

TINGLEY'S PROBLEM ON FUNCTION SPACES

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1. INTRODUCTION

Tingley's problem, introduced by Tingley in 1987 [32], asks whether every surjective isometry between the unit spheres of Banach spaces is extended to a surjective isometry between the whole spaces. Various solutions have been proposed for Tingley's problem. Wang suggested the first solution [33], who worked on the space of all continuous functions that vanish at infinity on a locally compact Hausdorff space (see [34]). Many interesting results have shown that Tingley's problem has been solved in specific spaces, and no counterexample has been found.

According to [36, p.730], Ding was the first to consider Tingley's problem in different types of spaces [13]. Ding [14, Corollary 2] proved that the real Banach space consisting of all null sequences of real numbers satisfies the Mazur-Ulam property. Liu [20] also contributed early to Tingley's problem on different types of spaces. Later, Cheng and Dong [8] formally introduced the concept of the Mazur-Ulam property.

Definition 1.1. A real or complex Banach space B has the *Mazur-Ulam property* if any surjective isometry from the unit sphere of B onto the unit sphere of another real or complex Banach space B' admits an extension to a surjective real-linear isometry from B onto B' .

In their studies, Tan [26, 27, 28] demonstrated that the space $L^p(\mathbb{R})$ for σ -finite positive measure space has the Mazur-Ulam property. Additionally, Boyko, Kadets, Martín, and Werner introduced two concepts, C -richness [4, Definition 2.3] and lushness [4, Definition 2.1], for subspaces of continuous functions. They also proved that a C -rich subspace is lush [4, Theorem 2.4]. In another study, Tan, Huang and Liu [29] introduced the notion of local GL (generalized lush) spaces and showed that every local GL space has the Mazur-Ulam property.

Tanaka [31] introduced a new direction in investigating Tingley's problem by presenting a positive solution for the Banach algebra of complex matrices. Mori and Ozawa [22] demonstrated that the Mazur-Ulam property holds for unital C^* -algebras and real von Neumann algebras. Cueto-Avellaneda and Peralta [11] proved that the complex

(or real) Banach space of all continuous maps taking values in a complex (or real) Hilbert space has the Mazur-Ulam property (cf. [12]). The results of Becerra-Guerrero, Cueto-Avellaneda, Fernández-Polo, and Peralta [2] and Kalenda and Peralta [19] demonstrated that any JBW*-triple has the Mazur-Ulam property. Peralta and Švarc [25] extended the results of Mori and Ozawa [22] for unital JB*-algebras.

The Mazur-Ulam property for a Banach space of dimension 2 remained unresolved for many years. The final solution was presented by the remarkable and outstanding advance of Banach [1], who proved that any Banach space of dimension 2 has the Mazur-Ulam property. The problem regarding a Banach space of finite dimension greater than 2 remains open. The study of the Mazur-Ulam property is currently a challenging subject (cf. [5, 35]). Jiménez-Vargas, Morales-Compoy, Peralta, and Ramírez [18, Theorems 3.8, 3.9] likely provided the first example of complex Banach spaces that have the complex Mazur-Ulam property (cf. [24]). Hatori [16] formally introduced the concept of the complex Mazur-Ulam property.

Definition 1.2. A complex Banach space B is said to have the *complex Mazur-Ulam property*, emphasizing the term ‘complex’, if for any surjective isometry from the unit sphere of B onto the unit sphere of another complex Banach space B' admits an extension to a surjective real-linear isometry from B onto B'

Note that a complex Banach space has the complex Mazur-Ulam property provided that it has the Mazur-Ulam property as a real Banach space since a complex Banach space is a real Banach space simultaneously.

In the paper [16], the complex Mazur-Ulam property for uniform algebras is proved. It is shown that the existence of a unit in a uniform algebra is crucial for the proof of this property. The problem of the complex Mazur-Ulam property for a uniformly closed algebra on a locally compact Hausdorff space is discussed in the same paper.

