

ON CRITICAL VALUES OF THE TENSOR PRODUCT L -FUNCTIONS FOR $\mathrm{GSp}_4 \times \mathrm{GSp}_4$

SHIH-YU CHEN

1. MAIN RESULT

Let Σ and Π be cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$. We denote by

$$L(s, \Sigma \times \Pi) = \prod_v L(s, \Sigma_v \times \Pi_v), \quad \mathrm{Re}(s) \gg 0$$

the degree 16 tensor product L -function associated to Σ and Π . It admits meromorphic continuation to the whole complex plane and satisfies a functional equation relating $L(s, \Sigma \times \Pi)$ to $L(1-s, \Sigma^\vee \times \Pi^\vee)$. Assume Σ_∞ and Π_∞ are holomorphic discrete series representations of weights $(\underline{\kappa}; \mathbf{u})$ and $(\underline{\kappa}'; \mathbf{u}')$ respectively in $\mathbb{Z}^2 \times \mathbb{Z}$ with $\underline{\kappa} = (\kappa_1, \kappa_2)$, $\underline{\kappa}' = (\kappa'_1, \kappa'_2)$ such that

$$\kappa_1 \geq \kappa_2 \geq 3, \quad \kappa'_1 \geq \kappa'_2 \geq 3, \quad \kappa_1 + \kappa_2 \equiv \mathbf{u} \pmod{2}, \quad \kappa'_1 + \kappa'_2 \equiv \mathbf{u}' \pmod{2}.$$

In this case, for each $\sigma \in \mathrm{Aut}(\mathbb{C})$, the irreducible admissible representations

$${}^\sigma \Sigma := \Sigma_\infty \otimes {}^\sigma \Sigma_f, \quad {}^\sigma \Pi := \Pi_\infty \otimes {}^\sigma \Pi_f$$

are cuspidal automorphic. Let Σ_+ and Π_+ be the space of cusp forms in Σ and Π whose archimedean component are highest weight vectors in the minimal $\mathrm{U}(2)$ -type of Σ_∞ and Π_∞ respectively. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we then have σ -linear $\mathrm{GSp}_4(\mathbb{A}_f)$ -equivariant isomorphisms

$$\begin{aligned} \Sigma_+ &\longrightarrow {}^\sigma \Sigma_+, & \varphi &\longmapsto {}^\sigma \varphi \\ \Pi_+ &\longrightarrow {}^\sigma \Pi_+, & \varphi &\longmapsto {}^\sigma \varphi \end{aligned}$$

defined via conjugation of σ on the Fourier coefficients of cusp forms along the Siegel parabolic subgroup of GSp_4 . A critical point for $L(s, \Sigma \times \Pi)$ is an integer m such that the local factors $L(s, \Sigma_\infty \times \Pi_\infty)$ and $L(1-s, \Sigma_\infty^\vee \times \Pi_\infty^\vee)$ are holomorphic at $s = m$. Note that there exist critical points if and only if

$$\{\kappa_1 + \kappa_2 - 2, \kappa_1 - \kappa_2 + 2\} \cap \{\kappa'_1 + \kappa'_2 - 2, \kappa'_1 - \kappa'_2 + 2\} = \emptyset.$$

In particular, we must have $\kappa_1 - \kappa_2 \neq \kappa'_1 - \kappa'_2$. Without loss of generality, we may assume $\kappa_1 - \kappa_2 > \kappa'_1 - \kappa'_2$. We then have three critical types according to the relative position of $\kappa_1 + \kappa_2 - 2, \kappa_1 - \kappa_2 + 2, \kappa'_1 + \kappa'_2 - 2, \kappa'_1 - \kappa'_2 + 2$:

$$\begin{cases} \kappa'_1 - \kappa'_2 + 2 < \kappa_1 - \kappa_2 + 2 < \kappa'_1 + \kappa'_2 - 2 < \kappa_1 + \kappa_2 - 2 & \text{Case 1,} \\ \kappa'_1 - \kappa'_2 + 2 < \kappa_1 - \kappa_2 + 2 < \kappa_1 + \kappa_2 - 2 < \kappa'_1 + \kappa'_2 - 2 & \text{Case 2,} \\ \kappa'_1 - \kappa'_2 + 2 < \kappa'_1 + \kappa'_2 - 2 < \kappa_1 - \kappa_2 + 2 < \kappa_1 + \kappa_2 - 2 & \text{Case 3.} \end{cases}$$

