Lévy measure.

2. On an infinitely divisible characteristic function

We first show (1). Suppose that α is a positive constant. We see by change of variable, $u = e^{t/\alpha}$ that

$$\begin{aligned}
 P\{Y \leq x\} &= P\{\log X \leq \frac{x}{\alpha}\} \\
 &= P\{X \leq \exp(\frac{x}{\alpha})\} \\
 &= \int_0^{\exp(\frac{x}{\alpha})} \frac{\beta^\xi}{\Gamma(\xi)} u^{\xi-1} \exp(-\beta u) du \\
 &= \frac{\beta^\xi}{\Gamma(\xi)} \int_{-\infty}^x \exp\left\{\frac{\xi}{\alpha}t - \beta e^{\frac{t}{\alpha}}\right\} \frac{dt}{\alpha}. \tag{4}
 \end{aligned}$$

Next, suppose that α is a negative constant. Let us set $\alpha' = -\alpha$. We see by change of variable, $u = e^{-t/\alpha'}$ that

$$\begin{aligned}
 P\{Y \leq x\} &= P\{\log X \geq -\frac{x}{\alpha'}\} = P\{X \geq \exp(-\frac{x}{\alpha'})\} \\
 &= 1 - P\{X < \exp(-\frac{x}{\alpha'})\} \\
 &= 1 - \int_0^{\exp(-\frac{x}{\alpha'})} \frac{\beta^\xi}{\Gamma(\xi)} u^{\xi-1} \exp(-\beta u) du \\
 &= 1 - \frac{\beta^\xi}{\Gamma(\xi)} \int_{-\infty}^x \left(e^{-\frac{t}{\alpha'}}\right)^\xi \exp(-\beta e^{-\frac{t}{\alpha'}}) \left(-\frac{dt}{\alpha'}\right) \\
 &= 1 - \frac{\beta^\xi}{\Gamma(\xi)} \int_x^\infty \exp\left\{-\frac{\xi}{\alpha'}t - \beta e^{-\frac{t}{\alpha'}}\right\} \frac{dt}{\alpha'} \\
 &= 1 + \frac{\beta^\xi}{\Gamma(\xi)} \int_x^\infty \exp\left\{\frac{\xi}{\alpha}t - \beta e^{\frac{t}{\alpha}}\right\} \frac{dt}{\alpha}. \tag{5}
 \end{aligned}$$

Suppose that α is a positive constant. Next, we will get a characteristic function of distribution function of Y . We see that

$$\begin{aligned}
 Ee^{izY} &= \int_{-\infty}^{\infty} e^{izy} \frac{\beta^\xi}{\Gamma(\xi)} e^{\frac{\xi}{\alpha}y} e^{-\beta e^{\frac{y}{\alpha}}} \frac{dy}{\alpha} \\
 &= \frac{\beta^\xi}{\Gamma(\xi)} \int_{-\infty}^{\infty} e^{(iz + \frac{\xi}{\alpha})y - \beta e^{\frac{y}{\alpha}}} \frac{dy}{\alpha} \\
 &= \frac{\beta^\xi}{\Gamma(\xi)} \int_{+0}^{\infty} x^{(\xi + i\alpha z)} e^{-\beta x} \frac{dx}{x} \\
 &= \frac{\beta^\xi}{\Gamma(\xi)} \int_{+0}^{\infty} v^{(\xi + i\alpha z - 1)} e^{-v} dv \frac{1}{\beta^{\xi + i\alpha z}}
 \end{aligned}$$

$$= \frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}}. \quad (6)$$

Next, suppose that α is a negative constant. Let us set $\alpha' = -\alpha$. We see that

$$\begin{aligned} Ee^{izY} &= \int_{-\infty}^{\infty} e^{izy} \frac{\beta^\xi}{\Gamma(\xi)} e^{-\frac{\xi}{\alpha'}y} e^{-\beta e^{-\frac{y}{\alpha'}}} \frac{dy}{\alpha'} \\ &= \frac{\beta^\xi}{\Gamma(\xi)} \int_{-\infty}^{\infty} e^{i(z-\frac{\xi}{\alpha'})y - \beta e^{-\frac{y}{\alpha'}}} \frac{dy}{\alpha'} \\ &= \frac{\beta^\xi}{\Gamma(\xi)} \int_{-\infty}^{+0} x^{(\xi-i\alpha'z)} e^{-\beta x} (-1) \frac{dx}{x} \\ &= \frac{\beta^\xi}{\Gamma(\xi)} \int_{+0}^{\infty} v^{(\xi-i\alpha'z-1)} e^{-v} dv \frac{1}{\beta^{\xi-i\alpha'z-1}\beta} \\ &= \frac{\Gamma(\xi - i\alpha'z)}{\Gamma(\xi)\beta^{-i\alpha'z}} = \frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}}. \end{aligned} \quad (7)$$