In a recent paper by Cueto-Avellaneda, Hirota, Miura and Peralta [10], it is demonstrated that every surjective isometry between the unit spheres of two uniformly closed algebras on locally compact Hausdorff spaces, which separate the points without common zeros, can be extended to a surjective real linear isometry between these algebras. Cabezas, Cueto-Avellaneda, Hirota, Miura and Peralta [6] have recently proved the complex Mazur-Ulam property for a commutative JB*-triple. Both results concern the spaces of continuous functions

without constants. Peralta [23] gives the first example of an infinite-dimensional non-commutative C^* -algebra containing no unitaries and with the Mazur-Ulam property.

In this paper, we further study the problem of the complex Mazur-Ulam property. We introduce a separation condition named $(**)$ for a Banach space in section 4. We prove that a real (resp. complex) Banach space, which satisfies the condition $(**)$, has the (resp. complex) Mazur-Ulam property. An extremely C -regular subspace satisfies the condition $(*)$. Fleming and Jamison introduced it cite[Definition 2.3.9]fj1, which is a generalization of an extremely regular subspace coined by Cengiz [9].

2. NOTATIONS AND TERMINOLOGIES

Throughout this paper, we will use the following notations: B , B_1 , and B_2 will always refer to real or complex Banach spaces. For a real or complex Banach space B the unit sphere $\{a \in B : \|a\| = 1\}$ of B is denoted by $S(B)$ and the closed unit ball $\{a \in B : \|a\| \leq 1\}$ by $\text{Ball}(B)$. The set of all maximal convex subsets of $S(B)$ is denoted by $\mathfrak{F}_B$. We denote by $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{C}$) the set of all real (resp. complex) numbers. We denote the open unit disk in $\mathbb{C}$ by D , the closed unit disk by $\bar{D}$, and $\mathbb{T} = \{z \in \mathbb{K} : |z| = 1\}$. Throughout the paper, Y denotes a locally compact Hausdorff space, and X is a compact Hausdorff space. The space of all $\mathbb{K}$ -valued continuous functions on Y , which vanish at infinity, is denoted by $C_0(Y, \mathbb{K})$. If Y is compact, then we simply denote $C(Y, \mathbb{K})$ instead of $C_0(Y, \mathbb{K})$. The supremum norm on a subset W of Y is denoted by $\|\cdot\|_{\infty(W)}$ or $\|\cdot\|_{\infty}$. For a function $f \in C_0(Y, \mathbb{K})$ and $S \subset Y$, we denote the restriction of f on S by $f|_S$. For $A \subset C_0(Y, \mathbb{K})$ and $S \subset Y$, we denote $A|_S = \{f|_S : f \in A\}$.

We will also use $T: S(B_1) \rightarrow S(B_2)$ to refer to a surjective isometry without assuming any linearity. We write $\mathbb{T} = \{\lambda \in \mathbb{K} : |\lambda| = 1\}$ for a $\mathbb{K}$ -Banach space, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We say that the set of all extreme points of the unit ball of the dual space B^* of B by the Choquet boundary for B , that is,

$$\text{Ch}(B) = \{p \in B^* : p \text{ is an exteme point of } \text{Ball}(B^*)\}.$$

This notation may not be familiar, but we can assume that every Banach space can be viewed as a closed space of $C_0(\text{Ball}(B^*) \setminus \{0\})$. This means that the set of all extreme points of the closed unit ball of the dual space can be considered the Choquet boundary for B . In the following, we suppose that a $\mathbb{K}$ -Banach space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is a closed subspace of $C_0(\text{Ball}(B^*) \setminus \{0\}, \mathbb{K})$.

