

On a theorem of de Franchis

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1 Introduction

Let X be a compact Riemann surface of genus $g (> 1)$. De Franchis [1] stated the following:

Theorem 1 (de Franchis) (a) *For a fixed compact Riemann surface Y of genus > 1 , the number of nonconstant holomorphic maps $X \rightarrow Y$ is finite.*

(b) *There are only finitely many compact Riemann surfaces Y_i of genus > 1 which admit a nonconstant holomorphic map from X .*

The second statement (b) is often attributed to Severi. After knowing the finiteness of maps, we may ask if there exists an upper bound depending only on some topological invariant, for example, the genus g . Related to the statement (a), the author [4] showed that the bound is smaller than $(cg)^{2g}$ for some constant c .

Now, we consider a bound for holomorphic maps when Y is not fixed, that is, we estimate the number of all nonconstant holomorphic maps from X to other Riemann surfaces. Let $f_i : X \rightarrow Y_i$ be nonconstant holomorphic maps for $i = 1, 2$. We say that f_1 and f_2 are isomorphic if and only if there is a conformal map $h : Y_1 \rightarrow Y_2$ such that $h \circ f_1 = f_2$. Let $\mathcal{I}_\gamma(X)$ denote the set of all isomorphic classes of nonconstant holomorphic maps into compact Riemann surfaces of genus $\gamma > 1$, and denote $\mathcal{I}(X) = \bigcup_{\gamma > 1} \mathcal{I}_\gamma(X)$. By the theorem of de Franchis, we see that $\#\mathcal{I}(X)$ is finite. In 1983 Howard and Sommese [2] first showed that there is a bound on $\#\mathcal{I}(X)$ depending only on g .

Let

$$M(g) = \max_X \{\#\mathcal{I}(X)\},$$

where the maximum is taken over all Riemann surfaces X of genus g . It is an interesting problem to determine the exact rate of growth of $M(g)$. The author [5] showed

$$M(g) \leq (cg)^{5g}$$

for some constant c and it was the best upper bound depending only on g .

In this note we will improve the bound and show

$$M(g) \leq (cg)^{2g}$$

for some constant c .

On the other hand, Kani [3] also constructed a sequence of Riemann surfaces of genera $g_1 < g_2 < \dots < g_n < \dots$, such that the number of isomorphic classes of nonconstant holomorphic maps of each Riemann surface is larger than $\exp(c(\log(g_n))^2)$ for some constant $c > 0$ (independent of n). It implies that $M(g)$ cannot be bounded by any polynomial in g .

2 The bound

In the following, we will refer to [5] for all of the notation and lemmata. In [5], the leading term of the upper bound was depend on Lemma 3 (p.3060) and the Proposition (p.3062). We improve them as follows.

Lemma 3' *Let $f_1 : X \rightarrow Y_1$ be a holomorphic map of degree d , and $\mathfrak{f}_1 : \mathcal{J}(X) \rightarrow \mathcal{J}(Y_1)$ be the homomorphism induced by f_1 . Take an arbitrary $u \in {}^t\mathfrak{f}_1(\widehat{\mathcal{J}(Y_1)})$. Then, the number of isomorphic classes of holomorphic maps $f_i : X \rightarrow Y_i$ of degree d such that the dual map ${}^t\mathfrak{f}_i : \widehat{\mathcal{J}(Y_i)} \rightarrow \widehat{\mathcal{J}(X)}$ of the induced homomorphism $\mathfrak{f}_i$ satisfies $u \in {}^t\mathfrak{f}_i(\widehat{\mathcal{J}(Y_i)})$ is at most $\binom{2g-2}{d} \times \binom{4g-4}{d}$.*

In [5], the conclusion was $\binom{2g-2}{d} \times (2g-1)^d$ which is now replaced by $\binom{2g-2}{d} \times \binom{4g-4}{d}$.

Proof. The assumption means that there exist holomorphic differentials ϕ_1 on Y_1 and ϕ_i on Y_i such that their pull backs satisfy $f_{1*}\phi_1 = f_{i*}\phi_i$.