Let $z = -iu$ and u real in an interval of the real line which includes the origin such that $\xi + \alpha u > 0$ if ξ is a positive constant. Let us take the principal logarithm such that

$$\log \left\{ \frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}} \right\} = 0$$

for $z = -iu = 0$ and let us denote

$$\Psi(u) = \log \Gamma(\xi + \alpha u) - \log \Gamma(\xi)\beta^{\alpha u}.$$

Theorem 1. *The characteristic function of the distribution function (4) or (5) is given in the following form.*

$$\begin{aligned} &\frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}} \\ &= \exp \left[iz \left\{ \alpha \frac{\Gamma'(\xi)}{\Gamma(\xi)} - \alpha \log \beta + \alpha^3 \int_{-\infty}^{-0} \frac{x^2}{1 + \alpha^2 x^2} \frac{e^{\xi x}}{(1 - e^x)} dx \right\} \right. \\ &\quad \left. + \int_{-\infty}^{-0} \left(e^{izx} - 1 - \frac{izx}{1 + x^2} \right) \frac{e^{(\xi/\alpha)x}}{(1 - e^{x/\alpha})|x|} dx \right] \end{aligned} \quad (8)$$

Proof. By the result in [2] we see that

$$\begin{aligned} \frac{d\Psi(u)}{du} &= (\log \Gamma(\xi + \alpha u) - \log \Gamma(\xi)\beta^{\alpha u})' \\ &= \frac{\Gamma'(\xi + \alpha u)}{\Gamma(\xi + \alpha u)} \alpha - \alpha \log \beta \\ &= \alpha \int_{+0}^{\infty} \left\{ \frac{e^{-x}}{x} - \frac{e^{-(\xi+\alpha u)x}}{1 - e^{-x}} \right\} dx - \alpha \log \beta. \end{aligned} \quad (9)$$

Consider the case that α is positive. Integrating from 0 to u , we have

$$\Psi(u) - \Psi(0) = \alpha \int_0^u \left(\int_{+0}^{\infty} \left\{ \frac{e^{-x}}{x} - \frac{e^{-(\xi+\alpha t)x}}{1-e^{-x}} \right\} dx \right) dt - \alpha u \log \beta \quad (10)$$

and so we obtain

$$\begin{aligned} \Psi(u) &= \alpha \int_{+0}^{\infty} \left(\int_{+0}^u \left\{ \frac{e^{-x}}{x} - \frac{e^{-(\xi+\alpha t)x}}{1-e^{-x}} \right\} dt \right) dx - \alpha u \log \beta \\ &= \alpha \int_{+0}^{\infty} \left(u \frac{e^{-x}}{x} - \frac{e^{-\xi x}}{1-e^{-x}} \frac{e^{-\alpha u x} - 1}{-\alpha x} \right) dx - \alpha u \log \beta \\ &= \alpha u \int_{+0}^{\infty} \left(\frac{e^{-x}}{x} - \frac{e^{-\xi x}}{1-e^{-x}} \right) dx \\ &\quad + \alpha u \int_{+0}^{\infty} \left(1 - \frac{1}{1+\alpha^2 x^2} \right) \frac{e^{-\xi x}}{1-e^{-x}} dx \\ &\quad + \int_{+0}^{\infty} \left(e^{-\alpha u x} - 1 + \frac{\alpha u x}{1+\alpha^2 x^2} \right) \frac{e^{-\xi x}}{(1-e^{-x})x} dx - \alpha u \log \beta. \end{aligned} \quad (11)$$

and by the facts that

$$\frac{\Gamma'(\xi)}{\Gamma(\xi)} = \int_{+0}^{\infty} \left(\frac{e^{-x}}{x} - \frac{e^{-\xi x}}{1-e^{-x}} \right) dx$$