Based on

- Deligne's conjecture [Del79] on critical values of motivic L -functions, in the special case for the tensor product of two motives of rank 4;
- the computation of Yoshida [Yos01, §3 and §4] on the fundamental periods of a motive of rank 4 associated to a Siegel cusp eigenform of degree two;

we expect that the algebraicity of critical values for $L(s, \Sigma \times \Pi)$ should be expressed in terms of a monomial (an automorphic analogue of Deligne's periods) in

$$\begin{aligned} & 2\pi\sqrt{-1}, \quad G(\omega_\Sigma), \quad G(\omega_\Pi) \quad (\text{Gauss sums}) \\ & (\sqrt{-1})^u \cdot \langle f_\Sigma, f_\Sigma \rangle, \quad (\sqrt{-1})^{u'} \cdot \langle f_\Pi, f_\Pi \rangle \quad (\text{Petersson norms}) \\ & (\sqrt{-1})^u \cdot L^{(\infty)}(1, \Sigma, \text{Ad}) \cdot \langle f_\Sigma, f_\Sigma \rangle^{-1} \quad (\text{adjoint } L\text{-value / Petersson norm}). \end{aligned}$$

More precisely, we propose the following conjecture.

Conjecture. Fix non-zero $f_\Sigma \in \Sigma_+$ and $f_\Pi \in \Pi_+$. For a critical point $m \in \mathbb{Z}$, we have

$$\sigma \left(\frac{L^{(\infty)}(m, \Sigma \times \Pi)}{(2\pi\sqrt{-1})^{8m} \cdot q(\Sigma \times \Pi)} \right) = \frac{L^{(\infty)}(m, {}^\sigma\Sigma \times {}^\sigma\Pi)}{(2\pi\sqrt{-1})^{8m} \cdot q({}^\sigma\Sigma \times {}^\sigma\Pi)}, \quad \sigma \in \text{Aut}(\mathbb{C}).$$

Here

$$\begin{aligned} & q(\Sigma \times \Pi) \\ &= (2\pi\sqrt{-1})^{4u+4u'} \cdot G(\omega_\Sigma)^4 G(\omega_\Pi)^4 \\ & \times \begin{cases} (2\pi\sqrt{-1})^{\kappa'_1+\kappa'_2-6} \cdot (\sqrt{-1})^{u'} \cdot L^{(\infty)}(1, \Sigma, \text{Ad}) \cdot \langle f_\Pi, f_\Pi \rangle & \text{Case 1,} \\ (2\pi\sqrt{-1})^{-\kappa_1-\kappa_2+2\kappa'_1+2\kappa'_2-6} \cdot (\sqrt{-1})^u \cdot L^{(\infty)}(1, \Sigma, \text{Ad}) \cdot \langle f_\Sigma, f_\Sigma \rangle^{-1} \cdot \langle f_\Pi, f_\Pi \rangle^2 & \text{Case 2,} \\ (2\pi\sqrt{-1})^{-2\kappa_1-2\kappa_2-12} \cdot L^{(\infty)}(1, \Sigma, \text{Ad})^2 \cdot \langle f_\Sigma, f_\Sigma \rangle^{-2} & \text{Case 3.} \end{cases} \end{aligned}$$

In particular, if f_Σ and f_Π are rational over some number field E , then

$$\frac{L^{(\infty)}(m, \Sigma \times \Pi)}{(2\pi\sqrt{-1})^{8m} \cdot q(\Sigma \times \Pi)} \in E.$$

Our main result is the following theorem.