3. THE HOMOGENEOUS EXTENSION IS THE UNIQUE CANDIDATE

The homogeneous extension $\tilde{T} : B_1 \rightarrow B_2$ of T is ;

$$\tilde{T}(f) = \begin{cases} \|f\|T\left(\frac{f}{\|f\|}\right), & f \neq 0 \\ 0, & f = 0 \end{cases}$$

$\tilde{T}$ is a surjective function that preserves the norm. Thus, to solve Tingley's problem, we need to prove that $\tilde{T}$ is additive; that is, $\tilde{T}(f + g) = \tilde{T}(f) + \tilde{T}(g)$ for all $f, g \in B_1$. Any subset of $S(B_j)$, which is a singleton, is convex. By Zorn's lemma, we have a non-empty family $\mathfrak{F}_B$ consisting of all maximal convex subsets of $S(B_j)$ such that $\bigcup_{F \in \mathfrak{F}_B} F = S(B)$. Each $F \in \mathfrak{F}_B$ can be indexed by $\text{Ch}(B) \times \mathbb{T}$, but it needs not to be unique.

Lemma 3.1. *For every $F \in \mathfrak{F}_B$, there exists a $(p, \lambda) \in \text{Ch}(B) \times \mathbb{T}$ such that $F = \{f \in S(B) : f(p) = \lambda\}$.*

By the axiom of choice, there exists $P \subset \text{Ch}(B)$ such that

$$\begin{array}{ccc} L_B : P \times \mathbb{T} & \longrightarrow & \mathfrak{F}_B \\ \Psi & & \Psi \\ (p, \lambda) & \longrightarrow & F_{p,\lambda} \end{array}$$

is a bijection. Here we denote $F_{p,\lambda} = \{f \in S(B) : f(p) = \lambda\}$. We call P a set of representatives for $\mathfrak{F}_B$. Note that a set of representatives does not need to be unique. A key gradient is

Theorem 3.2. *T preserves maximal convex subsets in both directions.*

$$F \in \mathfrak{F}_{B_1} \longleftrightarrow T(F) \in \mathfrak{F}_{B_2}$$

This is originally exhibited by Cheng and Dong [8]. But a crystal clear proof is given by Tanaka [30]. It induces the bijection

$$\mathbf{T} : \mathfrak{F}_{B_1} \rightarrow \mathfrak{F}_{B_2},$$

which defines the lift Φ of $T : S(B_1) \rightarrow S(B_2)$ by

$$\begin{array}{ccc} P_1 \times \mathbb{T} & \xrightarrow{\Phi} & P_2 \times \mathbb{T} \\ L_{B_1} \downarrow & \circlearrowleft & \downarrow L_{B_2} \\ \mathfrak{F}_{B_1} & \xrightarrow{\mathbf{T}} & \mathfrak{F}_{B_2} \end{array}$$

Denoting

$$\Phi(p, \lambda) = (\phi(p, \lambda), \tau(p, \lambda)), \quad (p, \lambda) \in P_1 \times \mathbb{T},$$

we have

$$\mathbf{T}(F_{p,\lambda}) = F_{\phi(p,\lambda),\tau(p,\lambda)}, \quad (p, \lambda) \in P_1 \times \mathbb{T}.$$

Rewriting it, we arrive at an important equality

$$(3.1) \quad (T(f))(\phi(p, \lambda)) = \tau(p, \lambda), \quad f \in F_{p,\lambda}.$$

Let $p \in P_1$ be fixed. Suppose that we can prove :

- [1] $\phi(p, \lambda)$ does not depend on λ : say $\phi(p, \lambda) = \phi(p)$,
- [2] $\tau(p, \lambda) = \tau(p, 1)\lambda$ or $\tau(p, 1)\bar{\lambda}$.

The equation (3.1) is as

$$(T(f))(\phi(p)) = \tau(p, 1)\lambda \text{ or } \tau(p, 1)\bar{\lambda}, \quad f \in F_{p,\lambda}$$

for any $\lambda \in \mathbb{T}$. As $\lambda = f(p)$ for $f \in F_{p,\lambda}$ we infer that

$$(3.2) \quad (T(f))(\phi(p)) = \tau(p, 1)f(p) \text{ or } \tau(p, 1)\overline{f(p)}, \quad f \in S(B_1) \text{ with } |f(p)| = 1$$

under the conditions

- [1] $\phi(p, \lambda)$ does not depend on λ : say $\phi(p, \lambda) = \phi(p)$,
- [2] $\tau(p, \lambda) = \tau(p, 1)\lambda$ or $\tau(p, 1)\bar{\lambda}$.

