

An example of one-dimensional Cohen-Macaulay local rings that possess only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules

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In our paper [GN] with K. Nishida we proved the following

Theorem 1. Let $P = k[[X_1, X_2, \dots, X_n]]$ be a formal power series ring over an algebraically closed field k of $\text{ch } k \neq 2$. Let $R = P/I$, where I is an ideal of P and suppose $\dim R = d \geq 2$. Then the following two conditions are equivalent.

- (1) R is a regular local ring.
- (2) R is a Cohen-Macaulay ring that possesses only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules.

When this is the case, the syzygy modules of the residue class field k of R are the representatives of indecomposable maximal Buchsbaum modules and so there are exactly d non-isomorphic indecomposable maximal Buchsbaum modules over R .

Our contribution in the above theorem is the implication (2) $\Rightarrow$ (1). The last assertion and the implication (1) $\Rightarrow$ (2) are due to [G] (see also [EG, Theorem 3.2]), where some consequences of the result are discussed too.

I would like to note here that the assumption $\dim R \geq 2$ in Theorem 1 is not superfluous. When $\dim R = 1$, maximal Buchsbaum R -modules M are characterized by the condition that

$$\dim_R M = 1 \quad \text{and} \quad \underline{m} \cdot H_{\underline{m}}^0(M) = (0)$$

(here $H_{\underline{m}}^i(\cdot)$ denotes the i th local cohomology functor of R relative to the maximal ideal $\underline{m}$ of R). This condition (is of course not too much weak but) seems not quite strong. Nevertheless in some sense surprisingly, there exist such Cohen-Macaulay local rings R of $\dim R = 1$ that are non-regular but possess only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules. In my lecture I will explore the typical example $R = k[[t^2, t^3]]$.

Now let k be a field and $S = k[[t]]$ a formal power series ring over k . We put $R = k[[t^2, t^3]]$. Then R and S are the only indecomposable maximal Cohen-Macaulay R -modules (cf. [H, Satz 1.6]) and the R -module S has a resolution of the following form

$$\dots \rightarrow R^2 \xrightarrow{\begin{pmatrix} t^3 & t^4 \\ -t^2 & -t^3 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} t^3 & t^4 \\ -t^2 & -t^3 \end{pmatrix}} R^2 \xrightarrow{\varepsilon} S \rightarrow 0,$$

where $\varepsilon\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = a + bt$. Therefore we have an embedding

$\sigma : S \rightarrow R^2$ which sends 1 (resp. t) to $\begin{pmatrix} t^3 \\ -t^2 \end{pmatrix}$ (resp. $\begin{pmatrix} t^4 \\ -t^3 \end{pmatrix}$)

and makes the diagram

$$\begin{array}{ccccc} S & \xrightarrow{\sigma} & R^2 & \xrightarrow{\varepsilon} & S \\ \downarrow t & & \downarrow \rho & & \downarrow -t \\ S & \xrightarrow{\sigma} & R^2 & \xrightarrow{\varepsilon} & S \end{array}$$

commutative, where $\rho = \begin{pmatrix} 0 & -t^2 \\ -1 & 0 \end{pmatrix}$. Let $A = (a_{ij})$ and $B =$

(b_{ij}) be $m \times m$ matrices with entries in k and let

$\phi : (R^2)^m \rightarrow (R^2)^m$ be the homomorphism defined by

$$\phi((x_i)) = \left(\sum_{j=1}^m a_{ij} x_j + \sum_{j=1}^m b_{ij} \rho(x_j) \right).$$

Then we clearly have

Lemma 2. *The diagram*

$$\begin{array}{ccccc} S^m & \xrightarrow{\sigma^m} & (R^2)^m & \xrightarrow{\epsilon^m} & S^m \\ \downarrow A+tB & & \downarrow \phi & & \downarrow A-tB \\ S^m & \xrightarrow{\sigma^m} & (R^2)^m & \xrightarrow{\epsilon^m} & S^m \end{array}$$

is commutative (here σ^m and ϵ^m respectively denote the direct sum of m copies of σ and ϵ).

Let $\underline{m}$ ($= t^2 S$) denote the maximal ideal of R and let N be an R -submodule of S such that $\underline{m}$ is contained in N . We put $M = R^2/\sigma(N)$. Then

Proposition 3. *M is an indecomposable maximal Buchsbaum R -module with $H_{\underline{m}}^0(M) = S/N$.*

Proof. Considering the exact sequence

$$0 \rightarrow S/N \rightarrow M \rightarrow S \rightarrow 0,$$

we get $H_{\underline{m}}^0(M) = S/N$ as $\underline{m} \cdot (S/N) = (0)$ and as S is Cohen-Macaulay; so M is a maximal Buchsbaum R -module. Assume that $M = M_1 \oplus M_2$ for some non-zero submodules M_1 and M_2 . Then M_i 's are cyclic, since M is generated by two elements. If $\dim_{R/\underline{m}} M_i = 1$ for $i = 1, 2$, the isomorphisms $S \cong M/H_{\underline{m}}^0(M) \cong M_1/H_{\underline{m}}^0(M_1) \oplus M_2/H_{\underline{m}}^0(M_2)$ claim that S is decomposable. Hence

$\dim_R M_i = 0$ for some i , say $i = 2$. Then M_2 is contained in $H_{\underline{m}}^0(M)$ and so S is a homomorphic image of M_1 — this is impossible, because M_1 is cyclic while S is not. Thus we see M is indecomposable.