Theorem. The conjecture holds under the following regularity conditions:

- (1) $\kappa_1 - \kappa_2 > \kappa'_1 - \kappa'_2 \geq 1$.
- (2) $\min \{ |\kappa_1 + \varepsilon(\kappa_2 - 2) - \kappa'_1 - \varepsilon'(\kappa'_2 - 2)|, 2\kappa_2 - 4, 2\kappa'_2 - 4 \mid \varepsilon, \varepsilon' \in \{\pm 1\} \} \geq \begin{cases} 4 & \text{if } u \text{ and } u' \text{ are odd,} \\ 5 & \text{if } u + u' \text{ is odd,} \\ 6 & \text{if } u \text{ and } u' \text{ are even.} \end{cases}$
- (3) In Case 1, $\kappa_2 \geq 7$, $\kappa'_2 \geq 7$, and $\min\{\kappa_1 + \kappa_2 - \kappa'_1 - \kappa'_2, \kappa_1 - \kappa_2 - \kappa'_1 + \kappa'_2\} \geq 5$.
- (4) In Case 2, $\kappa_2 \geq 7$, $\kappa'_2 \geq 7$, and $\kappa'_1 + \kappa'_2 - \kappa_1 - \kappa_2 \geq 5$.
- (5) In Case 3, $\kappa_2 \geq 7$.

2. SKETCH OF PROOF

Let Ψ and Φ be the functorial lifts of Σ and Π to $\text{GL}_4(\mathbb{A}_{\mathbb{Q}})$ with respect to the standard representation $\text{GSp}_4(\mathbb{C}) \rightarrow \text{GL}_4(\mathbb{C})$. In particular,

$$L(s, \Sigma \times \Pi) = L(s, \Psi \times \Phi),$$

where the right-hand side is the Rankin-Selberg L -function for $\text{GL}_4 \times \text{GL}_4$. Since $\kappa_1 > \kappa_2 \geq 4$ and $\kappa'_1 > \kappa'_2 \geq 4$, Σ and Π are non-CAP (cf. [ACI⁺23, Proposition A.3-(2)]). This implies that Ψ and Φ are regular algebraic in the sense of Clozel [Clo90] with essentially tempered

archimedean components. Consider two auxiliary isobaric automorphic representations of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$

$$\Psi' := \Pi_1 \boxplus \Pi_2, \quad \Phi' := \Pi_3 \boxplus \Pi_4,$$

where $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ are cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. Assume further that $\Psi_{\infty} \simeq \Psi'_{\infty}$ and $\Phi_{\infty} \simeq \Phi'_{\infty}$. By the cross-ratio formula [Che23, Theorem 4.3], under the regularity conditions (1) and (2), for any critical point $m \in \mathbb{Z}$ for $L(s, \Sigma \times \Pi)$ we have

$$\frac{L(m, \Psi \times \Phi) \cdot L(m, \Psi' \times \Phi')}{L(m, \Psi \times \Phi') \cdot L(m, \Psi' \times \Phi)} \sim 1,$$

provided the denominator is non-zero. The formula can be rewrite as

$$(2.1) \quad L(m, \Sigma \times \Pi) \sim \frac{L(m, \Sigma \times \Pi_3) \cdot L(m, \Sigma \times \Pi_4) \cdot L(m, \Pi \times \Pi_1) \cdot L(m, \Pi \times \Pi_2)}{\prod_{1 \leq i \leq 2, 3 \leq j \leq 4} L(m, \Pi_i \times \Pi_j)}.$$

For the denominator of the right-hand side of (2.1), we have the algebraicity result of Shimura [Shi76] for $\mathrm{GL}_2 \times \mathrm{GL}_2$. Therefore, we are reduced to the study of algebraicity of critical values for $\mathrm{GSp}_4 \times \mathrm{GL}_2$.

Let Σ^{gen} and Π^{gen} be the globally generic cuspidal automorphic representations which are nearly equivalent to Σ and Π respectively. By comparing the rational structures via the global Whittaker models and the interior coherent cohomology of the Siegel modular threefold, we can define the de Rham–Whittaker periods

$$p_1(\Sigma^{\mathrm{gen}}), p_2(\Sigma^{\mathrm{gen}}), p_1(\Pi^{\mathrm{gen}}), p_2(\Pi^{\mathrm{gen}})$$

associated to Σ^{gen} and Π^{gen} . Here the subscripts refer to the cohomological degree. Based on the integral representation for $\mathrm{GSp}_4 \times \mathrm{GL}_2$ suggested by Novodvorsky in [Nov79, § 3] (see also [PSS84, Theorem 1.1]) and its cohomological interpretation, we prove that the algebraicity of the numerator of the right-hand side of (2.1) can be expressed in terms of the de Rham–Whittaker periods. Note that in the unbalanced case for $\mathrm{GSp}_4 \times \mathrm{GL}_2$, the algebraicity follows from our result [Che23, Theorem 5.7] applied to $\mathrm{GL}_4 \times \mathrm{GL}_2$. Finally, by