Suppose that (3.2) holds for any $f \in S(B_1)$. We will arrive at the final destination. Then, under the conditions

- [1] $\phi(p, \lambda)$ does not depend on λ : say $\phi(p, \lambda) = \phi(p)$,
- [2] $\tau(p, \lambda) = \tau(p, 1)\lambda$ or $\tau(p, 1)\bar{\lambda}$
- [3] $(T(f))(\phi(p)) = \tau(p, 1)f(p)$ or $\tau(p, 1)\overline{f(p)}$, $\forall f \in S(B_1)$

we infer that

$$(3.3) \quad (\tilde{T}(f))(\phi(p)) = \tau(p, 1)f(p) \text{ or } \tau(p, 1)\overline{f(p)} \quad \forall f \in B_1$$

Recall that the homogeneous extension $\tilde{T} : B_1 \rightarrow B_2$ is defined by

$$\tilde{T}(f) = \begin{cases} \|f\|T\left(\frac{f}{\|f\|}\right), & f \neq 0, \\ 0, & f = 0. \end{cases}$$

By (3.3) we have for $f, g \in B_1$

$$\begin{aligned} (\tilde{T}(f+g))(\phi(p)) &= \tau(p, 1)(f+g)(p) \text{ or } \tau(p, 1)\overline{(f+g)(p)} \\ &= \tau(p, 1)(f(p) + g(p)) \text{ or } \tau(p, 1)\overline{f(p) + g(p)} \\ &= (\tilde{T}(f))(\phi(p)) + (\tilde{T}(g))(\phi(p)) \\ &= (\tilde{T}(f) + \tilde{T}(g))(\phi(p)). \end{aligned}$$

If the set of all $p \in P_1$, which satisfies the above equality, is a norming family for B_1 , we have

$$(3.4) \quad \tilde{T}(f+g) = \tilde{T}(f) + \tilde{T}(g), \quad f, g \in B_1.$$

We arrive at the final destination. We conclude that

Proposition 3.3. *Suppose that the set of all $p \in P_1$ which satisfies*

- [1] $\phi(p, \lambda)$ does not depend on λ : say $\phi(p, \lambda) = \phi(p)$,
- [2] $\tau(p, \lambda) = \tau(p, 1)\lambda$ or $\tau(p, 1)\bar{\lambda}$,
- [3] $(T(f))(\phi(p)) = \tau(p, 1)f(p)$ or $\tau(p, 1)\overline{f(p)}$, $\forall f \in S(B_1)$

is a norming family of B_1 . Then, the homogeneous extension $\tilde{T}$ of T is additive, which means that B_1 has the Mazur-Ulam property.

4. LOOKING FORWARD A NEAT, SUFFICIENT CONDITION FOR [1], [2] AND [3]

In [17], we consider the condition (*).

Definition 4.1 (Definition 5.1 in [17]). We say that B satisfies the condition (*) whenever there exists a set of representative P for $\mathfrak{F}_B$ with the condition : for every $p \in P, \varepsilon > 0$, and a closed subset F of P with respect to the relative topology induced by the weak*-topology on B^* such that $p \notin F$, there exists $a \in S(B)$ such that $p(a) = 1$ and $|q(a)| \leq \varepsilon$ for all $q \in F$.

Fleming and Jamison [15, Definition 2.3.9] introduced the extremely C -regular space, which is a generalization of an extremely regular space defined by Cengiz [9]. We denote the supremum norm of $f \in C_0(Y, \mathbb{K})$ by $\|f\|_\infty$.