and

$$\begin{aligned} &\int_{+0}^{\infty} \left((e^{-\alpha u x} - 1 + \frac{\alpha u x}{1+\alpha^2 x^2}) \frac{e^{-\xi x}}{(1-e^{-x})x} \right) dx \\ &= \int_{+0}^{\infty} \left(e^{-ut} - 1 + \frac{ut}{1+t^2} \right) \frac{e^{-(\xi/\alpha)t}}{(1-e^{-t/\alpha})t} dt \\ &= \int_{-\infty}^{-0} \left(e^{uy} - 1 - \frac{uy}{1+y^2} \right) \frac{e^{(\xi/\alpha)y}}{(1-e^{y/\alpha})|y|} dy \end{aligned} \quad (12)$$

we obtain

$$\begin{aligned} \Psi(u) &= \alpha u \frac{\Gamma'(\xi)}{\Gamma(\xi)} + \alpha^3 u \int_{+0}^{\infty} \frac{x^2}{1+\alpha^2 x^2} \frac{e^{-\xi x}}{1-e^{-x}} dx - \alpha u \log \beta \\ &\quad + \int_{-\infty}^{-0} \left(e^{uy} - 1 - \frac{uy}{1+y^2} \right) \frac{e^{(\xi/\alpha)y}}{(1-e^{y/\alpha})|y|} dy. \end{aligned} \quad (13)$$

Next, suppose that α is negative. In the same way as the case that α is positive, we obtain

$$\begin{aligned} \Psi(u) &= \alpha u \frac{\Gamma'(\xi)}{\Gamma(\xi)} + \alpha^3 u \int_{+0}^{\infty} \frac{x^2}{1+\alpha^2 x^2} \frac{e^{-\xi x}}{1-e^{-x}} dx - \alpha u \log \beta \\ &\quad + \int_{+0}^{\infty} \left(e^{uy} - 1 - \frac{uy}{1+y^2} \right) \frac{e^{(\xi/\alpha)y}}{(1-e^{y/\alpha})y} dy \end{aligned} \quad (14)$$

and hence the Lévy representation (8). q.e.d

3. On Thorin's representation of characteristic function

We will show Thorin's representation of the characteristic function (2).

Theorem 2. *We obtain Thorin's representation in the following form.*

$$\begin{aligned} & \frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}} \\ &= \exp\left[i\alpha z\left(\frac{\Gamma'(\xi)}{\Gamma(\xi)} - \log \beta + \sum_{k=0}^{\infty} \frac{\alpha^2}{(\alpha^2 + (\xi + k)^2)(\xi + k)}\right)\right. \\ & \quad \left.+ \sum_{k=0}^{\infty} \left\{-\log \frac{i\alpha z + \xi + k}{\xi + k} + iz \frac{(\xi + k)\alpha}{\alpha^2 + (\xi + k)^2}\right\}\right] \end{aligned} \quad (15)$$

Proof. Suppose that α is positive. We see that

$$\begin{aligned} & \int_{-\infty}^{-0} \left(e^{ut} - 1 - \frac{ut}{1+t^2}\right) \frac{e^{(\xi/\alpha)t}}{(1 - e^{t/\alpha})|t|} dt \\ &= \int_{-\infty}^{-0} \left(\frac{e^{ut} - 1}{-t} + \frac{u}{1+t^2}\right) \frac{e^{(\xi/\alpha)t}}{(1 - e^{t/\alpha})} dt \\ &= - \int_{-\infty}^{-0} \left(\int_{+0}^u e^{ty} dy - \frac{u}{1+t^2}\right) e^{(\xi/\alpha)t} \sum_{k=0}^{\infty} e^{k/\alpha t} dt \\ &= - \sum_{k=0}^{\infty} \left\{ \int_{+0}^u \left(\int_{-\infty}^{-0} (e^{(y+(\xi+k)/\alpha)t} dt)\right) dy \right. \\ & \quad \left. - u \int_{-\infty}^{-0} \frac{1}{1+t^2} e^{(\xi+k)/\alpha t} dt \right\} \\ &= - \sum_{k=0}^{\infty} \left\{ \int_{+0}^u \frac{dy}{y + (\xi+k)/\alpha} - u \frac{(\xi+k)/\alpha}{1 + (\xi+k)^2/\alpha^2} \right\} \\ & \quad - u \sum_{k=0}^{\infty} \left\{ \frac{(\xi+k)/\alpha}{1 + (\xi+k)^2/\alpha^2} - \int_{-\infty}^{-0} \frac{1}{1+t^2} e^{(\xi+k)/\alpha t} dt \right\}. \end{aligned} \quad (16)$$