- comparing with the algebraicity results of Morimoto [Mor14], [Mor18] for $\mathrm{GSp}_4 \times \mathrm{GL}_2$;
- the algebraicity of the Petersson norms of cusp forms in Σ^{gen} and Π^{gen} in terms of their adjoint L -values proved in [CI23] and [Che22];

we obtain period relations between the de Rham–Whittaker periods and $\langle f_{\Sigma}, f_{\Sigma} \rangle, \langle f_{\Pi}, f_{\Pi} \rangle, L(1, \Sigma, \mathrm{Ad})$. The regularity conditions (3)–(5) are imposed so that various assumptions in the above mentioned algebraicity results have non-empty intersection.

Acknowledgement. This report is based on a talk given by the author at the RIMS conference “Arithmetic aspects of automorphic forms and automorphic representations”, held at the Research Institute for Mathematical Sciences, Kyoto University, in January 2025. The author would like to express his sincere gratitude to Professor Kenichi Namikawa and Professor Keiichi Gunji for the kind invitation and the opportunity to present at the conference.

The research of the author was supported in part by the National Science and Technology Council of R.O.C. [113-2115-M-007-002-MY3].

REFERENCES

- [ACI⁺23] H. Atobe, M. Chida, T. Ibukiyama, H. Katsurada, and T. Yamauchi. Harder’s conjecture I. *J. Math. Soc. Japan*, 75(4):1339–1408, 2023.
- [Che22] S.-Y. Chen. Algebraicity of critical values of adjoint L -functions for GSp_4 . *Ramanujan J.*, 59:883–931, 2022.

- [Che23] S.-Y. Chen. Algebraicity of ratios of Rankin–Selberg L -functions and applications to Deligne’s conjecture. 2023. arXiv:2205.15382.
- [CI23] S.-Y. Chen and A. Ichino. On Petersson norms of generic cusp forms and special values of adjoint L -functions for GSp_4 . *Amer. J. Math.*, 145:899–993, 2023.
- [Clo90] L. Clozel. Motifs et Formes Automorphes: Applications du Principe de Fonctorialité. In *Automorphic Forms, Shimura Varieties, and L -functions, Vol. I*, Perspectives in Mathematics, pages 77–159, 1990.
- [Del79] P. Deligne. Valeurs de fonctions L et périodes d’intégrales. In *Automorphic Forms, Representations and L -Functions*, volume 33, Part 2 of *Proceedings of Symposia in Pure Mathematics*, pages 313–346. Amer. Math. Soc., 1979.
- [Mor14] K. Morimoto. On L -functions for quaternion unitary groups of degree 2 and $\mathrm{GL}(2)$ (with an Appendix by M. Furusawa and A. Ichino). *Int. Math. Res. Not.*, 2014(7):1729–1832, 2014.
- [Mor18] K. Morimoto. On tensor product L -functions for quaternion unitary groups and $\mathrm{GL}(2)$ over totally real fields: Mixed weight cases. *Adv. Math.*, 337:317–362, 2018.
- [Nov79] M. E. Novodvorsky. Automorphic L -functions for symplectic group $\mathrm{GSp}(4)$. In *Automorphic Forms, Representations and L -Functions*, volume 33, Part 2 of *Proceedings of Symposia in Pure Mathematics*, pages 87–95. Amer. Math. Soc., 1979.
- [PSS84] I. I. Piatetski-Shapiro and D. Soudry. L and ε functions for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$. *Proc. Natl. Acad. Sci. USA*, 81:3924–3927, 1984.
- [Shi76] G. Shimura. The special values of the zeta functions associated with cusp forms. *Comm. Pure Appl. Math.*, pages 783–804, 1976.
- [Yos01] H. Yoshida. Motives and Siegel modular forms. *Amer. J. Math.*, 123(6):1171–1197, 2001.

DEPARTMENT OF MATHEMATICS, NATIONAL TSING HUA UNIVERSITY, 101, SECTION 2, KUANG-FU ROAD, HSINCHU 300, TAIWAN, ROC

Email address: sychen.math@gmail.com