Definition 4.2 (Definition 2.3.9 in [15], [9]). A $\mathbb{K}$ -linear subspace E of $C_0(Y, \mathbb{K})$ for a locally compact Hausdorff space Y is called an extremely C -regular space (resp. regular) if for each $x \in \text{Ch}(E)$ (resp. $x \in Y$) satisfies the condition that for each $\varepsilon > 0$ and each open neighborhood U of x , there exists $f \in E$ such that $f(x) = 1 = \|f\|_\infty$, and $|f| < \varepsilon$ on $Y \setminus U$.

Note that for a uniformly closed extremely C -regular $\mathbb{K}$ -linear subspace E of $C_0(Y, \mathbb{K})$ which separates the point of Y , $x \in \text{Ch}(E)$ if and only if x is a strong boundary point for E if and only if the representing measure for the point evaluation τ_x for x on E is only the Dirac measure at x (see [17, Theorem 3.11]). Note also that a uniformly closed subalgebra of $C_0(Y, \mathbb{K})$ which separates the point of Y is extremely C -regular.

Example 4.3 (Example 5.2 in [17]). Suppose that E is a uniformly closed C -regular $\mathbb{K}$ -linear subspace of $C_0(Y, \mathbb{K})$ for a locally compact Hausdorff space Y which separates the point of Y . Suppose that $P = \{\tau_x : x \in \text{Ch}(E)\}$. Then, P is a set of representatives, and the condition

(*) holds with P . Thus a uniformly closed subalgebra of $C_0(Y, \mathbb{K})$ which separates the point of Y satisfies the condition (*).

Theorem 4.4 (Theorem 6.3 in [17]). *Suppose that a real (resp. complex) Banach space satisfies the condition (*). Then B has the (resp. complex) Mazur-Ulam property.*

We have the following from Theorem 4.4. Cabezas, Cueto-Avellaneda, Hirota, Miura and Peralta [6] studies the complex Mazur-Ulam property for commutative JB^* -triples. In particular, [6, Corollary 3.2] exhibits the same result for $C_0(Y, \mathbb{C})$. Recall that a uniform algebra A on a compact Hausdorff space X is a closed subalgebra of $C(X, \mathbb{C})$ which separates the points of X and contains constants.

Corollary 4.5 (Corollaries 6.4 in [17]). *A uniformly closed extremely C -regular real (resp. complex) space has the (resp. complex) Mazur-Ulam property. In particular, a closed subalgebra of $C_0(Y, \mathbb{C})$ and a uniform algebra have the complex Mazur-Ulam property.*

Suppose that $P(\bar{D})$ be a disk algebra on the closed unit disk $\bar{D}$ in the complex plane; i.e.,

$$P(\bar{D}) = \{f \in C(\bar{D}, \mathbb{C}) : f \text{ is analytic on the open unit disk } D\}.$$

Then $\overline{\operatorname{Re} P(\bar{D})}$ is a space of harmonic function on D . As the unit circle $\mathbb{T}$ is the Šilov boundary for $\overline{\operatorname{Re} P(\bar{D})}$ and $\overline{\operatorname{Re} P(\bar{D})}|_{\mathbb{T}} = C(\mathbb{T}, \mathbb{R})$, we infer that $\overline{\operatorname{Re} P(\bar{D})}$ has the Mazur-Ulam property. In general, as a result, we see the following.

Corollary 4.6. *Let A be a uniform algebra. Then, the uniform closure $\overline{\operatorname{Re} A}$ of the real part $\operatorname{Re} A$ of A has the Mazur-Ulam property.*

For a compact subset K in the complex plane, $R(K)$ (resp. $A(K)$) denotes the uniform algebra, which consists of complex-valued continuous functions that are uniformly approximated on K by rational functions with poles off K (resp. analytic on the interior of K). By Corollary 4.6, the space of harmonic functions on the interior of K , $\overline{\operatorname{Re} R(K)}$ and $\overline{\operatorname{Re} A(K)}$ have the Mazur-Ulam property. Inspired by Corollary 4.6, we may consider the following problem about the Mazur-Ulam property for space of harmonic functions.