From the above (13) we see that

$$\begin{aligned} \Psi(u) &= \alpha u \frac{\Gamma'(\xi)}{\Gamma(\xi)} + \sum_{k=0}^{\infty} \alpha^3 u \int_{+0}^{\infty} \frac{x^2}{1 + \alpha^2 x^2} e^{-(\xi+k)x} dx - \alpha u \log \beta \\ & \quad - \sum_{k=0}^{\infty} \left\{ \int_{+0}^u \frac{dy}{y + (\xi+k)/\alpha} dy - u \frac{(\xi+k)/\alpha}{1 + (\xi+k)^2/\alpha^2} \right\} \\ & \quad - u \sum_{k=0}^{\infty} \left\{ \frac{(\xi+k)/\alpha}{1 + (\xi+k)^2/\alpha^2} - \int_{-\infty}^{-0} \frac{1}{1+t^2} e^{(\xi+k)/\alpha t} dt \right\} \end{aligned}$$

$$\begin{aligned}
&= \alpha u \frac{\Gamma'(\xi)}{\Gamma(\xi)} - \alpha u \log \beta \\
&\quad - \sum_{k=0}^{\infty} \left\{ \int_{+0}^u \frac{dy}{y + (\xi + k)/\alpha} dy - u \frac{(\xi + k)/\alpha}{1 + (\xi + k)^2/\alpha^2} \right\} \\
&\quad - u \sum_{k=0}^{\infty} \left\{ \frac{(\xi + k)/\alpha}{1 + (\xi + k)^2/\alpha^2} - \alpha \int_{-\infty}^{-0} e^{(\xi+k)t} dt \right\} \\
&= \alpha u \frac{\Gamma'(\xi)}{\Gamma(\xi)} - \alpha u \log \beta \\
&\quad + \sum_{k=0}^{\infty} \left\{ -\log \frac{u + (\xi + k)/\alpha}{(\xi + k)/\alpha} + u \frac{(\xi + k)/\alpha}{1 + (\xi + k)^2/\alpha^2} \right\} \\
&\quad - \alpha u \sum_{k=0}^{\infty} \left\{ \frac{\xi + k}{\alpha^2 + (\xi + k)^2} - \frac{1}{\xi + k} \right\} \\
&= \alpha u \frac{\Gamma'(\xi)}{\Gamma(\xi)} - \alpha u \log \beta + \alpha u \sum_{k=0}^{\infty} \frac{\alpha^2}{(\alpha^2 + (\xi + k)^2)(\xi + k)} \\
&\quad + \sum_{k=0}^{\infty} \left\{ -\log \frac{u + (\xi + k)/\alpha}{(\xi + k)/\alpha} + u \frac{(\xi + k)/\alpha}{1 + (\xi + k)^2/\alpha^2} \right\}. \tag{17}
\end{aligned}$$

Therefore we obtain Thorin's representation

$$\begin{aligned}
&\frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}} \\
&= \exp \left[i\alpha z \left(\frac{\Gamma'(\xi)}{\Gamma(\xi)} - \log \beta + \sum_{k=0}^{\infty} \frac{\alpha^2}{(\alpha^2 + (\xi + k)^2)(\xi + k)} \right) \right. \\
&\quad \left. + \sum_{k=0}^{\infty} \left\{ -\log \frac{i\alpha z + \xi + k}{\xi + k} + iz \frac{(\xi + k)\alpha}{\alpha^2 + (\xi + k)^2} \right\} \right]. \tag{18}
\end{aligned}$$

For the case that α is negative, in the same way as the above we obtain the same expression as the above formula. q.e.d