Then, for a zero p_{01} of ϕ_1 , the number of possible $f_1^{-1}(p_{01})$ (counting multiplicities) that can occur is at most $\binom{2g-2}{d}$. After determining $\phi = f_{1*}\phi_1$ and $f_1^{-1}(p_{01})$, we can show that there are at most $\binom{4g-4}{d}$ possible isomorphic classes of holomorphic maps of degree d as follows.

Let $f_i : X \rightarrow Y_i$ be holomorphic maps ($i = 1, 2$). Suppose that there are holomorphic differentials ϕ_1 and ϕ_2 on Y_1 and Y_2 , respectively, with $f_{1*}\phi_1 = f_{2*}\phi_2$, and there is a zero p_{01} (resp. p_{02}) of ϕ_1 (resp. ϕ_2) satisfying $f_1^{-1}(p_{01}) = f_2^{-1}(p_{02})$. We put $\phi = f_{1*}\phi_1 = f_{2*}\phi_2$.

Let $\tilde{p}_0 \in f_1^{-1}(p_{01}) = f_2^{-1}(p_{02})$. Take a sufficiently small neighbourhood $U_{\tilde{p}_0}$ (resp. $U_{p_{0i}}$) of $\tilde{p}_0$ (resp. p_{0i}) so that there is no zero of ϕ (resp. ϕ_i) on $U_{\tilde{p}_0}$ (resp. $U_{p_{0i}}$) except $\tilde{p}_0$ (resp. p_{0i}), and that $f_i(U_{\tilde{p}_0}) \subset U_{p_{0i}}$ ($i = 1, 2$). We may take a local coordinate z (resp. z_i) on $U_{\tilde{p}_0}$ (resp. $U_{p_{0i}}$) such that $z(\tilde{p}_0) = 0$ (resp. $z_i(p_{0i}) = 0$) and the differential is written as

$$\phi = z^m dz \quad (\text{resp. } \phi_i = z_i^{n_i} dz_i).$$

Recalling that $f_1^{-1}(p_{01}) = f_2^{-1}(p_{02})$, we see $n_1 = n_2$ and we will denote it by n for brevity. We take two real lines $\gamma_i : [0, a) \rightarrow U_{p_{0i}}$ with $\gamma_i(t) = t \in \mathbb{R}$ in the local coordinates z_i ($i = 1, 2$). For an arbitrary $\tilde{p} \in U_{\tilde{p}_0} \setminus \{\tilde{p}_0\}$,

$$\int_0^{\tilde{p}} z^m dz = \int_0^{f_1(\tilde{p})} z_1^n dz_1 = \int_0^{f_2(\tilde{p})} z_2^n dz_2,$$

hence the number of possible positions for the set of lifts of γ_1 (thus also those of γ_2) in $U_{\tilde{p}_0}$ is at most $m + 1$. Accordingly, the total number of possible positions for the set of all the lifts of γ_1 is at most $\binom{4g-4}{d}$.

Let $\{\tilde{p}_{0j}\}_{j=1}^N = f_1^{-1}(p_{01}) (= f_2^{-1}(p_{02}))$. Suppose that, for every $\tilde{p}_{0j} \in f_1^{-1}(p_{01})$, $U_{\tilde{p}_{0j}} \cap f_1^{-1}(\gamma_1) = U_{\tilde{p}_{0j}} \cap f_2^{-1}(\gamma_2)$, that is, the set of lifts of γ_1 coincide with that of γ_2 . Then, it is easy to see that we can define a local conformal map $h : f_1(U_{\tilde{p}_{0j}}) \rightarrow f_2(U_{\tilde{p}_{0j}})$ such that $h \circ f_1|_{U_{\tilde{p}_{0j}}} = f_2|_{U_{\tilde{p}_{0j}}}$. We want to extend it to a global conformal map from Y_1 to Y_2 , and actually it is possible. Indeed, for an arbitrary point $p \in Y_1$, we will draw a curve c from p_{01} to p avoiding branch points of f_1 other than possibly at p_{01} and p . Let $\tilde{c}$ and $\tilde{c}'$ be two lifts of c by f_1 . Then, we see that $f_2(\tilde{c}) = f_2(\tilde{c}')$ since $h \circ f_1$ is well-defined near $\tilde{p}_{0j}$ ($j = 1, \dots, N$). It implies that h is well-defined on Y_1 . It is easy to see that h is invertible. $\square$