We define

$$M_1 = R^2/\sigma(\underline{m}), \quad M_2 = R^2/\sigma(R), \quad \text{and} \quad M_3 = R^2/\sigma(tS).$$

By Proposition 3 M_i 's are indecomposable maximal Buchsbaum R -modules and $M_1 \not\cong M_i$ ($i = 2, 3$), since

$$\begin{aligned} \dim_k H_{\underline{m}}^0(M_i) &= 2 & (i = 1), \\ &= 1 & (i = 2, 3). \end{aligned}$$

M_2 is of homological dimension 1 but M_3 is not; so $M_2 \not\cong M_3$. The goal of my lecture is the following

Theorem 4. M_1, M_2, M_3, S and R are the indecomposable maximal Buchsbaum R -modules.

To prove this theorem we need one more lemma 5, the proof of which is routine (use the induction on the size of matrices C) and shall be omitted.

Lemma 5. Let C be an $m \times n$ matrix with entries in S/t^2S . Then there exist an invertible $m \times m$ matrix P with entries in S/t^2S and an invertible $n \times n$ matrix Q with entries in k such that PCQ has the following form

with exact rows and columns. Here we consider N to be an R -submodule of S^m and the homomorphism $i : N \rightarrow S^m$ to be the inclusion map. Hence $\underline{m}.S^m$ is contained in N , as $V \cong S^m/N$. Let $\tau : S^m \rightarrow S^m/\underline{m}S^m = (S/t^2S)^m$ denote the canonical epimorphism. We put $U = \tau(N)$ and $n = \dim_k U$. If $n = 0$, then $N = \underline{m}S^m$ and so $M = (R^2/\sigma(\underline{m}))^m$. Consequently, we get $m = 1$ and $M = M_1$.

Now suppose that $n \geq 1$ and let $v_1, v_2, \dots, v_n$ be a k -basis of U . Let us apply Lemma 5 to the $m \times n$ matrix $C = (v_1, v_2, \dots, v_n)$. Then Lemma 5 asserts that by some automorphism P of $(S/t^2S)^m$, U is mapped onto the k -subspace U' which is spanned by the columns of an $m \times n$ matrix of the following form:

$$(\#) \quad \begin{pmatrix} 1 & & & & & & & t \\ & \ddots & & & & & 0 & \ddots \\ & & \ddots & & & & 0 & \ddots \\ & & & \ddots & & & 0 & \ddots \\ & & & & \ddots & & 0 & \ddots \\ & & & & & \ddots & 0 & \ddots \\ & & & & & & 0 & \ddots \\ & & & & & & 0 & \ddots \end{pmatrix} \pmod{t^2S}.$$

Let L be the R -submodule of S^m generated by the columns of the above matrix $(\#)$ and put $N' = \underline{m}S^m + L$. Then clearly $U' = \tau(N')$.

We write $P = A + tB \pmod{t^2S}$ with $m \times m$ matrices A

and B with entries in k . Then since the following diagram

$$\begin{array}{ccc} S^m & \xrightarrow{\tau} & (S/t^2S)^m \\ \downarrow A+tB & & \downarrow P \\ S^m & \xrightarrow{\tau} & (S/t^2S)^m \end{array}$$

is commutative and since $U' = \tau(N')$, we get that $N' = (A+tB)N$.

Let us now recall the diagram in Lemma 2:

$$\begin{array}{ccccccc} 0 & \rightarrow & S^m & \xrightarrow{\sigma^m} & (R^2)^m & \xrightarrow{\epsilon^m} & S^m \rightarrow 0 \\ & & \downarrow A+tB & & \downarrow \phi & & \downarrow A-tB \\ 0 & \rightarrow & S^m & \xrightarrow{\sigma^m} & (R^2)^m & \xrightarrow{\epsilon^m} & S^m \rightarrow 0 \end{array} .$$

Then as the rows of this diagram are exact and as both the matrices $A + tB$ and $A - tB$ are invertible, the middle ϕ has to be an isomorphism whence, via ϕ , we find that

$$\begin{aligned} M &= (R^2)^m / \sigma^m(N) \\ &\cong (R^2)^m / \sigma^m(N') . \end{aligned}$$

Consequently we may assume that $N = N'$. The condition that M is indecomposable now causes a very tight restriction on the form of the matrix (#) above. We readily see that $m = 1$ and the matrix (#) must be one of

$$(1 \ t), (1) \text{ and } (t) .$$

Thus $M = R^2/\sigma(S)$ ($= S$), $M = R^2/\sigma(R)$ ($= M_2$), or $M = R^2/\sigma(tS)$ ($= M_3$) as claimed. This completes the proof of Theorem 4.

References

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