4. The duplication formula of Legendre and Gauss's multiplication formula

Let $\alpha = 1$, $z = \xi + i\eta$ and $m = 2, 3, \dots$. The duplication formula of Legendre is

$$\Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1-2z} \sqrt{\pi} \Gamma(2z)$$

and Gauss's multiplication formula is

$$\prod_{k=0}^{m-1} \Gamma(z + \frac{k}{m}) = m^{1/2-mz} (2\pi)^{1/2(m-1)} \Gamma(mz). \quad (19)$$

In what follows, we show that Gauss' multiplication formula can be deduced from a property of the Lévy measure in the Lévy representation (3). We see that the left hand side of (19) can be written in the following form:

$$\begin{aligned} \prod_{k=0}^{m-1} \Gamma(\xi + i\eta + \frac{k}{m}) &= \Gamma(\xi) \beta^{i\eta} \Gamma(\xi + \frac{1}{m}) \beta^{i\eta} \cdots \Gamma(\xi + \frac{m-1}{m}) \beta^{i\eta} \\ &\cdot \exp \left[i\eta \left\{ \frac{\Gamma'(\xi)}{\Gamma(\xi)} - \log \beta + \int_{-\infty}^{-0} \frac{x^2}{1+x^2} \frac{e^{\xi x}}{(1-e^x)} dx \right\} \right. \\ &+ \left. \int_{-\infty}^{-0} \left(e^{i\eta x} - 1 - \frac{i\eta x}{1+x^2} \right) \frac{e^{\xi x}}{(1-e^x)|x|} dx \right] \\ &\exp \left[i\eta \left\{ \frac{\Gamma'(\xi + 1/m)}{\Gamma(\xi + 1/m)} - \log \beta + \int_{-\infty}^{-0} \frac{x^2}{1+x^2} \frac{e^{(\xi+1/m)x}}{(1-e^x)} dx \right\} \right. \\ &+ \left. \int_{-\infty}^{-0} \left(e^{i\eta x} - 1 - \frac{i\eta x}{1+x^2} \right) \frac{e^{(\xi+1/m)x}}{(1-e^x)|x|} dx \right] \\ &\cdots \exp \left[i\eta \left\{ \frac{\Gamma'(\xi + (m-1)/m)}{\Gamma(\xi + (m-1)/m)} - \log \beta \right. \right. \\ &+ \left. \left. \int_{-\infty}^{-0} \frac{x^2}{1+x^2} \frac{e^{(\xi+(m-1)/m)x}}{(1-e^x)} dx \right\} \right. \\ &+ \left. \int_{-\infty}^{-0} \left(e^{i\eta x} - 1 - \frac{i\eta x}{1+x^2} \right) \frac{e^{(\xi+(m-1)/m)x}}{(1-e^x)|x|} dx \right] \\ &= \Gamma(\xi) \Gamma(\xi + \frac{1}{m}) \cdots \Gamma(\xi + \frac{m-1}{m}) \beta^{im\eta} \\ &\cdot \exp \left[i\eta \left\{ \frac{\Gamma'(\xi)}{\Gamma(\xi)} + \frac{\Gamma'(\xi + 1/m)}{\Gamma(\xi + 1/m)} + \cdots + \frac{\Gamma'(\xi + (m-1)/m)}{\Gamma(\xi + (m-1)/m)} \right\} \right. \\ &- i\eta m \log \beta + i\eta \int_{-\infty}^{-0} \frac{x^2}{1+x^2} \frac{e^{\xi x}}{(1-e^{x/m})} dx \left. \right\} \\ &+ \int_{-\infty}^{-0} \left(e^{i\eta x} - 1 - \frac{i\eta x}{1+x^2} \right) \frac{e^{\xi x}}{(1-e^{x/m})|x|} dx \left. \right]. \end{aligned}$$

By change of variable we see that

$$\begin{aligned} &\int_{-\infty}^{-0} \left(e^{i\eta x} - 1 - \frac{i\eta x}{1+x^2} \right) \frac{e^{\xi x}}{(1-e^{x/m})|x|} dx \\ &= \int_{-\infty}^{-0} \left(e^{im\eta x} - 1 - \frac{im\eta x}{1+x^2} \right) \frac{e^{m\xi x}}{(1-e^x)|x|} dx \end{aligned}$$

$$+ \int_{-\infty}^{-0} \left(\frac{im\eta x}{1+x^2} - \frac{im\eta x}{1+m^2x^2} \right) \frac{e^{m\xi x}}{(1-e^x)|x|} dx$$

and

$$\int_{-\infty}^{-0} \frac{x^2}{1+x^2} \frac{e^{\xi x}}{(1-e^{x/m})} dx = \int_{-\infty}^{-0} \frac{m^3 x^2}{1+m^2 x^2} \frac{e^{m\xi x}}{(1-e^x)} dx.$$