Problem 4.7. *Let K be a non-empty compact subset of the complex plane. Let*

- (i) $H_0(K) = \{u \in C(K, \mathbb{R}) : u \text{ is harmonic on the interior of } K\}$,
- (ii) $H(K) = \{u \in C(K, \mathbb{R}) : u \text{ is uniformly approximated on } K \text{ by harmonic functions on open sets which include } K\}$.

Do $H_0(K)$ and $H(K)$ have the Mazur-Ulam property?

Note that

$$\overline{\text{Re}P}(\bar{D}) = H_0(\bar{D}) = H(\bar{D}).$$

It is known that $H_0(K)$ nor $H(K)$ do not satisfy the condition (*). We version up the condition (*). In general, it is known that if every boundary point of K is a peak point for $R(K)$, then $H(K)|_{\partial K} = C(\partial K, \mathbb{R})$, where ∂K denotes the boundary of K . In this case, $H(K)$, hence and $H_0(K)$ have the Mazur-Ulam property. In particular, if $\mathbb{C} \setminus K$ has a finite number of components, then every boundary point of K is a peak point; hence, $R(K)$ has the Mazur-Ulam property. The necessary and sufficient condition for K such that every boundary point of K is a peak point for $R(K)$ is known as a theorem of Melnikov (see. [37]).

Definition 4.8. Suppose that P is a set of representatives for $\mathfrak{F}_B$. Let $Q \subset P$. We say that $p \in Q$ is a strong boundary point for Q in the sense of Fleming and Jamison if the following holds: for any open neighborhood U of p and positive ε , there exists $f \in S(B)$ such that $f(p) = 1$, $|f| < \varepsilon$ on $Q \setminus U$.

Definition 4.9. B satisfies the condition (**) if and only if there exists $\emptyset \neq Q \subset P$ such that every $p \in Q$ is a strong boundary point for Q in the sense of Fleming and Jamison, and Q is a norming family for B ; i.e. $\|f\| = \sup_{p \in Q} |f(p)|$ for every $f \in B$.

In general, if B satisfies the condition (*), then it satisfies the condition (**) with $Q = P$. A closed subalgebra of $C_0(Y)$, which separates the points of Y , particularly a uniform algebra, is an extremely C -regular space. Hence, it satisfies (**). A space of harmonic functions

$$\{u \in C(K, \mathbb{R}) : u \text{ is harmonic in the interior of } K\}$$

for a plane compacts K need not be an extremely C -regular space, but satisfies (**).

Theorem 4.10. Suppose that a real (resp. complex) Banach space satisfies the condition (**). Then B has the (resp. complex) Mazur-Ulam property.

We can prove Theorem 4.10 in a similar way as Theorem 4.4. We omit proof. It is known that the set of all peak points for $R(K)$ is dense in the boundary ∂K of K . Hence $H(K)$ and $H_0(K)$ satisfies the condition (**). We have

Corollary 4.11. The spaces $H(K)$ and $H_0(K)$ have the Mazur-Ulam property.

5. APPENDIX

In this section, A denotes a uniform algebra on a compact Hausdorff space X . We present an idea for proving that a uniform algebra has the complex Mazur-Ulam property. We focus on the Hausdorff distance between the maximal convex sets. The Hausdorff distance $d_H(\cdot, \cdot)$ between these sets is crucial for proving [1] and [2]. Recall that

- [1] $\phi(p, \lambda)$ does not depend on λ : say $\phi(p, \lambda) = \phi(p)$,
- [2] $\tau(p, \lambda) = \tau(p, 1)\lambda$ or $\tau(p, 1)\bar{\lambda}$

The Hausdorff distance $d_H(\cdot, \cdot)$ between $F_{p,\lambda}$ and $F_{q,\mu}$ in $\mathfrak{F}_B$ is

$$d_H(F_{p,\lambda}, F_{q,\mu}) = \max\left\{ \sup_{f \in F_{p,\lambda}} \left(\inf_{h \in F_{q,\mu}} \|f - h\| \right), \sup_{h \in F_{q,\mu}} \left(\inf_{f \in F_{p,\lambda}} \|h - f\| \right) \right\}$$