Proposition' *Let $f_i : X \rightarrow Y_i$ be nonconstant holomorphic maps, and $\mathcal{F}_i$ be the rational representations of the endomorphisms associated with f_i ($i = 1, 2$). Suppose*

that, for some $k < 2g$,

$$\begin{cases} {}^t\mathcal{F}_1 a_1 = \dots = {}^t\mathcal{F}_1 a_{k-1} = 0, \\ {}^t\mathcal{F}_2 a_1 = \dots = {}^t\mathcal{F}_2 a_{k-1} = 0, \end{cases}$$

and that there exists some integer $l > 2g - 2$ such that ${}^t\mathcal{F}_1 a_k \equiv {}^t\mathcal{F}_2 a_k \pmod{l}$ holds. Then ${}^t\mathcal{F}_1 a_k = {}^t\mathcal{F}_2 a_k$.

If, in addition, Y_1 and Y_2 are of the same genus γ , then the assumption $l > 2g - 2$ can be replaced by $l > (2g - 2)/(\gamma - 1)$.

In [5], we assumed $l > (2g - 2)^2$. But in Proposition', we only need $l > 2g - 2$.

Proof. Let $D = \mathcal{F}_1 - \mathcal{F}_2$. Then, D is the rational representation of some endomorphism of $J(X)$. By an easy calculation, we see ${}^tD' = {}^tD$. We note that ${}^tD' x, a_1, \dots, a_{k-1}$ are linearly independent for any vector $x \in \mathbb{R}^{2g}$ if ${}^tD' x$ is not zero. Indeed, using Lemma 1 in [5], we see $({}^tD' x, a_j)_X = (x, {}^tD a_j)_X = 0$ for $j = 1, \dots, k-1$ by the assumption. Thus, ${}^tD' x, a_1, \dots, a_{k-1}$ are linearly independent. By the assumption, ${}^tD' a_k \equiv 0 \pmod{l}$ thus the vector ${}^tD' a_k$ can be written in the form ${}^tD' a_k = l \times n$, where $n \in \mathbb{Z}^{2g}$. Thus, if it is not 0, then

$$||{}^tD a_k|| \geq l\lambda_k.$$

We also have

$$||{}^tD a_k|| \leq ||{}^t\mathcal{F}_1 a_k|| + ||{}^t\mathcal{F}_2 a_k|| \leq d_1 ||a_k|| + d_2 ||a_k||,$$

where d_i is the degree of f_i ($i = 1, 2$). The first inequality is just the triangle inequality, and the second one is obtained by Lemma 2.

Therefore, we have

$$||{}^tD a_k|| \leq ||a_k|| (d_1 + d_2) = (d_1 + d_2) \lambda_k.$$

By Riemann-Hurwitz formula, $d_i \leq g - 1$ and we see that ${}^tD a_k$ must be 0 since $l > 2(g - 1)$.

A little modification of above argument lead us to the conclusion for the case Y_1 and Y_2 are of the same genus γ . $\square$

Now we will get the improved bound. Just the same consideration as in [5, p.3063], we have

$$\#\mathcal{I}_\gamma(X) < \sum_{d>1} (2g - 2\gamma + 1) \times \left\{ \left(\frac{2g - 2}{\gamma - 1} \right) + 1 \right\}^{2g} \times \binom{2g - 2}{d} \times \binom{4g - 4}{d}.$$

Observing $\binom{m}{d} \leq 2^m$, we see that the right hand side is smaller than

$$\left\{\left(\frac{2g-2}{\gamma-1}\right) + 1\right\}^{2g} \times 2^{2g-2} \times 2^{4g-4} \times (2g-2\gamma+1)(g-\gamma)/(\gamma-1).$$

Summing up for all possible γ , we get

$$M(g) \leq (cg)^{2g}$$

for some constant c .

References

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