Therefore we obtain

$$\begin{aligned} & \int_{-\infty}^{-0} \left(\frac{im\eta x}{1+x^2} - \frac{im\eta x}{1+m^2x^2} \right) \frac{e^{m\xi x}}{(1-e^x)|x|} dx \\ & + i\eta \int_{-\infty}^{-0} \frac{m^3 x^2}{1+m^2 x^2} \frac{e^{m\xi x}}{(1-e^x)} dx = im\eta \int_{-\infty}^{-0} \frac{x^2}{1+x^2} \frac{e^{m\xi x}}{(1-e^x)} dx \end{aligned}$$

and so we obtain

$$\begin{aligned} & \prod_{k=0}^{m-1} \Gamma(\xi + i\eta + \frac{k}{m}) = \Gamma(\xi) \Gamma(\xi + \frac{1}{m}) \cdots \Gamma(\xi + \frac{m-1}{m}) \beta^{im\eta} \\ & \cdot \exp \left[i\eta \left\{ \frac{\Gamma'(\xi)}{\Gamma(\xi)} + \frac{\Gamma'(\xi + 1/m)}{\Gamma(\xi + 1/m)} + \cdots + \frac{\Gamma'(\xi + (m-1)/m)}{\Gamma(\xi + (m-1)/m)} \right\} \right. \\ & \left. - i\eta m \log \beta + im\eta \int_{-\infty}^{-0} \frac{x^2}{1+x^2} \frac{e^{m\xi x}}{(1-e^x)} dx \right] \\ & + \int_{-\infty}^{-0} \left(e^{im\eta x} - 1 - \frac{im\eta x}{1+x^2} \right) \frac{e^{m\xi x}}{(1-e^x)|x|} dx. \end{aligned} \quad (20)$$

We obtain

$$\begin{aligned} & \frac{m\Gamma'(m\xi)}{\Gamma(m\xi)} - \frac{\Gamma'(\xi)}{\Gamma(\xi)} + \frac{\Gamma'(\xi + 1/m)}{\Gamma(\xi + 1/m)} + \cdots + \frac{\Gamma'(\xi + (m-1)/m)}{\Gamma(\xi + (m-1)/m)} \\ & = m \int_{+0}^{\infty} \frac{e^{-t} - e^{-mt}}{t} dt \\ & = m \log m \end{aligned} \quad (21)$$

and

$$\begin{aligned} & \frac{\Gamma(\xi) \Gamma(\xi + 1/m) \cdots \Gamma(\xi + (m-1)/m)}{\Gamma(m\xi)} \\ & = \frac{\Gamma(1/m) \Gamma(2/m) \cdots \Gamma((m-1)/m)}{m^{m\xi-1}} = (2\pi)^{(m-1)/2} m^{1/2-m\xi}. \end{aligned} \quad (22)$$

From the above results we see that

$$\begin{aligned}
\Pi_{k=0}^{m-1} \Gamma(\xi + i\eta + \frac{k}{m}) &= (2\pi)^{(m-1)/2} m^{1/2-m(\xi+i\eta)} \Gamma(m\xi) \beta^{im\eta} \\
&\cdot \exp \left[im\eta \left\{ \frac{\Gamma'(m\xi)}{\Gamma(m\xi)} - \log \beta + \int_{-\infty}^{-0} \frac{x^2}{1+x^2} \frac{e^{m\xi x}}{(1-e^x)} dx \right\} \right. \\
&+ \left. \int_{-\infty}^{-0} \left(e^{im\eta x} - 1 - \frac{im\eta x}{1+x^2} \right) \frac{e^{m\xi x}}{(1-e^x)|x|} dx \right] \\
&= (2\pi)^{(m-1)/2} m^{1/2-m(\xi+i\eta)} \Gamma(m(\xi + i\eta))
\end{aligned} \tag{23}$$

for a positive number ξ . By analytic continuation we obtain Gauss's formula (19).

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