One can prove that any two-point-subset $\{p, q\} \subset \text{Ch}(A)$ is a peak interpolation set for A . In particular, for any $\lambda, \mu \in \mathbb{T}$ there exists a function $f \in S(A)$ such that $f(p) = \lambda$ and $f(q) = \mu$. It follows that

$$(5.1) \quad d_H(F_{p,\lambda}, F_{q,\mu}) = 2, \quad \lambda, \mu \in \mathbb{T}$$

for every pair of different points $p, q \in \text{Ch}(A)$ Through a simple calculation

$$d_H(F_{p,\lambda}, F_{p,\lambda'}) = |\lambda - \lambda'|, \quad p \in P, \lambda, \lambda' \in \mathbb{T},$$

with which (5.1) ensures [1]. Suppose that [1] does not hold: There exists (p, λ) and (p, λ') such that $\phi(p, \lambda) \neq \phi(p, \lambda')$. We may assume that $|\lambda - \lambda'| < 2$. By (5.1)

$$\sharp\{(q, \mu) \in P_1 \times \mathbb{T} : d_H(F_{p,\lambda}, F_{q,\mu}) = |\lambda - \lambda'| \} = 2,$$

say, $(q, \mu) = (p, \lambda')$, (p, μ) where μ is the symmetric point of λ' with respect to λ .

On the other hand, since $\mathbf{T}$ preserves the Hausdorff distance

$$\sharp\{(q, \mu) \in P_2 \times \mathbb{T} : d_H(F_{\phi(p,\lambda), \tau(p,\lambda)}, F_{q,\mu}) = |\lambda - \lambda'| \} \geq 3.$$

In fact, at least $(q, \mu) = (\phi(p, \lambda'), \tau(p, \lambda'))$, $(\phi(p, \lambda), \lambda \bar{\lambda}' \tau(p, \lambda))$, and $(\phi(p, \lambda), \lambda'' \tau(p, \lambda))$, where λ'' is the symmetric point of $\lambda \bar{\lambda}' \tau(p, \lambda)$ with respect to $\tau(p, \lambda)$. This is a contradiction because $\mathbb{T}$ preserves the Hausdorff distance. Furthermore

Proposition 5.1. *We have*

$$[1] \quad \phi(p, \lambda) = \phi(p, \lambda') \text{ for every } p \in P_1 \text{ and } \lambda, \lambda' \in \mathbb{T}.$$

Letting

$$P_1^+ = \{p \in P_1 : \tau(p, i) = i\tau(p, 1)\}$$

and

$$P_1^- = \{p \in P_1 : \tau(p, i) = -i\tau(p, 1)\},$$

we have $P_1^+ \cup P_1^- = P_1$ and

$$[2] \quad \begin{cases} \tau(p, \lambda) = \lambda\tau(p, 1), & p \in P_1^+, \lambda \in \mathbb{T} \\ \tau(p, \lambda) = \bar{\lambda}\tau(p, 1), & p \in P_1^-, \lambda \in \mathbb{T}. \end{cases}$$

We obtain a so-called the additive Bishop lemma, and we conclude that

$$\begin{aligned} & \{f \in S(A) : f(p) = \alpha\} \\ &= \left\{ f \in S(A) : d\left(f, F_{p, \frac{\alpha}{|\alpha|}}\right) \leq 1 - |\alpha|, d\left(f, F_{p, \frac{\alpha}{|\alpha|}}\right) \leq 1 + |\alpha| \right\}. \end{aligned}$$

for every $|\alpha| \leq 1$, where

$$d(f, F_{p, \lambda}) = \inf\{\|f - g\| : g \in F_{p, \lambda}\}.$$

Then we have [3]. It follows by Proposition 3.3 that a uniform algebra has the Mazur-Ulam property.

Acknowledgments. This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University. The author was supported by JSPS KAKENHI Grant Number JP19K03536.

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