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A refinement of quantum mechanics by algorithmic randomness (extended abstract)

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The notion of probability plays a crucial role in quantum mechanics. It appears in quantum mechanics as the so-called *Born rule*, i.e., *the probability interpretation of the wave function*. In modern mathematics which describes quantum mechanics, however, probability theory means nothing other than measure theory, and therefore any operational characterization of the notion of probability is still missing in quantum mechanics. In this sense, the current form of quantum mechanics is considered to be imperfect as a physical theory which must stand on operational means.

We present an alternative rule to the Born rule based on the toolkit of *algorithmic randomness* without reference to the notion of probability for the purpose of making quantum mechanics perfect. Algorithmic randomness, also known as *algorithmic information theory*, is a field of mathematics which enables us to consider the randomness of an *individual* object. It originated in the groundbreaking works of Solomonoff [11], Kolmogorov [7], and Chaitin [1] in the mid-1960s. They independently introduced the notion of *program-size complexity*, also known as *Kolmogorov complexity*, in order to quantify the randomness of an individual object. Around the same time, Martin-Löf [8] introduced a measure theoretic approach to characterize the randomness of an individual infinite binary sequence. This approach, called *Martin-Löf randomness* nowadays, is one of the major notions in algorithmic randomness as well as program-size complexity. Later on, in the 1970s Schnorr and Chaitin showed that Martin-Löf randomness is equivalent to the randomness defined by program-size complexity in characterizing random infinite binary sequences. In the 21st century, algorithmic randomness makes remarkable progress through close interaction with recursion theory. See [10, 5] for the recent development as well as the historical detail of algorithmic randomness.

We use the notion of *Martin-Löf randomness with respect to Bernoulli measure* to state the alternative rule to the Born rule for specifying the property of the results of quantum measurements in an operational way. As the first step of the research of this line, we only consider, for simplicity, the case where the set of all possible outcomes of a quantum measurement is finite, while the state space itself of the quantum system is allowed to have an infinite dimension.

Recall that the Born rule of quantum mechanics is given as the following postulate:

Postulate 1 (The Born rule). *Quantum measurement is described by an* observable, *M, a Hermitian operator on the state space of the system being measured. The observable has a spectral decomposition*

$$M = \sum_{k=1}^{N} \lambda_k E_k,$$

where E_k is the projector onto the eigenspace of M with eigenvalue λ_k. The possible outcomes of the measurement correspond to the eigenvalues, λ_k, of the observable. If the state of the quantum system is $|\Psi\rangle$ immediately before the measurement then the probability *that result λ_k occurs is given by $\langle\Psi|E_k|\Psi\rangle$, and the state of the system after the measurement is*

$$\frac{E_k|\Psi\rangle}{\sqrt{\langle\Psi|E_k|\Psi\rangle}}.$$

□

Thus, the Born rule uses the notion of probability. However, the operational characterization of the notion of probability is not given in the Born rule, and therefore its relation to an specific infinite sequence of outcomes of quantum measurements which are being generated by an infinitely repeated measurements is unclear. We try to fix this point.

Let $\Omega = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ be a finite alphabet, which serves as *the set of all possible measurement outcomes*, and Ω^∞ the set of all infinite sequences over Ω. We introduce the notion of *Martin-Löf P-randomness* for an infinite sequence over Ω, where $P = (p_1, p_2, \ldots, p_N)$ is an N-tuple of non-negative reals such that $p_1 + p_2 + \cdots + p_N = 1$. *Intuitively*, a Martin-Löf P-random sequence is an individual 'typical' infinite sequence which is obtained as a result by performing an infinitely repeated Bernoulli trials where the 'probability' p_k is associated with the outcome λ_k for each $k = 1, 2, \ldots, N$.

Let us identify the form of the postulate of quantum measurements as it ought to be, from a general point of view. Consider the sequence of the outcomes of quantum measurements, which is an element of Ω^∞. All that the experimenter of quantum measurements can obtain through the measurements about quantum system is such a specific infinite sequence of outcomes of the measurements which are being generated by infinitely repeated measurements. Thus, the object about which the postulate of quantum measurements makes a statement should be the properties of an specific infinite sequence of outcomes of the measurements. Suggested by this consideration, we propose to replace the Born rule, Postulate 1, by the following postulate:

Postulate 2. *Quantum measurement is described by an* observable, *M, a Hermitian operator on the state space of the system being measured. The observable has a spectral decomposition*

$$M = \sum_{k=1}^{N} \lambda_k E_k,$$

where E_k is the projector onto the eigenspace of M with eigenvalue λ_k. The possible outcomes of the measurement is in the spectrum $\Omega = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of M. Suppose that the measurements are repeatedly performed over identical quantum systems whose states are all $|\Psi\rangle$, and the infinite sequence $\alpha \in \Omega^\infty$ of measurement outcomes is being

generated. Then α is Martin-Löf P-random, where $P = (\langle\Psi|E_1|\Psi\rangle, \ldots, \langle\Psi|E_N|\Psi\rangle)$. For each of the measurements, the state of the system immediately after the measurement is

$$\frac{E_k|\Psi\rangle}{\sqrt{\langle\Psi|E_k|\Psi\rangle}}, \tag{1}$$

where λ_k is the corresponding measurement outcome. □

We show that Postulate 2 is certainly a refinement of the Born rule from the point of view of our intuitive understanding of the notion of probability. For example, according to Postulate 2 we can show that the law of large numbers, i.e., the frequency interpretation, holds for the infinite sequence $\alpha \in \Omega^\infty$ in Postulate 2. On the other hand, we verify the *self-consistency* of Postulate 2 on some level, which suggests that Postulate 2 is not too strong.

So far we have only considered the case of pure states. According to Postulate 2, the result of the quantum measurements forms a Martin-Löf P-random sequence of states each of which is of the form (1). In the conventional quantum mechanics, this result is described as a mixed state. Suggested by this, we give a mathematical definition of the notion of a mixed state in terms of Martin-Löf P-randomness. We then propose to replace the Born rule about mixed states by a rule based on algorithmic randomness.

Finally, we consider the validity of our new rules, in particular, based on *the many-worlds interpretation of quantum mechanics* (*MWI*, for short) [6]. MWI is more than just an interpretation of quantum mechanics. It aims to recover the predictions of quantum mechanics without assuming the Born rule, Postulate 1. For that purpose, MWI usually assumes that our world is 'typical' or 'random' among many coexisting worlds. However, the proposal of MWI by Everett was nearly a decade earlier than the advent of algorithmic randomness, and this assumption of 'typicality' was not rigorous. The notion of 'typicality' or 'randomness' is just the research object of algorithmic randomness. Based on a generalization of the notion of Martin-Löf P-randomness, we introduce a postulate, called *the principle of typicality*, which is a refinement of the assumption of 'typicality' by Everett. We then show that all of our new rules can be derived from the principle of typicality in a unified way. In particular, the principle of typicality is equivalent to Postulate 2 in the case of pure states.

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